Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.10, (2021), Issue 1, pp.91-101

ON THREE DIMENSIONAL f-KENMOTSU MANIFOLDS WITH A CERTAIN CONNECTION

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ABSTRACT. The object of the present paper is to study 3-dimensional f-Kenmotsu manifolds with respect to the semi-symmetric metric connection satisfying certain curvature conditions. Finally, we give an example of 3-dimensional f-Kenmotsu manifolds.

1. INTRODUCTION

In 1991, the notion of an f-Kenmotsu manifold was introduced by Olszak and Rosca [18] which are normal locally conformal and almost cosymplectic manifolds. Further, they give a geometric interpretation of f-Kenmotsu manifold and proved that a Ricci symmetric f-Kenmotsu manifold is an Einstein manifold. Recently, f-Kenmotsu manifolds have been studied by various authors in several ways to a different extent such as ([4], [6]-[8], [13], [16]) and many others. In 1924, the notion of the semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [1]. Later in 1932, Hayden [10] introduced the idea of metric connection with a torsion on a Riemannian manifold. A semi-symmetric connection on a Riemannian manifold was systematically studied by Yano [12], which was further studied by Haseeb [2], Haseeb and Prasad [3], Sharfuddin and Hussain [5], Amur and Pujar [11], Binh [14], De and Biswas [15] and many others.

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one to one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes, the projective curvature tensor is defined by [17]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{1.1}$$

where $X, Y, Z \in \chi(M)$, R is the curvature tensor and S is the Ricci tensor with respect to the Levi-Civita connection, respectively.

2010 *Mathematics Subject Classification.* Primary 53C05; Secondary 53C25, 53D15.

Key words and phrases. f-Kenmotsu manifold, semi-symmetric metric connection, projectively curvature tensor, $\eta-$Einstein manifold, cyclic parallel Ricci tensor.

Motivated by the above studies, in this paper we study certain curvature conditions on 3-dimensional f-Kenmotsu manifolds with respect to the semi-symmetric metric connection. The paper is organized as follows: After the introduction, section 2 is concerned with some preliminaries. Pseudoprojectively flat and ϕ-projectively flat 3-dimensional f-Kenmotsu manifolds with respect to the semi-symmetric metric connection have studied in sections 3 and 4, respectively. Section 5 deals with the study of 3-dimensional f-Kenmotsu manifolds with respect to the semi-symmetric metric connection admitting cyclic parallel Ricci tensor. In section 6, we study ϕ-Ricci symmetric 3-dimensional f-Kenmotsu manifolds with respect to the semi-symmetric metric connection.

2. PRELIMINARIES

Let M be a (2n+1)-dimensional differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) which satisfies the following equations [9]:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \tag{2.1}$$

$$\eta(X) = g(X, \xi), \quad \eta \circ \phi = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields $X, Y \in \chi(M)$; where $\chi(M)$ is a set of all smooth vector fields on M, I is the identity map on the tangent bundle TM, η is a 1-form, ξ is a vector field, g is a metric tensor field and ϕ is a vector field of type $(1,1)$. We say that (M, ϕ, ξ, η, g) is an f-Kenmotsu manifold if the Levi-Civita connection of g satisfy

$$(\nabla_X \phi)Y = f[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.4}$$

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$. If $f = 0$, then the manifold is cosymplectic [18]. An f-Kenmotsu manifold is said to be regular if $f^2 + f' \neq 0$, where $f' = \xi f$.
In an f-Kenmotsu manifold, from (2.4) we have

$$\nabla_X \xi = f[X - \eta(X)\xi]. \tag{2.5}$$

The condition $df \wedge \eta = 0$ holds if dim $M \geq 5$. This does not hold in general if dim $M = 3$ [6]. For a 3-dimensional Riemannian manifold, we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad -S(X,Z)Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.6}$$

In a 3-dimensional f-Kenmotsu manifold M, we have [18]

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2f^2 + 2f')[g(Y,Z)X - g(X,Z)Y] \\ &\quad -(\frac{r}{2} + 3f^2 + 3f')[g(Y,Z)\eta(X)\xi \\ &\quad -g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y], \end{aligned} \tag{2.7}$$

$$S(X,Y) = (\frac{r}{2} + f^2 + f')g(X,Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y), \tag{2.8}$$

$$R(X,Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y], \tag{2.9}$$

$$R(\xi, X)Y = -(f^2 + f')[g(X,Y)\xi - \eta(Y)X], \tag{2.10}$$

$$\eta(R(X,Y)Z) = -(f^2 + f')[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \tag{2.11}$$

for all vector fields $X, Y, Z \in \chi(M)$, where R, S, Q and r are the Riemannian curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature, respectively on M. Also from (2.8), we get

$$S(X, \xi) = -2(f^2 + f')\eta(X), \tag{2.12}$$

$$Q\xi = -2(f^2 + f')\xi. \tag{2.13}$$

Using (2.5), we have

$$(\nabla_X \eta)Y = f[g(X, Y) - \eta(X)\eta(Y)]. \tag{2.14}$$

Also from (2.8) it follows that

$$S(\phi X, \phi Y) = S(X, Y) + 2(f^2 + f')\eta(X)\eta(Y) \tag{2.15}$$

for all vector fields $X, Y \in \chi(M)$.

Definition 2.1. *An f-Kenmotsu manifold M is said to be an η-Einstein manifold if its non-vanishing Ricci tensor S is of the form*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where a, b are smooth functions on M. If $b = 0$, then M is said to be an Einstein manifold.

Let (M, g) be a Riemannian manifold with the Levi-Civita connection ∇. A linear connection $\overline{\nabla}$ on (M, g) is said to be semi-symmetric [12] if its torsion tensor T is given by

$$T(X, Y) = \pi(Y)X - \pi(X)Y,$$

where π is a 1-form on M and associated with the vector field ρ by

$$\pi(X) = g(X, \rho)$$

for all vector fields $X \in \chi(M)$.

A semi-symmetric connection $\overline{\nabla}$ is called a semi-symmetric metric if it satisfies the condition

$$\overline{\nabla} g = 0.$$

On an almost contact metric manifold, a semi-symmetric metric connection is defined by replacing 1-form π by the contact 1-form η, i.e.,

$$T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where $g(X, \xi) = \eta(X)$ for all $X \in \chi(M)$.

A relation between a semi-symmetric metric connection $\overline{\nabla}$ and the Levi-Civita connection ∇ on M and is given by [12]

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi, \tag{2.16}$$

where $\eta(X) = g(X, \xi)$.

A relation between the curvature tensors R and $\overline{R}$ of the connections ∇ and $\overline{\nabla}$, respectively is given by [1]

$$\begin{aligned} \overline{R}(X, Y)Z &= R(X, Y)Z + K(X, Z)Y - K(Y, Z)X \\ &\quad + g(X, Z)FY - g(Y, Z)FX, \end{aligned} \tag{2.17}$$

where K is a tensor field of type $(0, 2)$ and F is a $(1, 1)$-tensor field which is given by

$$K(Y, Z) = g(FY, Z) = (\nabla_Y \eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}g(Y, Z). \tag{2.18}$$

For a 3-dimensional f-Kenmotsu manifold, (2.18) takes the form

$$K(Y,Z) = g(FY,Z) = -(f+1)\eta(Y)\eta(Z) + (f+\frac{1}{2})g(Y,Z). \tag{2.19}$$

It yields

$$FY = -(f+1)\eta(Y)\xi + (f+\frac{1}{2})Y. \tag{2.20}$$

Thus by using (2.19), (2.20) in (2.17), we get

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z - (f+1)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \\ &\quad -(f+1)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi \\ &\quad +(2f+1)[g(X,Z)Y - g(Y,Z)X].\end{aligned} \tag{2.21}$$

Contracting (2.21) over X, we get

$$\overline{S}(Y,Z) = S(Y,Z) + (f+1)\eta(Y)\eta(Z) - (3f+1)g(Y,Z), \tag{2.22}$$

where $\overline{S}$ and S are the Ricci tensors of the connections $\overline{\nabla}$ and ∇, respectively on M. Again contracting Y and Z in (2.22), it follows that

$$\overline{r} = r - 8f - 2, \tag{2.23}$$

where $\overline{r}$ and r are the scalar curvatures of the connections $\overline{\nabla}$ and ∇, respectively on M.

Lemma 2.1. *Let M be a 3-dimensional f-Kenmotsu manifold with respect to the semi-symmetric metric connection. Then we have*

$$\bar{S}(X,\xi) = -2(f^2 + f' + f)\eta(X), \tag{2.24}$$

$$\bar{Q}\xi = -2(f^2 + f' + f)\xi, \tag{2.25}$$

$$\bar{\nabla}_X\xi = (1+f)(X - \eta(X)\xi), \tag{2.26}$$

$$(\bar{\nabla}_X\eta)Y = (1+f)g(\phi X, \phi Y), \tag{2.27}$$

$$\begin{aligned}\bar{S}(\phi X, \phi Y) &= S(X,Y) - (3f+1)g(X,Y) \\ &\quad +(2f^2 + 2f' + 3f + 1)\eta(X)\eta(Y).\end{aligned} \tag{2.28}$$

for all $X, Y \in \chi(M)$.

3. PSEUDOPROJECTIVELY FLAT 3-DIMENSIONAL f-KENMOTSU MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION

Definition 3.1. *An f-Kenmotsu manifold is said to be pseudoprojectively flat with respect to the semi-symmetric metric connection if*

$$g(\overline{P}(\phi X, Y)Z, \phi W) = 0, \tag{3.1}$$

for all $X, Y, Z, W \in \chi(M)$, where $\bar{P}(X,Y)Z$ is the projective curvature tensor with respect to the semi-symmetric metric connection connection and is given by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{2n}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y]. \tag{3.2}$$

Let M be a 3-dimensional pseudoprojectively flat f-Kenmotsu manifold with respect to the connection $\bar{\nabla}$. Then from the equation (3.1) and (3.2), it follows that

$$g(\overline{R}(\phi X, Y)Z, \phi W) = \frac{1}{2}(\overline{S}(Y, Z)g(\phi X, \phi W) - \overline{S}(\phi X, Z)g(Y, \phi W)). \tag{3.3}$$

Let $\{e_1, e_2, e_3 = \xi\}$ be a local orthonormal basis of the vector fields in M. Then $\{\phi e_1, \phi e_2, \xi\}$ is also local orthonormal basis of the vector field in this manifold. Putting $Y = Z = e_i$ in the equation (3.3) and taking summation over $1 \leq i \leq 3$, we obtain

$$\overline{S}(\phi X, \phi W) = \frac{1}{3}\overline{r}g(\phi X, \phi W), \tag{3.4}$$

which in view of (2.3), (2.22) and (2.23) becomes

$$S(\phi X, \phi W) = \frac{1}{3}(r + f + 1)(g(X, W) - \eta(X)\eta(W)). \tag{3.5}$$

Replacing X by ϕX, W by ϕW in (3.5) and using (2.1), we get

$$S(X, W) - \eta(W)S(X, \xi) - \eta(X)S(W, \xi) - 2(f^2 + f')\eta(X)\eta(W) = \frac{1}{3}(r + f + 1)g(\phi X, \phi W)$$

which by using (2.3) and (2.12) takes the form

$$\begin{aligned} S(X, W) &= \left\{\frac{(r + f + 1)}{3}\right\}g(X, W) \\ &\quad - \left\{\frac{(r + f + 1)}{3} + 2(f^2 + f')\right\}\eta(X)\eta(W). \end{aligned} \tag{3.6}$$

Thus we can state the following:

Theorem 3.1. *A 3-dimensional pseudoprojectively flat f-Kenmotsu manifold with respect to semi-symmetric metric connection is an $\eta-$Einstein manifold of the form (3.6).*

4. ϕ-PROJECTIVELY FLAT 3-DIMENSIONAL f-KENMOTSU MANIFOLD WITH RESPECT TO THE SEMI- SYMMETRIC METRIC CONNECTION

Definition 4.1. *An f-Kenmotsu manifold is said to be ϕ-projectively flat with respect to the semi-symmetric metric connection if*

$$g(\overline{P}(\phi X, \phi Y)\phi Z, \phi W) = 0 \tag{4.1}$$

for all $X, Y, Z, W \in \chi(M)$.

Let M be a 3-dimensional ϕ-projectively flat f-Kenmotsu manifold with respect to the semi-symmetric metric connection. Then from (3.2) and (4.1), it follows that

$$g(\overline{R}(\phi X, \phi Y)\phi Z, \phi W)) = \frac{1}{2}\{\overline{S}(\phi Y, \phi Z)g(\phi X, \phi W) - \overline{S}(\phi X, \phi Z)g(\phi Y, \phi W)\}. \tag{4.2}$$

Let $\{e_1, e_2, e_3 = \xi\}$ be a local orthonormal basis of the vector fields in M. Then $\{\phi e_1, \phi e_2, \xi\}$ is also local orthonormal basis of the vector fields in this manifold. Putting $Y = Z = e_i$ in (4.2) and taking summation from $i = 1$ to $i = 2$, we get

$$\overline{S}(\phi X, \phi W) = \frac{\overline{r}}{3}g(\phi X, \phi W) \tag{4.3}$$

which in view of (2.3), (2.22) and (2.23) becomes

$$S(\phi X, \phi W) = \frac{1}{3}(r + f + 1)[g(X, W) - \eta(X)\eta(W)]. \tag{4.4}$$

Now replacing X by ϕX, W by ϕW in (4.4) and using (2.1), (2.2), we get

$$S(X, W) - \eta(W)S(X, \xi) - \eta(X)S(W, \xi) - 2(f^2 + f')\eta(X)\eta(W) = \frac{1}{3}(r + f + 1)g(\phi X, \phi W)$$

which by using (2.3) and (2.12) turns to

$$\begin{aligned} S(X, W) &= \left\{\frac{(r + f + 1)}{3}\right\} g(X, W) \\ &\quad - \left\{\frac{(r + f + 1)}{3} + 2(f^2 + f')\right\} \eta(X)\eta(W). \end{aligned} \tag{4.5}$$

Next, by comparing (4.5) and (2.8), we get $r = 2(f + 1) - 6(f^2 + f')$. Thus we have the following:

Theorem 4.1. *A* 3*-dimensional ϕ-projectively flat f-Kenmotsu manifold with respect to the semi-symmetric metric connection is an η-Einstein manifold of the form* (4.5) *with the scalar curvature* $2(f + 1) - 6(f^2 + f')$.

5. 3-DIMENSIONAL f-KENMOTSU MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION ADMITTING CYCLIC PARALLEL RICCI TENSOR

Definition 5.1. *An f-Kenmotsu manifold with respect to the semi-symmetric metric connection is said to have cyclic parallel Ricci tensor if its Ricci tensor $\overline{S}$ of type* $(0, 2)$ *is non-zero and satisfies the following condition*

$$(\overline{\nabla}_X \overline{S})(Y, Z) + (\overline{\nabla}_Y \overline{S})(Z, X) + (\overline{\nabla}_Z \overline{S})(X, Y) = 0 \tag{5.1}$$

for all $X, Y, Z \in \chi(M)$.

Let a 3-dimensional f-Kenmotsu manifold with respect to the semi-symmetric metric connection admits cyclic parallel Ricci tensor, then (5.1) holds. From (2.8) and (2.22), we have

$$\bar{S}(Y, Z) = (\frac{r}{2} + f^2 + f' - 3f - 1)g(Y, Z) - (\frac{r}{2} + 3f^2 + 3f' - f - 1)\eta(Y)\eta(Z). \tag{5.2}$$

Taking covariant derivative of (5.2) with respect to X, we have

$$\begin{aligned} (\overline{\nabla}_X \overline{S})(Y, Z) &= [\frac{dr(X)}{2} + (2f - 3)(Xf) + (Xf')]g(Y, Z) \\ &\quad -[\frac{dr(X)}{2} + (6f - 1)(Xf) + 3(Xf')]\eta(Y)\eta(Z) \\ &\quad -(1 + f)[\frac{r}{2} + 3f^2 + 3f' - f - 1](g(\phi X, \phi Y)\eta(Z) + g(\phi X, \phi Z)\eta(Y)). \end{aligned} \tag{5.3}$$

Similarly, we can find

$$(\overline{\nabla}_Y\overline{S})(Z,X) = [\frac{dr(Y)}{2} + (2f-3)(Yf) + (Yf')]g(Z,X) \tag{5.4}$$
$$-[\frac{dr(Y)}{2} + (6f-1)(Yf) + 3(Yf')]\eta(Z)\eta(X)$$
$$-(1+f)[\frac{r}{2} + 3f^2 + 3f' - f - 1](g(\phi Y,\phi Z)\eta(X) + g(\phi Y,\phi X)\eta(Z))$$

and

$$(\overline{\nabla}_Z\overline{S})(X,Y) = [\frac{dr(Z)}{2} + (2f-3)(Zf) + (Zf')]g(X,Y) \tag{5.5}$$
$$-[\frac{dr(Z)}{2} + (6f-1)(Zf) + 3(Zf')]\eta(X)\eta(Y)$$
$$-(1+f)[\frac{r}{2} + 3f^2 + 3f' - f - 1](g(\phi Z,\phi X)\eta(Y) + g(\phi Z,\phi Y)\eta(X)).$$

Adding the equations (5.3)-(5.5), we find $(\overline{\nabla}_X\overline{S})(Y,Z) + (\overline{\nabla}_Y\overline{S})(Z,X) + (\overline{\nabla}_Z\overline{S})(X,Y) = 0$, if $r = -6f^2 + 2f + 2$, f being constant. Thus we have the following:

Theorem 5.1. *A* 3*-dimensional f-Kenmotsu manifold with respect to the semi- symmetric metric connection admits cyclic parallel Ricci tensor if the scalar curvature given by $-6f^2 + 2f + 2$ is constant, provided f constant.*

6. ϕ-RICCI SYMMETRIC 3-DIMENSIONAL f-KENMOTSU MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION

Let M be a ϕ-Ricci symmetric 3-dimensional f-Kenmotsu manifold with respect to the semi-symmetric metric connection, i.e., $\phi^2((\bar{\nabla}_X\bar{Q})Y) = 0$, which by virtue of (2.1) turns to

$$-(\bar{\nabla}_X\bar{Q})Y + \eta((\bar{\nabla}_X\bar{Q})Y)\xi = 0. \tag{6.1}$$

Taking the inner product of (6.1) with Z and using (2.2), we find

$$g((\bar{\nabla}_X\bar{Q})Y,Z) - \eta((\bar{\nabla}_X\bar{Q})Y)\eta(Z) = 0$$

from which it follows that

$$g(\bar{\nabla}_X\bar{Q}Y,Z) - S(\bar{\nabla}_XY,Z) - \eta((\bar{\nabla}_X\bar{Q})Y)\eta(Z) = 0. \tag{6.2}$$

Now putting $Y = \xi$ in (6.2) then using (2.25) and (2.26), we obtain

$$-2[\{(2f+1)(Xf) + (Xf')\}\,\eta(Z) + (f^2 + f' + f)(1+f)g(X,Z)] \tag{6.3}$$
$$-(1+f)\bar{S}(X,Z) - \eta((\bar{\nabla}_X\bar{Q})\xi)\eta(Z) = 0.$$

Replacing X by ϕX and Z by ϕZ in (6.3) yields

$$\bar{S}(\phi X,\phi Z) + 2(f^2 + f' + f)g(\phi X,\phi Z) = 0, \qquad (1+f) \neq 0. \tag{6.4}$$

In view of (2.3) and (2.28), (6.4) turns to

$$S(X,Z) = -(2f^2 + 2f' - f - 1)g(X,Z) - (f+1)\eta(X)\eta(Z). \tag{6.5}$$

Contracting (6.5) over X and Z, we get $r = -6(f^2 + f') + 2(f+1)$. Thus we can state the following:

Theorem 6.1. *A ϕ-Ricci symmetric 3-dimensional f-Kenmotsu manifold with respect to the semi-symmetric metric connection is an η-Einstein manifold of the form (6.5) with the scalar curvature $-6(f^2+f')+2(f+1)$.*

Next, from (2.8) and (2.22), we have

$$\overline{Q}Y = QY + (f+1)\eta(Y)\xi - (3f+1)Y, \tag{6.6}$$

$$QY = (\frac{r}{2} + f^2 + f')Y - (\frac{r}{2} + 3f^2 + 3f')\eta(Y)\xi, \tag{6.7}$$

respectively. By using (6.7) in (6.6), we have

$$\overline{Q}Y = (\frac{r}{2} + f^2 + f' - 3f - 1)Y - (\frac{r}{2} + 3f^2 + 3f' - f - 1)\eta(Y)\xi. \tag{6.8}$$

By covariant differentiation of (6.8) with respect to X and using (2.26), (2.27), we find

$$\begin{aligned}(\overline{\nabla}_X\overline{Q})Y &= [\frac{dr(X)}{2} + (2f-3)(Xf) + (Xf')]Y \\ &-[\frac{dr(X)}{2} + (6f-1)(Xf) + 3(Xf')]\eta(Y)\xi \\ -(\frac{r}{2} + 3f^2 + 3f' - f - 1)(1+f)&[(g(X,Y) - \eta(X)\eta(Y))\xi + \eta(Y)X - \eta(X)\eta(Y)\xi].\end{aligned} \tag{6.9}$$

Operating ϕ^2 to both sides of (6.9) and using (2.1), we have

$$\begin{aligned}\phi^2(\overline{\nabla}_X\overline{Q})Y &= [\frac{dr(X)}{2} + (2f-3)(Xf) + (Xf')](-Y + \eta(Y)\xi) \\ &-(\frac{r}{2} + 3f^2 + 3f' - f - 1)(1+f)\eta(Y)\phi^2X.\end{aligned} \tag{6.10}$$

Suppose that Y is orthogonal to ξ, then from the last equation it follows that

$$\phi^2(\overline{\nabla}_X\overline{Q})Y = -[\frac{dr(X)}{2} + (2f-3)(Xf) + (Xf')]Y. \tag{6.11}$$

In particular, if f is constant, then from (6.11), we have

$$\phi^2(\overline{\nabla}_X\overline{Q})Y = -\frac{dr(X)}{2}. \tag{6.12}$$

Thus we can state the following:

Theorem 6.2. *A 3-dimensional f-Kenmotsu manifold with respect to the semi-symmetric metric connection is locally ϕ-Ricci symmetric if and only if the scalar curvature r is constant, provided f is a constant.*

A 3-dimensional f-Kenmotsu manifold with respect to the semi-symmetric metric connection is said to have $\eta-$parallel Ricci tensor if $(\overline{\nabla}_X\overline{S})(\phi Y, \phi Z) = 0$ for any X, Y, Z on M [18].

Replacing Y by ϕY and Z by ϕZ in (5.3), we have

$$(\overline{\nabla}_X\overline{S})(\phi Y, \phi Z) = [\frac{dr(X)}{2} + (2f-3)(Xf) + (Xf')]g(\phi Y, \phi Z). \tag{6.13}$$

In particular, if f is constant, then from (6.13), we have

$$(\overline{\nabla}_X\overline{S})(\phi Y, \phi Z) = \frac{dr(X)}{2}. \tag{6.14}$$

Thus we can state the following:

Theorem 6.3. *In a 3-dimensional f-Kenmotsu manifold with respect to the semi-symmetric metric connection, the Ricci tensor is η-parallel if and only if the scalar curvature is constant, provided f is a constant.*

From the Theorems 6.2 and 6.3, we can state the following:

Theorem 6.4. *In a 3-dimensional f-Kenmotsu manifold with respect to the semi-symmetric metric connection, the Ricci tensor is η-parallel if and only if it is locally ϕ-Ricci symmetric, povided f is a constant.*

Example. We consider the 3-dimensional manifold $\widetilde{M} = \{(x, y, z) \in R^3 : y \neq 0\}$, where (x, y, z) are the standard coordinates in R^3. Define the vector fields

$$e_1 = e^y \frac{\partial}{\partial x}, \quad e_2 = e^y \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y} = \xi,$$

which are linearly independent at each point of $\widetilde{M}$ and form basis of tangent space at each point of $\widetilde{M}$. Let g be the Riemannian metric defined by

$$g_{ij} = \left\{ \begin{array}{ll} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{array} \right\}.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_3)$ for any $X \in \chi(\widetilde{M})$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = -e_2, \ \phi(e_2) = e_1, \ \phi(e_3) = 0.$$

Then from the linearity of ϕ and g, we have

$$\eta(e_3) \ = \ 1, \ \ \phi^2(X) = -X + \eta(X)e_3, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any $X, Y \in \chi(\widetilde{M})$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on $\widetilde{M}$. Now by computation, we obtain

$$[e_1, \ e_2] = 0, \quad [e_1, \ e_3] = -\, e_1, \quad [e_3, \ e_2] = e_2.$$

The Levi-Civita connection ∇ of the metric tensor g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]).$$

From the above formula, we get

$$\begin{array}{rcl} \nabla_{e_1} e_1 & = & e_3, \ \ \nabla_{e_3} e_1 = 0, \ \ \nabla_{e_2} e_1 = 0, \\ \nabla_{e_1} e_3 & = & -e_1, \ \nabla_{e_3} e_3 = 0, \ \ \nabla_{e_2} e_3 = -e_2, \\ \nabla_{e_1} e_2 & = & 0, \ \ \nabla_{e_3} e_2 = 0, \ \ \nabla_{e_2} e_2 = e_3. \end{array}$$

It can be easily verifed that the manifold satisfies $\nabla_X \xi = f[X - \eta(X)\xi]$ for $\xi = e_3$, where $f = -1$. Hence we conclude that M is an f-Kenmotsu manifold. Also $f^2 + f' \neq 0$. Hence M is a regular f-Kenmotsu manifold. It known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

With the help of the above formula it can be easily obtain

$$R(e_1, \ e_2)e_3 = 0, \quad R(e_2, \ e_3)e_3 = -e_2, \quad R(e_1, \ e_3)e_3 = -e_1,$$

$$R(e_1, \ e_2)e_2 = -e_1, \ \ R(e_3, \ e_2)e_2 = -\, e_3, \ \ R(e_1, \ e_3)e_2 = 0,$$

$$R(e_2, \ e_1)e_1 = -\, e_2, \quad R(e_2, \ e_3)e_1 = 0, \quad R(e_3, \ e_1)e_1 = -\, e_3.$$

From the above expressions of the curvature tensors, we obtain

$$S(e_{1,}e_1) = S(e_2, e_2) = S(e_3, e_3) = -2.$$

Thus, we have $S(X,Y) = -2g(X,Y)$. Hence we get $r = -6$. Now, we consider a linear connection $\overline{\nabla}$ such that

$$\overline{\nabla}_{e_i}e_j = \nabla_{e_i}e_j + \eta(e_j)e_i - g(e_i, e_j)e_3 \quad \text{for all } i, j = 1, 2, 3.$$

It can be easily seen that $\overline{\nabla}_{e_i}e_j = 0$ $(1 \leq i, j \leq 3)$ from which it follows that $\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z = 0$ $(1 \leq i, j \leq 3)$. Thus the manifold is a flat with respect to the semi-symmetric metric connection.
Hence, this is an example 3-dimensional f-Kenmotsu manifold which is an Einstein manifold with respect to the Levi-Civita connection and is flat with respect to the semi-symmetric metric connection. For $f = -1$, it can be seen that the given example satisfies all the theorems of the paper.

Acknowledgments. The authors are thankful to the editor and anonymous referees for their valuable suggestions in the improvement of the paper.

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