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ON RECTANGULAR HYPERBOLAS CIRCUMSCRIBING A TRIANGLE

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ABSTRACT. In this article we study rectangular hyperbolas circumscribing a triangle and their generation in which participate the Steiner lines of the triangle. Using elements of the related procedure we prove several properties of these hyperbolas. In addition we study a pencil of hyperbolas naturally associated to such a rectangular hyperbola and a related parabola carrying the centers of the members of the pencil.

1. INTRODUCTION

The aim in this article is to show that a given rectangular hyperbola circumscribing a triangle can be described by the intersections of two lines $\{\lambda_X, \eta_X\}$ passing through two fixed points $\{L, H\}$ and corresponding to each other by a homographic relation $f : L^* \longrightarrow H^*$ between the pencils of lines through L and H. A similar generation of the special case of the *"Jerabek hyperbola"* has been studied in [1]. Figure 1 shows our basic configuration of a triangle ABC, its orthocenter H and a fixed point L on its circumcircle κ. Line λ_X is simply LX for a

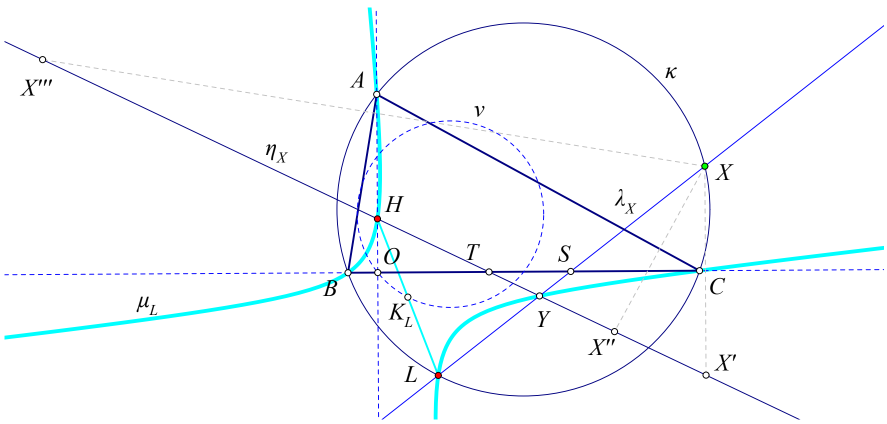

FIGURE 1. The hyperbola μ generated by the intersections $\{\lambda_X \cap \eta_X : X \in \kappa\}$

variable point X on the circumcircle κ of $\triangle ABC$. Line η_X is the *"Steiner line"* of X ([2, p.54]) containing per definition the reflected points $\{X', X'', X'''\}$ of $X \in \kappa$ on the sides of

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the triangle and proved to pass through H. This line is parallel to the Wallace-Simson line (WS-line) of X ([3, p.8]). We show that the intersection point $Y = \lambda_X \cap \eta_X$ describes the rectangular hyperbola μ_L passing through the vertices of the triangle and the points $\{L, H\}$. For the convenience of easy reference we formulate the basic facts known for this kind of hyperbolas in the form of a theorem ([4], [5, p.99], [6, II, p.248], [7, p.35], [8, p.290], [9], [10, p.287]).

Theorem 1. *Every rectangular hyperbola μ_L circumscribing the triangle ABC passes through the orthocenter H of the triangle and is uniquely determined by its forth intersection L with the circumcircle κ of the triangle. Its center K_L lies on the Euler circle ν of $\triangle ABC$ and is the middle of the segment HL. If $\widehat{A}$ is right, then the altitude from A is tangent to μ_L at A.*

2. The homography between the pencils $\{L^*, H^*\}$

Consider the system of Cartesian coordinates whose x-axis is the side BC and the y-axis is the altitude AO of the triangle (see Figure 1). In this system, the coordinates of the vertices are $\{A(0,a), B(b,0), C(c,0)\}$ and it is easily verified that the coordinates of the orthocenter H and the equation of the circumcircle are respectively:

$$H(0, -bc/a) \quad \text{and} \quad a(x^2+y^2) - a(b+c)x - (bc+a^2)y + abc = 0\,. \tag{2.1}$$

We fix point $L(m,n)$ on the circumcircle κ of $\triangle ABC$ and parameterize κ by a kind of stereographic projection from L through $S(s,0)$ on BC, the corresponding point $X(x,y) \in \kappa$ being the intersection of the circle with line LS and leading to:

$$x = \frac{s^2(c+b-m) + s(m^2+n^2-(b+c)m-bc) + bcm}{(m-s)^2+n^2}, \quad y = \frac{n(s-b)(s-c)}{(m-s)^2+n^2}.$$

The reflected X' of X w.r.t. BC has coordinates $(x, -y)$ and the Steiner line HX' of X intersects the x-axis at the point $T(t,0)$:

$$t = bc\frac{s^2(b+c-m) + s(m^2+n^2-(b+c)m-bc) + bcm}{s^2(bc-an) + s(an(b+c)-2bcm) + bc(m^2+n^2-an)}\,. \tag{2.2}$$

The two quadratics in s appearing in this quotient reduce to linear functions when point $L(m,n)$ obtains the position diametral to the vertex A w.r.t. κ for $\{m = b+c, an = bc\}$. This is a special case, which together with the other special cases in which L takes the position of the vertices of $\triangle ABC$ or becomes symmetric of H w.r.t. a side of the triangle, will be handled at the end of the section.

In some of the excepted positions, but also for all other points $L(m,n) \in \kappa$ the two quadratics are genuine. It is though easily verified that they have a common root

$$s_0 = \frac{bcm}{bc-an}\,.$$

Thus, dividing both with their common factor $(s - s_0)$, the relation between $\{s,t\}$ becomes a *"homographic"* one of the form

$$t = \frac{p\cdot s + q}{p'\cdot s + q'} = bc\frac{(b+c-m)s + (an-bc)}{-(an-bc)s + ((b+c)an - bcm)}\,. \tag{2.3}$$

The determinant $pq' - p'q$ of the relation (2.3) is

$$bc(an - bm + b^2)(an - cm + c^2)\,. \tag{2.4}$$

The case $bc = 0$, meaning that the triangle is right-angled, can be avoided by assuming the right angle at A. The two lines $an - bm + b^2 = 0$ and $an - cm + c^2 = 0$ define the quadrangle ABA_1C, with A_1 the diametral of A w.r.t. κ. Thus, the two lines intersect the circle at the points $\{B, C, A_1\}$ defining positions of L pertaining to the exceptional cases to be handled below. For all but these exceptional positions, the relation (2.3) is a genuine homographic one between the points $\{S, T\}$. It follows, that the correspoding lines $\{\lambda_X = XL\,,\ \eta_X = X'H\}$, which belong to the pencils $\{L^*, H^*\}$ of lines passing respectively through the points $\{L, H\}$, are also homographically related.

By the Chasles-Steiner principle of generation of conics ([11, p.5], [12, p.72], [13, p. 259]), their intersection $Y = \lambda_X \cap \eta_X$ generates a conic. It is then easy to see that this conic passes through the vertices of $\triangle ABC$ and, by the general properties of the Chasles-Steiner generation method, passes also from the centers of the pencils $\{H^*, L^*\}$ i.e. points $\{H, L\}$. This identifies the conic with the rectangular hyperbola passing through the vertices of $\triangle ABC$ and L.

Now to the special cases. Cases in which L takes the position of a vertex can be handled by assuming that L coincides with A (see Figure 2), i.e. $A = L(0, a)$ reducing relation (2.3) to

$$t \;=\; bc\frac{s(b+c) + (a^2 - bc)}{s(bc - a^2) + (a^2(b+c))} \quad \text{with determinant} \quad bc(b^2 + a^2)(c^2 + a^2)\,. \tag{2.5}$$

It is then easily seen that as X approaches A the corresponding point Y approaches A and XY takes the position of a common tangent to κ and the conic at A. In this case the center of

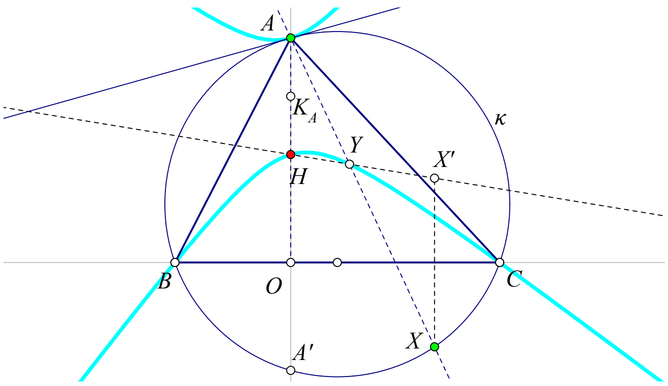

FIGURE 2. Rectangular hyperbola tangent to κ at A

the hyperbola is the middle K_A of the segment AH and its axes are parallel to the bisectors of the angle $\widehat{A}$. Also the tangents to the hyperbola at $\{B, C\}$ intersect on the symmedian of the triangle from A.

The case of the symmetrics of H w.r.t. the sides of $\triangle ABC$ can be handled by taking $L = A'(0, bc/a)$. This is a special case in which the three points $\{A, H, A'\}$ are on the conic, which is degenerate and reduces to the product of two lines $\{AH, BC\}$.

Finally, in the case of the diametral A_1 of A w.r.t. to κ : $A_1 = L(b+c, bc/a)$ (see Figure 3) the coefficients of the quadratic terms in s vanish, the quadratics reducing to linear functions in s and the relation in (2.2) taking then the form:

$$t \;=\; bc\frac{s(h^2 - bc) + (bc(b+c))}{s(-bc(b+c)) + (h^2 + bc + b^2 + c^2)}\,, \tag{2.6}$$

where we have set $h = -bc/a$ equal to the y-coordinate of H. The determinant of the homographic relation can be seen to be again non zero and the hyperbola in this case, passing through A_1 has its center at the middle M of side BC, its axes are parallel to the bisectors of $\widehat{A}$ and its tangents at $\{B,C\}$ are parallel to the tangent to κ at A_1 (see Figure 3). In this case it can be

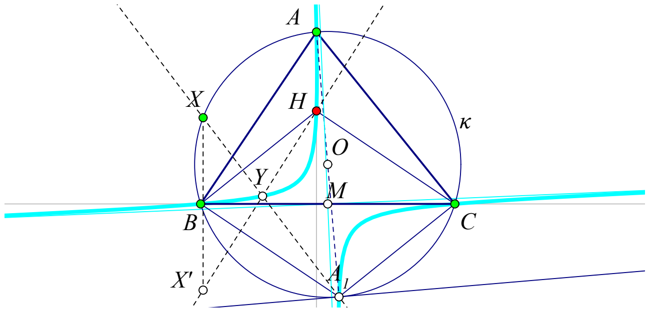

FIGURE 3. Rectangular hyperbola with center at the middle M of BC

also proved that the tangent to the hyperbola at A contains the symmedian point of $\triangle ABC$. The discussion shows that the generation of the hyperbola by the intersections $Y = \lambda_X \cap \eta_X$ is valid in all cases, except those for which the location of L is the reflected of the orthocenter H on a side of $\triangle ABC$, in which the hyperbola degenerates in a product of two lines and Y describes only one of them.

Theorem 2. *For all points L of the circumcircle κ of the triangle ABC, except the reflected of the orthocenter H w.r.t. the sides, the rectangular hyperbola passing through L is generated by the intersections $Y = \lambda_X \cap \eta_X$ of the lines $\lambda_X = LX$ and the Steiner line η_X for $X \in \kappa$. For the excepted positions of L the corresponding hyperbola μ_L degenerates to the product of a side-line and the orthogonal to it altitude-line of the triangle. The point Y in this case lies always on a side-line of the triangle.*

3. THE PROJECTIVITY MAPPING THE CIRCUMCIRCLE TO THE HYPERBOLA

Naturally connected to the generation of the hyperbola discussed in the preceding section is a projectivity mapping the circumcircle to the hyperbola.

Theorem 3. *With the notation and conventions adopted in the preceding section the transformation $f_L : X \mapsto Y$ is a projectivity mapping the circumcircle κ to the rectangular hyperbola μ_L circumscribing the triangle ABC and passing through $L \in \kappa$.*

Proof. The proof is very simple and results from the general properties of projectivities, according to which ([14, I, p.213]), given three distinct points $\{A,B,C\}$ on the conic κ and three other distinct points $\{A',B',C'\}$ on the conic κ', there is a unique projectivity $f : \kappa \to \kappa'$ with the property $\{f(A) = A', f(B) = B', f(C) = C'\}$. Considering the vertices of the triangle of reference ABC, as points of κ and also as points of μ_L, we deduce the existence of a projectivity mapping κ to μ_L and also fixing these three points. We show that this projectivity coincides with f_L. This follows at once from the preservation of the cross ratios by projectivities. In fact, referring to figure 4, the cross ratios (AB,CX) on κ and $(AB,CY) = (f(A)f(B), f(C)f(X))$ on μ_L are per definition both equal to the cross ratio of the pencil of lines through $\{L : LA, LB, LC, LX\}$, which is the same with the cross ratio $(A'B, CS)$ of their intersections with line BC. Thus, $f(X) = f_L(X) = Y$ for every $X \in \kappa$, which proves the claim. $\square$

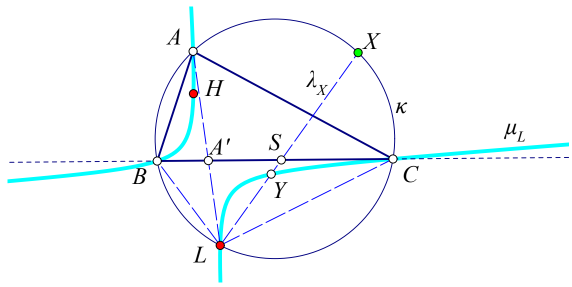

FIGURE 4. Equality of cross ratios

Remark. Notice that the argument appearing in the proof of the preceding theorem does not use any particular property of the hyperbola. It could be transferred verbatim to the more general case of an arbitrary triangle conic μ and its fourth intersection L with the circumcircle. This would produce an analogous projectivity $Y = f_L(X)$ mapping κ to μ and such that line XY passes through L for all $X \in \kappa$.

Remark. It is easy to see that every projectivity f between two conics $\{\kappa, \mu\}$ of the plane extends to a projectivity of the whole plane onto itself ([6, II, p.179]). It suffices to consider four points $\{A, B, C, D \in \kappa\}$ and their images $\{A', B', C', D' \in \mu\}$ under f. By the general properties of projectivities there is a projectivity of the plane f' mapping the first quadruple to the second and coinciding with f on κ. In the following we'll work with this extension of f_L to the whole plane, denoting the extension with the same symbol f_L.

In the rest of this section we discuss some consequences of theorems 2 and 3 expressing properties of rectangular hyperbolas, which we formulate as *"propositions"*. To start with, we consider the positions $\{D, E \in \kappa\}$ for which the corresponding intersection point $Y = \lambda_X \cap \eta_X$ goes to infinity, i.e. the two lines $\{\lambda_D, \eta_D\}$ corresponding to D say, become parallel defining the point at infinity of the hyperbola μ_L and the direction of one of its asymptotes (see Figure 5). Since the asymptotes are orthogonal the parallels to them $\{LD, LE\}$ are also orthogonal and ED is a diameter of the circumcircle. Since η_D is the Steiner line of D, the WS-line of D is the parallel to η_D passing through the middle K_L of LH. Thus the asymptote parallel to LD coincides with the WS-line of D. Analogous properties hold also for E. Since points $\{D, E\}$ map by the projectivity f_L to infinity, the tangents $\{t_D, t_E\}$ to κ respectively at $\{D, E\}$ map to the asymptotes and the line DE maps to infinity. Also, by the *"angle property"* of the pairs of points of κ by which their WS-lines intersect at half the angle and inverse orientation of their central angle ([15, p.207]), we deduce easily that line ζ is parallel to the WS-line of the diametral L' of L w.r.t. κ. All this proves next proposition.

Proposition 4. *The parallels $\{LD, LE\}$ to the asymptotes of μ_L define a diameter DE of the circle κ and the line $\zeta = DE$ maps via f_L to the line at infinity ε_∞ of the plane. The WS-lines of $\{D, E\}$ intersect at the middle K_L of HL, the center of the hyperbola, thus coinciding with the asymptotes which coincide also with the images via f_L of the tangents to κ correspondingly at $\{D, E\}$. The line ζ is parallel to the WS-line of the diametral point L' of L w.r.t. κ.*

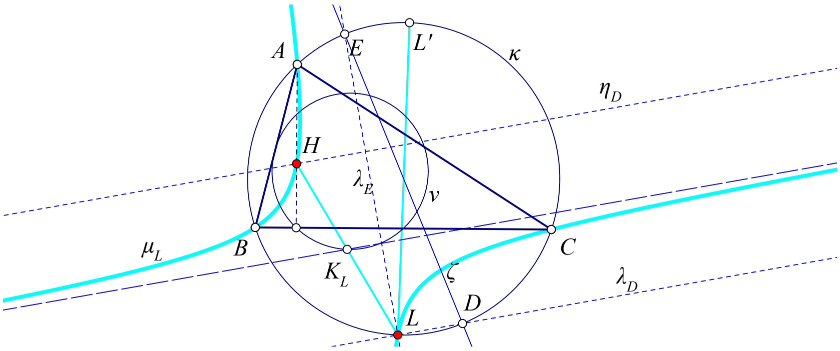

FIGURE 5. The directions $\{LD, LE\}$ of the asymptotes

Proposition 5. *Lines $\{\theta_P\}$ intersecting $\zeta = DE$ at a fixed point P map via f_L to parallel lines. In particular, all lines through D or E map to lines parallel to the asymptotes corresponding to the directions of LD or LE.*

Proof. In fact, lines passing through a point $P \in \zeta$ map to lines passing through the point $Q = f_L(P)$ at infinity. The tangent to κ at D maps via f_L to the tangent of μ_L at $f_L(D)$ which is a point at infinity of μ_L, hence its tangent there is the asymptote parallel to LD. □

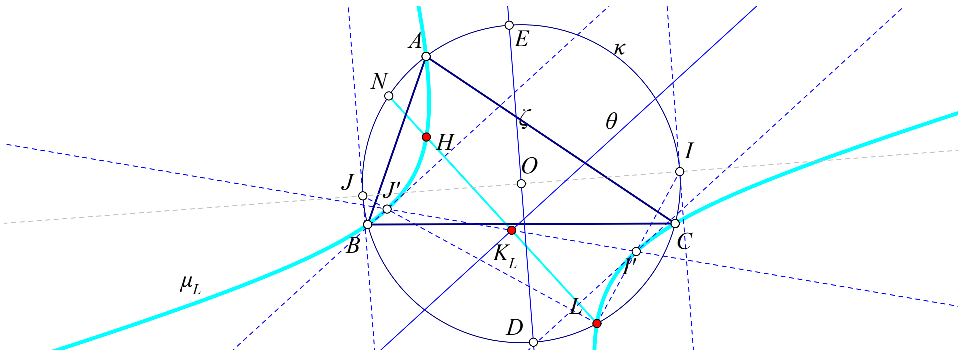

FIGURE 6. Two parallel tangents at $\{I', J'\}$

Proposition 6. *Lines parallel to $\zeta = DE$ map to parallel lines. In particular the tangents at the diametral points $\{I, J\}$ of the orthogonal to ζ diameter IJ map via f_L to parallel tangents of the hyperbola μ_L correspondingly at its points $\{I' = f_L(I), J' = f_L(J)\}$. The line $I'J'$ is the conjugate diameter to the direction of these parallel tangents of μ_L passing through the center K_L of μ_L (see Figure 6).*

Proof. This is a particular case of proposition 5. The intersection point P of a line ε parallel to ζ is the point at infinity of ζ which maps via f_L to a point at infinity too. The second claim is an immediate consequence of the first and the third claim is a consequence of the second. □

Proposition 7. *The directions of lines $\{LI, LJ\}$ are parallel to the axes of the hyperbola.*

Proof. This follows from the fact that $\widehat{DLE}$ is a right angle with sides parallel to the asymptotes and $\{LI, LJ\}$ are bisectors of this angle. □

Proposition 8. *The line at infinity ε_∞ maps via f_L to the middle-parallel θ of the tangents to μ_L respectively at $\{I', J'\}$.*

Proof. Since the line at infinity ε_∞ is parallel to every line of the plane, it is also a parallel to ζ. Thus, it maps to a line parallel to those tangents. The parallels $\{\zeta, \varepsilon_\infty\}$ are harmonic conjugate to the parallel tangents to κ at $\{I, J\}$. Since f_L preserves the cross ratio, the images $\{f_L(\zeta), f_L(\varepsilon_\infty)\}$ define on $I'J'$ harmonic conjugate points of the pair (I', J'). Since $f_L(\zeta) = \varepsilon_\infty$ its intersection with $I'J'$ is at infinity, hence the intersection of $f_L(\varepsilon_\infty)$ with $I'J'$ is the middle K_L of $I'J'$. □

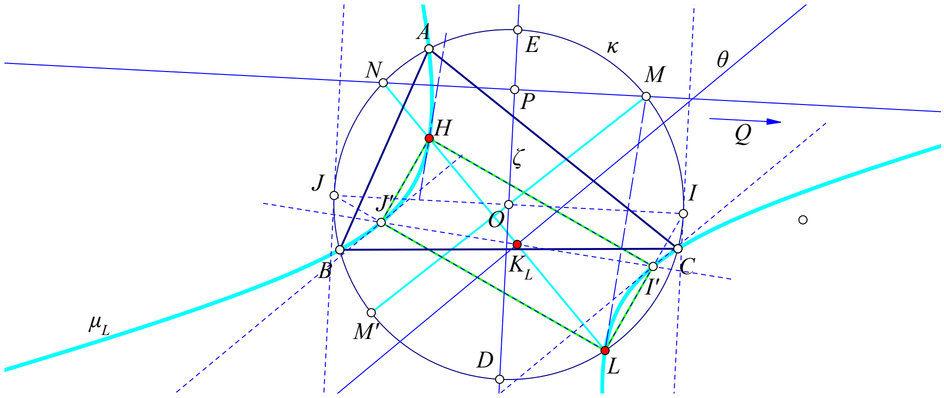

FIGURE 7. The rectangle with sides parallel to the axes

Proposition 9. *If $\{M, N\}$ are respectively the intersections of κ with the tangent to μ_L at L and with LH, then $\{f_L(N) = H, f_L(M) = L\}$. The quadrangle $HI'LJ'$ is a rectangle and the tangents at $\{L, H\}$ are parallel and orthogonal to $I'J'$. Line θ is orthogonal to LH. Further, line θ is parallel to the WS-line of the diametral M' of M w.r.t. κ.*

Proof. The first claim follows directly from the definition of f_L. The second claim follows from the parallelity of $\{LI', LJ'\}$ to the axes of μ_L. The orthogonality of LM to $I'J'$ follows from theorem 1. Since the triangles $\{HLI', HLJ'\}$ are isosceli the tangent to μ_L at L is orthogonal to $I'J'$ and the tangent at I' is orthogonal to LH. Last claim follows easily from the aforementioned *angle-property* of WS-lines. □

Proposition 10. *Points $\{M, N\}$ map via f_L correspondingly to $\{L, H\}$ and the middle P and the point at infinity Q of line MN map correspondingly to the point at infinity of line LH and to K_L.*

Proof. The first part was discussed in the preceding proposition. The rest follows from fact that the angles $\widehat{JIH} = \widehat{MLI}$ and P is the intersection $\zeta \cap NM$. Since f_L preserves cross ratios, $(NM, PQ) = -1$ and ζ maps via f_L to the line at infinity, $f_L(P)$ is the point at infinity of LH and consequently $f_L(Q) = K_L$. □

Proposition 11. *The line $L'L$ with $L' = f_L(L)$ is tangent to κ at L (see Figure 8).*

Proof. By the definition of f_L the image $Y = f_L(X)$ results as intersection of LX with the hyperbola μ_L. The result follows from the fact, that as X approaches L the corresponding line LX approaches the tangent to κ at L. □

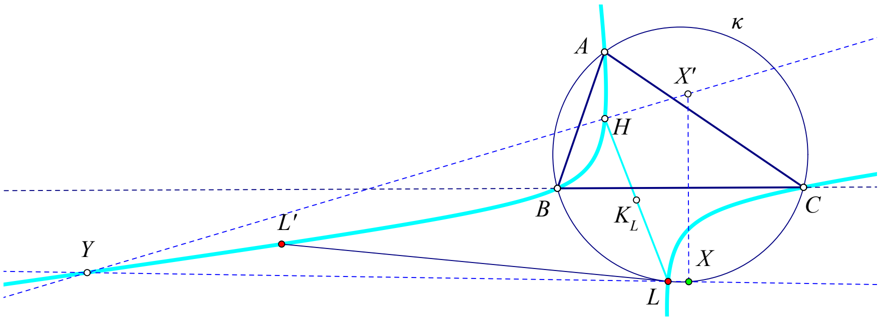

FIGURE 8. The line $L'L$ tangent to κ

Proposition 12. *The Steiner line $\eta_L = L'H$ is parallel to line MN (see Figure 9).*

Proof. In fact, the Steiner line $L'H$ is parallel to the WS-line K_LP of L which passes through the middle P of LL', which is also the middle of the segment of L_1L_2 cut on LL' by the two asymptotes $\{\alpha,\beta\}$. This implies the equality of the angles

$$\widehat{PK_LL_1} = \widehat{PL_1K_L} = \widehat{D_2LD} = \widehat{LED} = \widehat{DLD_1},$$

where D_1 is the projection of L on ED and $D_2 = LL' \cap ED$. This implies the parallelity of

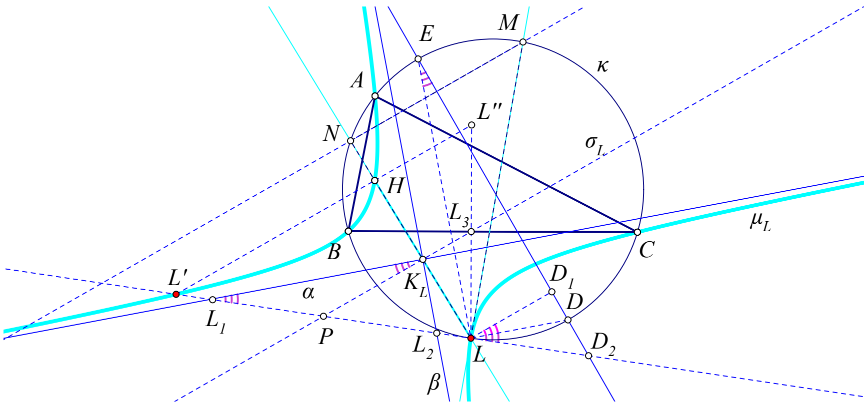

FIGURE 9. The Steiner line $L'H$ is parallel to MN

K_LP to LD_1 and consequently the orthogonality of K_LP to ED, which is also orthogonal to MN, hence latter is parallel to $\eta_L = L'H$. □

4. REPRESENTATIONS IN BARYCENTRICS

Concerning the representation of the various elements in coordinates, the equation of μ_L, since latter passes through the vertices of $\triangle ABC$, has in barycentrics $X(u,v,w)$ ([16], [17]) the form of the quadratic equation: $pvw + qwu + ruv = 0$. The coefficients can be easily found from the property of the conic to pass through $\{H,L\}$. This leads to a system of two homogeneous equations in $\{p,q,r\}$:

$$ph_2h_3 + qh_3h_1 + rh_1h_2 = 0 \quad \text{and} \quad pmn + qnl + rlm = 0\,,$$

where (l,m,n) are the barycentrics of L and $(h_1,h_2,h_3)=(1/S_A,1/S_B,1/S_C)$ those of the orthocenter H, with $\{a=|BC|,b=|CA|,c=|AB|\}$ and

$$S_A=(b^2+c^2-a^2)/2 \quad , \quad S_B=(c^2+a^2-b^2)/2 \quad , \quad S_C=(a^2+b^2-c^2)/2\,.$$

It turns out that the coefficients are multiples by a non-zero constant of the quantities:

$$p=l(mS_B-nS_C) \quad , \quad q=m(nS_C-lS_A) \quad , \quad r=n(lS_A-mS_B). \tag{4.1}$$

Because f_L fixes the vertices of $\triangle ABC$ its matrix is diagonal and can be determined by computing the barycentrics of one additional point and its image or an additional line and its image. For this it is convenient to use proposition 12, from which follows that the inverse projectivity g of f_L maps the tangent t_L of κ at L to the tangent of μ_L at L. The coefficients of the two tangents result by multiplying respectively the matrices of $\{\kappa,\mu_L\}$ with the barycentrics vector $L(l,m,n)$, leading to the coefficients of these two lines:

$$\begin{pmatrix} c^2m+b^2n \\ a^2n+c^2l \\ b^2l+a^2m \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} mn(nS_C-mS_B) \\ nl(lS_A-nS_C) \\ lm(mS_B-lS_A) \end{pmatrix} ,$$

and the expression of the diagonal matrix of f_L, which up to a non-zero constant multiplicative factor is:

$$\begin{pmatrix} \frac{mn(nS_C-mS_B)}{c^2m+b^2n} & 0 & 0 \\ 0 & \frac{nl(lS_A-nS_C)}{a^2n+c^2l} & 0 \\ 0 & 0 & \frac{lm(mS_B-lS_A)}{b^2l+a^2m} \end{pmatrix}.$$

5. A PENCIL OF RELATED HYPERBOLAS

Fixing a point L on the circumcircle κ of $\triangle ABC$ and the corresponding rectangular hyperbola μ_L, there emerges a naturally defined pencil of hyperbolas passing through the vertices of the triangle and an additional fourth point at infinity L_∞. This means that the hyperbolas of the

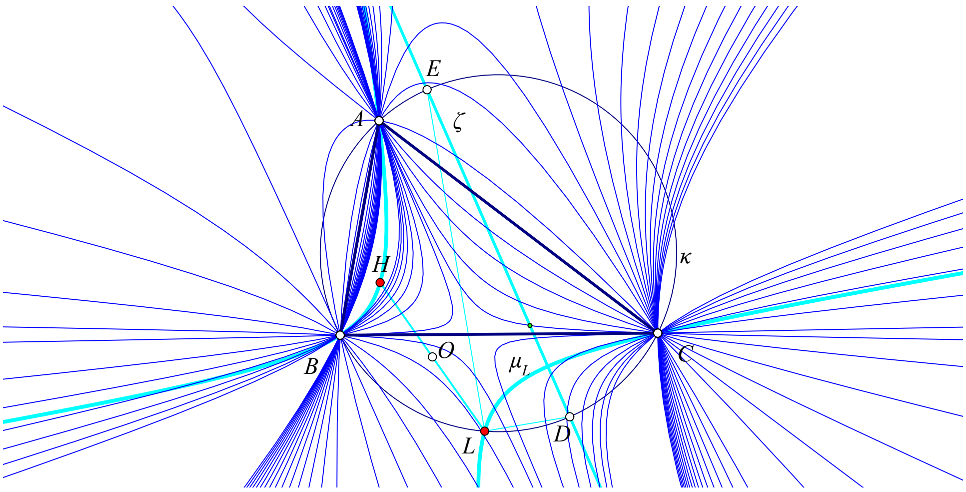

FIGURE 10. A pencil of hyperbolas defined by a rectangular hyperbola

pencil have one of their asymptotes pointing in the direction determined by the point at infinity L_∞ (see Figure 10). This point at infinity is determined by the direction of the line $\zeta = DE$, where $\{D, E\}$ are the diametral points of κ whose WS-lines are the asymptotes of μ_L. Next theorem formulates the definition of these hyperbolas (see Figure 11).

Theorem 13. *Let $P \in \zeta$ be fixed, $X \in \kappa$ be a variable point and $Y = f_L(X)$. Then the intersection point of lines $XP \cap f_L(XY) = Z$ describes a hyperbola $\zeta(L, P)$ passing through the vertices of the triangle of reference ABC, the point P and having one asymptote parallel to line ζ and the other asymptote parallel to the direction determined by the point at infinity $Q = f_L(P)$.*

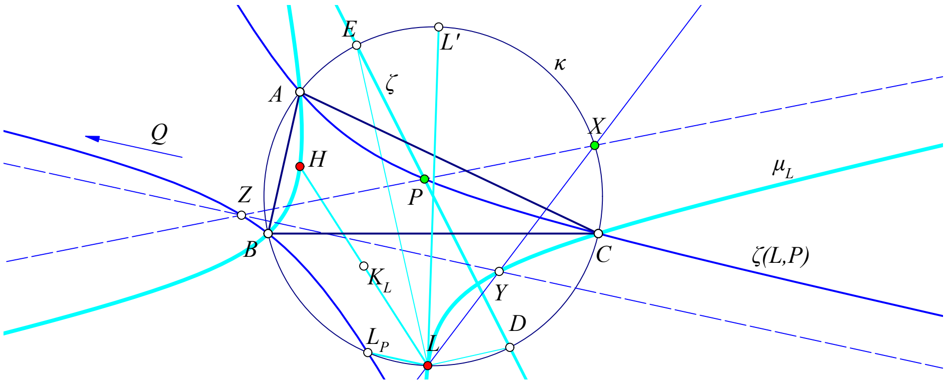

FIGURE 11. The hyperbola $\zeta(L, P)$ defined by point $P \in \zeta$

Proof. This is again an immediate consequence of the Chasles-Steiner principle. By theorem 5 the lines $\{YZ = f_L(XP), X \in \kappa\}$ are parallel to a fixed direction represented by the point at infinity $Q = f_L(P)$. The transformation f_L establishes a homography between the lines of the pencils $\{P^*, Q^*\}$. By the aforementioned principle, the intersections of pairs of lines of the two pencils corresponding under this homography $\{Z = PX \cap f_L(PX)\}$ describe a conic passing through the centers $\{P, Q\}$ of the two pencils. Since Q is at infinity, the conic is a hyperbola and the direction determined by Q coincides with the direction of one of its asymptotes, if we can show that there is also a second point at infinity. Latter is easily seen since, when X obtains the position of D, then $PX = \zeta$ and $f_L(\zeta)$ being the line at infinity implies that the corresponding point $Z = PX \cap f_L(PX)$ is at infinity in the direction of ζ. This shows that there is a second point at infinity and the other asymptote is parallel to ζ. Further, it is trivially seen that the hyperbola passes through the vertices of $\triangle ABC$ and the point $L_P \in \kappa$, which is the second intersection of κ with the parallel to Q through L and is also on line MP. □

Theorem 14. *The centers $\{K_P\}$ of the hyperbolas $\{\zeta(L, P) : P \in \zeta\}$ lie on a parabola ζ_L, whose axis is parallel to line ζ (see Figure 12). The parabola passes through the middles of the sides of the triangle ABC and also passes through the intersections $\{A', B', C'\}$ of the sides of the triangle with the parallels to ζ from the opposite vertices.*

Proof. It is well known ([18, p.153], [6, II, p.213]) that the centers $\{K, K', \ldots\}$ of conics $\{\kappa, \kappa', \ldots\}$ passing through four fixed points $\{A, B, C, D\}$ lie on a conic ν, called *"nine-point-conic"* of the four points (see Figure 13). This conic passes through the middles $\{E, F, G, H, I, J\}$

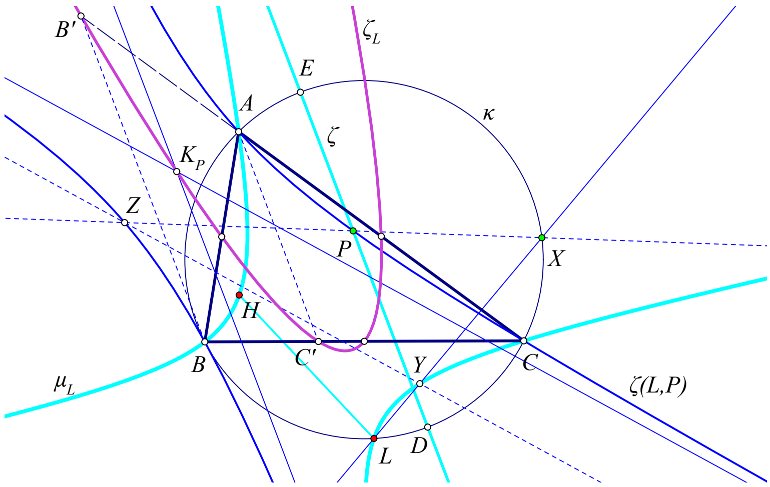

FIGURE 12. The parabola ζ_L containing the centers K_P of all $\{\zeta(L,P), P \in \zeta\}$

of the six segments joining pairs of these four points and also passes through the three intersections $\{M,N,O\}$ of *"opposite sides"*, i.e. intersections of (extensions of) pairs of segments with no common endpoint. In our case the fourth point D is at infinity and determines a di-

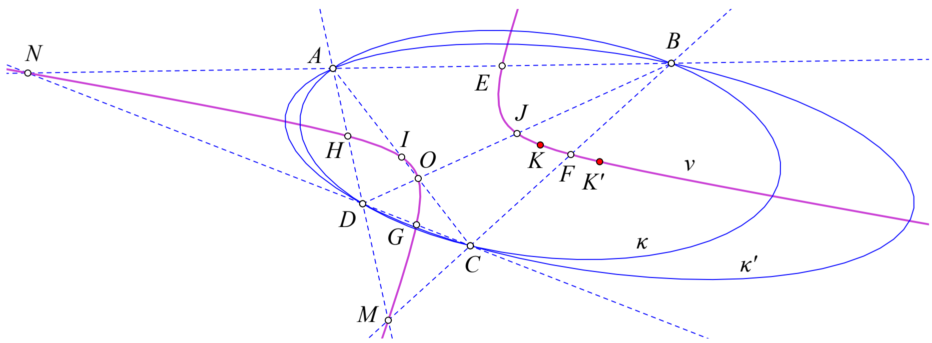

FIGURE 13. The *"nine-point-conic"* of $\{A,B,C,D\}$

rection of parallel lines. The points $\{M,N,O\}$ in this case are the intersections of the sides of $\triangle ABC$ with the lines $\{DA,DB,DC\}$, which are the parallels to the direction determined by D through the corresponding vertices of ABC. The fact that, in our case, this conic ν is a parabola as stated, follows from the following lemma (see Figure 14). □

Lemma 15. *Given a triangle ABC and a line ε, consider the triangle $A'B'C'$ formed by the parallels to the sides of ABC from the opposite vertices, called "anticomplementary" of ABC. The parallel projections $\{A'',B'',C''\}$ along ε of $\{A',B',C'\}$ respectively on the opposite sides of $A'B'C'$ together with $\{A,B,C\}$ are six points on a parabola, whose axis is parallel to ε.*

Proof. Consider the conic ν passing through the five points $\{A,B,C,B'',C''\}$. Then the segments $\{AB,CC''\}$ are parallel chords of ν and the line DD' joining their middles is parallel

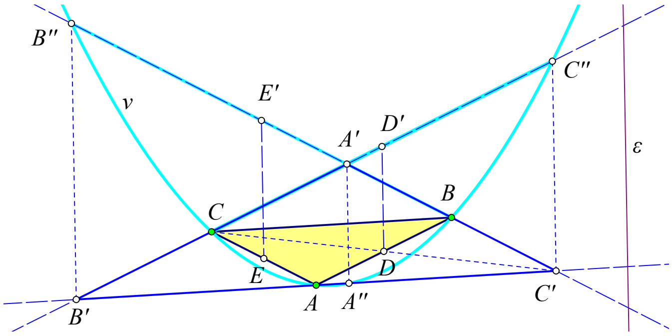

FIGURE 14. A property of the parabola

to ε. Analogously $\{AC, BB''\}$ are parallel chords of ν and the line EE' joining their middles is also parallel to ε. Thus, $\{DD', EE'\}$ are conjugate diameters of two different directions pointing in the same direction, which is possible only for parabolas with axis parallel to ε. Then $\{AA'', CB\}$ are segments whose middles $\{F, F'\}$ define a parallel to ε. Hence A'' is also on ν. □

Remark. Lemma 15 gives a convenient method to draw the parabola circumscribing $\triangle A'B'C'$ and having its axis parallel to a given direction ε, giving additional points on the parabola and allowing its construction as a conic passing through five points. All parabolas circumscribing a $\triangle A'B'C'$ can be constructed in this way.

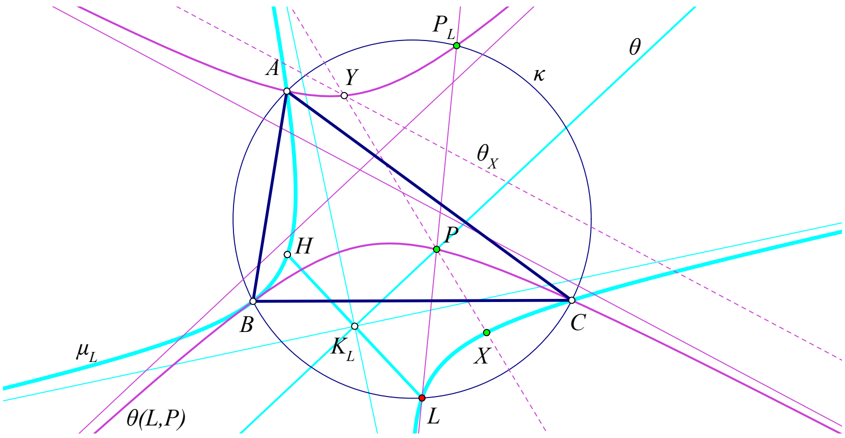

FIGURE 15. The hyperbola $\theta(L,P)$ defined by point $P \in \theta$

Considering the inverse transformation g_L of f_L we can define a pencil $\{\theta(L,P)\}$ of hyperbolas analogous to the pencil $\{\zeta(L,P)\}$ of theorem 13. The role of ζ plays now line θ. For each point $P \in \theta$ the lines PX through P map to lines θ_X parallel to the direction represented by the point at infinity $Q = g_L(P)$. Again, by the method of the aforementioned theorem, we can prove that the intersection points $\{PX \cap \theta_X, X \in \mu_L\}$ generate a hyperbola. Next theorem formulates the corresponding result (see Figure 15).

Theorem 16. *Let $P \in \theta$ be fixed, $X \in \mu_L$ be a variable point and $\theta_X = g_L(PX)$. Then the intersection point of lines $XP \cap \theta_X = Y$ describes a hyperbola $\theta(L,P)$ passing through the vertices of the triangle of reference ABC, the point P and having one asymptote parallel to line θ and the other asymptote parallel to the direction determined by the point at infinity $Q = g_L(P)$.*

Analogously also to theorem 14 the centers of the hyperbolas $\{\theta(L,P)\}$ vary on a parabola θ_L passing through the middles of the sides of $\triangle ABC$ and the parallel to θ projections $\{A', B', C'\}$ of the vertices of ABC on its opposite sides (see Figure 16).

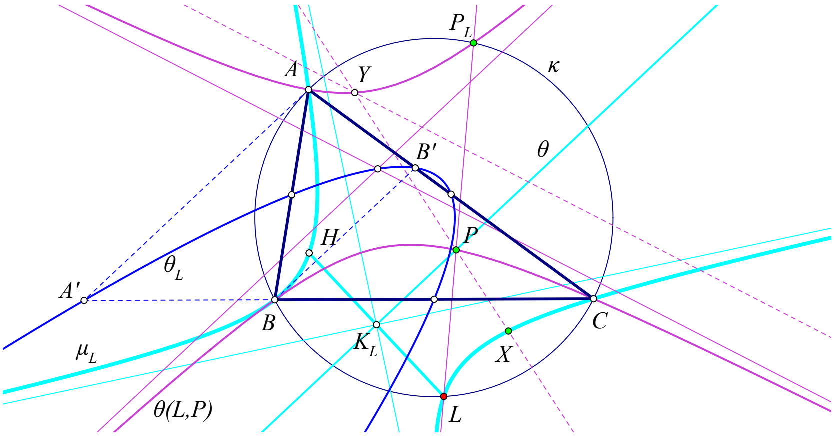

FIGURE 16. The parabola θ_L containing the centers of all $\{\theta(L,P), P \in \zeta\}$

Theorem 17. *Both parabolas $\{\zeta_L, \theta_L\}$ pass through the intersection point $U = \zeta \cap \theta$.*

Proof. We show this for ζ_L the proof for θ_L being completely analogous. The proof follows from propositions 5 and 8. In fact, we consider the hyperbola $\zeta(L,P)$ for P the point at infinity of the line ζ. This is generated by the intersections $\{Z = PX \cap f_L(PX)\}$, and X can be considered varying on line θ (see Figure 17). By the aforementioned propositions the lines XZ map via f_L to parallels θ_X to θ. Referring points Z to coordinates $\{X(x), Y(y)\}$ along the axes $\{\theta, \zeta\}$, the homographic relation between the lines $\{XZ, f_L(XZ) = YZ\}$ translates to a homographic relation between the coordinates

$$y = \frac{ax+b}{cx+d}.$$

Since by their definition, for X going to infinity, the corresponding θ_X goes to θ and for x tending to 0, the line θ_X tends to infinity, we see that the relation reduces to one of the form

$$y = \frac{k}{x} \quad \text{with} \quad k \neq 0.$$

It is then trivial to see that the hyperbola $\zeta(L,P)$ has the axes $\{\theta, \zeta\}$ as asymptotes and their intersection U as its center. □

Remark. From our discussion follows that the rectangular hyperbola μ_L defines a unique parabola ζ_L circumscribing the medial triangle $A'B'C'$ of the triangle of reference ABC. It is easily

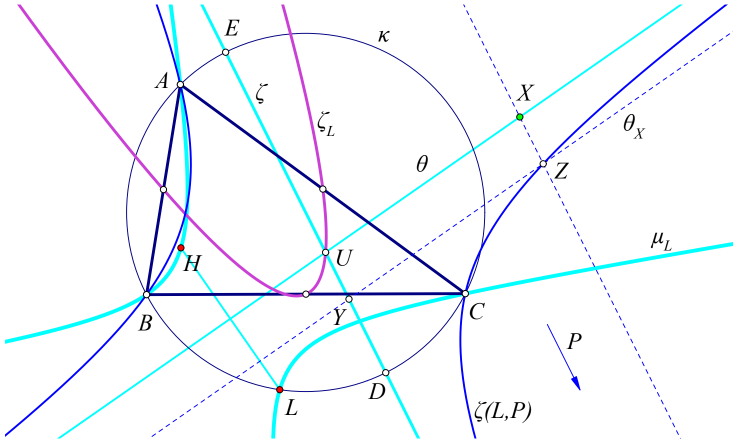

FIGURE 17. The special hyperbola $\zeta(L,P)$ for $P \in \zeta$ at infinity

seen, that conversely, given an arbitrary parabola ν circumscribing the triangle $A'B'C'$, there is associated a unique rectangular hyperbola μ_L circumscribing the anticomplementary triangle ABC for which the corresponding $\zeta_L = \nu$. To see this consider the diameter DE of the circumcircle κ of $\triangle ABC$, which is parallel to the axis of ν. If L' is the point on κ whose WS-line is parallel to DE, the diametral $L \in \kappa$, according to proposition 4, defines the point which in turn defines the hyperbola μ_L with the desired property.

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