ON THREE DIMENSIONAL $f$-KENMOTSU MANIFOLDS WITH A CERTAIN CONNECTION

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ABSTRACT. The object of the present paper is to study 3-dimensional $f$-Kenmotsu manifolds with respect to the semi-symmetric metric connection satisfying certain curvature conditions. Finally, we give an example of 3-dimensional $f$-Kenmotsu manifolds.

1. INTRODUCTION

In 1991, the notion of an $f$-Kenmotsu manifold was introduced by Olszak and Rosca [18] which are normal locally conformal and almost cosymplectic manifolds. Further, they give a geometric interpretation of $f$-Kenmotsu manifold and proved that a Ricci symmetric $f$-Kenmotsu manifold is an Einstein manifold. Recently, $f$-Kenmotsu manifolds have been studied by various authors in several ways to a different extent such as ([4], [6]-[8], [13], [16]) and many others. In 1924, the notion of the semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [1]. Later in 1932, Hayden [10] introduced the idea of metric connection with a torsion on a Riemannian manifold. A semi-symmetric connection on a Riemannian manifold was systematically studied by Yano [12], which was further studied by Haseeb [2], Haseeb and Prasad [3], Sharfuddin and Hussain [5], Amur and Pujar [11], Binh [14], De and Biswas [15] and many others.

The projective curvature tensor is an important tensor from the differential geometric point of view. Let $M$ be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one to one correspondence between each coordinate neighbourhood of $M$ and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then $M$ is said to be locally projectively flat. For $n \geq 1$, $M$ is locally projectively flat if and only if the well known projective curvature tensor $P$ vanishes, the projective curvature tensor is defined by [17]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y],$$  \hspace{1cm} (1.1)

where $X, Y, Z \in \chi(M)$, $R$ is the curvature tensor and $S$ is the Ricci tensor with respect to the Levi-Civita connection, respectively.

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Motivated by the above studies, in this paper we study certain curvature conditions on 3-dimensional $f$-Kenmotsu manifolds with respect to the semi-symmetric metric connection. The paper is organized as follows: After the introduction, section 2 is concerned with some preliminaries. Pseudoprojectively flat and $\phi$-projectively flat 3-dimensional $f$-Kenmotsu manifolds with respect to the semi-symmetric metric connection have studied in sections 3 and 4, respectively. Section 5 deals with the study of 3-dimensional $f$-Kenmotsu manifolds with respect to the semi-symmetric metric connection admitting cyclic parallel Ricci tensor. In section 6, we study $\phi$-Ricci symmetric 3-dimensional $f$-Kenmotsu manifolds with respect to the semi-symmetric metric connection.

2. PRELIMINARIES

Let $M$ be a $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, g)$ which satisfies the following equations [9]:

\[ \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad (2.1) \]

\[ \eta(X) = g(X, \xi), \quad \eta \circ \phi = 0, \quad (2.2) \]

\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.3) \]

for all vector fields $X, Y \in \chi(M)$; where $\chi(M)$ is a set of all smooth vector fields on $M$, $I$ is the identity map on the tangent bundle $TM$, $\eta$ is a 1-form, $\xi$ is a vector field, $g$ is a metric tensor field and $\phi$ is a vector field of type $(1,1)$. We say that $(M, \phi, \xi, \eta, g)$ is an $f$-Kenmotsu manifold if the Levi-Civita connection of $g$ satisfy

\[ (\nabla_X \phi) Y = f[g(\phi X, Y)\xi - \eta(Y)\phi X], \quad (2.4) \]

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$. If $f = 0$, then the manifold is cosymplectic [18]. An $f$-Kenmotsu manifold is said to be regular if $f^2 + f' \neq 0$, where $f' = \xi f$.

In an $f$-Kenmotsu manifold, from (2.4) we have

\[ \nabla_X \xi = f[X - \eta(X)\xi]. \quad (2.5) \]

The condition $df \wedge \eta = 0$ holds if $\dim M \geq 5$. This does not hold in general if $\dim M = 3$ [6]. For a 3-dimensional Riemannian manifold, we have

\[ R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y]. \quad (2.6) \]

In a 3-dimensional $f$-Kenmotsu manifold $M$, we have [18]

\[ R(X, Y)Z = \left( \frac{r}{2} + 2f^2 + 2f' \right)[g(Y, Z)X - g(X, Z)Y] \]

\[ -\left( \frac{r}{2} + 3f^2 + 3f' \right)\eta(X)\xi - g(X, Z)\eta(Y)\eta(Z)X - \eta(X)\eta(Y)\eta(Z)Y, \quad (2.7) \]

\[ S(X, Y) = \left( \frac{r}{2} + f^2 + f' \right)g(X, Y) - \left( \frac{r}{2} + 3f^2 + 3f' \right)\eta(X)\eta(Y), \quad (2.8) \]

\[ R(X, Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y], \quad (2.9) \]

\[ R(\xi, X)Y = -(f^2 + f')[g(X, Y)\xi - \eta(Y)X], \quad (2.10) \]

\[ \eta(R(X, Y)Z) = -(f^2 + f')[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \quad (2.11) \]

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for all vector fields \( X, Y, Z \in \chi(M) \), where \( R, S, Q \) and \( r \) are the Riemannian curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature, respectively on \( M \). Also from (2.8), we get

\[
S(X, \xi) = -2(f^2 + f')\eta(X),
\]

(2.12)

\[
Q\xi = -2(f^2 + f')\xi.
\]

(2.13)

Using (2.5), we have

\[
(\nabla_X \eta)Y = f[g(X, Y) - \eta(X)\eta(Y)],
\]

(2.14)

Also from (2.8) it follows that

\[
S(\phi X, \phi Y) = S(X, Y) + 2(f^2 + f')\eta(X)\eta(Y),
\]

(2.15)

for all vector fields \( X, Y \in \chi(M) \).

**Definition 2.1.** An \( f \)-Kenmotsu manifold \( M \) is said to be an \( \eta \)-Einstein manifold if its non-vanishing Ricci tensor \( S \) is of the form

\[
S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),
\]

where \( a, b \) are smooth functions on \( M \). If \( b = 0 \), then \( M \) is said to be an Einstein manifold.

Let \( (M, g) \) be a Riemannian manifold with the Levi-Civita connection \( \nabla \). A linear connection \( \nabla \) on \( (M, g) \) is said to be semi-symmetric [12] if its torsion tensor \( T \) is given by

\[
T(X, Y) = \pi(Y)X - \pi(X)Y,
\]

where \( \pi \) is a 1-form on \( M \) and associated with the vector field \( \rho \) by

\[
\pi(X) = g(X, \rho)
\]

for all vector fields \( X \in \chi(M) \).

A semi-symmetric connection \( \nabla \) is called a semi-symmetric metric if it satisfies the condition

\[
\nabla g = 0.
\]

On an almost contact metric manifold, a semi-symmetric metric connection is defined by replacing 1-form \( \pi \) by the contact 1-form \( \eta \), i.e.,

\[
T(X, Y) = \eta(Y)X - \eta(X)Y,
\]

where \( \eta(X, \xi) = \eta(X) \) for all \( X \in \chi(M) \).

A relation between a semi-symmetric metric connection \( \nabla \) and the Levi-Civita connection \( \nabla \) on \( M \) and is given by [12]

\[
\nabla_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi,
\]

(2.16)

where \( \eta(X) = g(X, \xi) \).

A relation between the curvature tensors \( R \) and \( \overline{R} \) of the connections \( \nabla \) and \( \nabla \), respectively is given by [1]

\[
\overline{R}(X, Y)Z = R(X, Y)Z + K(X, Z)Y - K(Y, Z)X + g(X, Z)FY - g(Y, Z)FX,
\]

(2.17)

where \( K \) is a tensor field of type \((0, 2)\) and \( F \) is a \((1, 1)\)-tensor field which is given by

\[
K(Y, Z) = g(FY, Z) = (\nabla_Y \eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}g(Y, Z).
\]

(2.18)
For a 3-dimensional $f$-Kenmotsu manifold, (2.18) takes the form

$$K(Y, Z) = g(FY, Z) = -(f + 1)\eta(Y)\eta(Z) + \left( f + \frac{1}{2} \right)g(Y, Z).$$  \hspace{1cm} (2.19)

It yields

$$FY = -(f + 1)\eta(Y)\xi + \left( f + \frac{1}{2} \right)Y.$$  \hspace{1cm} (2.20)

Thus by using (2.19), (2.20) in (2.17), we get

$$\bar{R}(X, Y)Z = R(X, Y)Z - (f + 1)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X]$$
$$- (f + 1)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]\xi$$
$$+ (2f + 1)[g(X, Z)Y - g(Y, Z)X].$$  \hspace{1cm} (2.21)

Contracting (2.21) over $X$, we get

$$S(Y, Z) = S(Y, Z) + (f + 1)\eta(Y)\eta(Z) - (3f + 1)g(Y, Z),$$  \hspace{1cm} (2.22)

where $S$ and $\bar{S}$ are the Ricci tensors of the connections $\nabla$ and $\bar{\nabla}$, respectively on $M$. Again contracting $Y$ and $Z$ in (2.22), it follows that

$$\tau = r - 8f - 2.$$  \hspace{1cm} (2.23)

where $\tau$ and $r$ are the scalar curvatures of the connections $\nabla$ and $\bar{\nabla}$, respectively on $M$.

**Lemma 2.1.** Let $M$ be a 3-dimensional $f$-Kenmotsu manifold with respect to the semi-symmetric metric connection. Then we have

$$\bar{S}(X, \xi) = -2(f^2 + f' + f)\eta(X),$$  \hspace{1cm} (2.24)

$$Q\xi = -2(f^2 + f' + f)\xi,$$  \hspace{1cm} (2.25)

$$\bar{\nabla}X\xi = (1 + f)(X - \eta(X)\xi),$$  \hspace{1cm} (2.26)

$$(\bar{\nabla}X\eta)Y = (1 + f)g(\phi X, \phi Y),$$  \hspace{1cm} (2.27)

$$\bar{S}(\phi X, \phi Y) = S(X, Y) - (3f + 1)g(X, Y)$$
$$+ (2f^2 + 2f' + 3f + 1)\eta(X)\eta(Y).$$  \hspace{1cm} (2.28)

for all $X, Y \in \chi(M)$.

### 3. PSEUDOPROJECTIVELY FLAT 3-DIMENSIONAL $f$-KENMOTSU MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION

**Definition 3.1.** An $f$-Kenmotsu manifold is said to be pseudoprojectively flat with respect to the semi-symmetric metric connection if

$$g(\bar{P}(\phi X, Y)Z, \phi W) = 0,$$  \hspace{1cm} (3.1)

for all $X, Y, Z, W \in \chi(M)$, where $\bar{P}(X, Y)Z$ is the projective curvature tensor with respect to the semi-symmetric metric connection and is given by

$$\bar{P}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{2h}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y].$$  \hspace{1cm} (3.2)
Let $M$ be a 3-dimensional pseudoprojectively flat $f$-Kenmotsu manifold with respect to the connection $\nabla$. Then from the equation (3.1) and (3.2), it follows that
\[
g(\mathcal{R}(\phi X, Y)Z, \phi W) = \frac{1}{2}(\mathcal{S}(Y, Z)g(\phi X, \phi W) - \mathcal{S}(\phi X, Z)g(Y, \phi W)).
\] (3.3)

Let $\{e_1, e_2, e_3 = \xi\}$ be a local orthonormal basis of the vector fields in $M$. Then $\{\phi e_1, \phi e_2, \xi\}$ is also local orthonormal basis of the vector field in this manifold. Putting $Y = Z = e_i$ in the equation (3.3) and taking summation over $1 \leq i \leq 3$, we obtain
\[
\mathcal{S}(\phi X, \phi W) = \frac{1}{3} \mathcal{R}_g(\phi X, \phi W),
\] (3.4)
which in view of (2.3), (2.22) and (2.23) becomes
\[
\mathcal{S}(\phi X, \phi W) = \frac{1}{3}(r + f + 1)(g(X, W) - \eta(X)\eta(W)).
\] (3.5)

Replacing $X$ by $\phi X, W$ by $\phi W$ in (3.5) and using (2.1), we get
\[
\mathcal{S}(X, W) - \eta(W)\mathcal{S}(X, \xi) - \eta(X)\mathcal{S}(W, \xi) - 2(f^2 + f')\eta(X)\eta(W) = \frac{1}{3}(r + f + 1)g(\phi X, \phi W)
\]
which by using (2.3) and (2.12) takes the form
\[
\mathcal{S}(X, W) = \left\{ \frac{(r + f + 1)}{3} \right\} g(X, W) - \left\{ \frac{(r + f + 1)}{3} + 2(f^2 + f') \right\} \eta(X)\eta(W).
\] (3.6)

Thus we can state the following:

**Theorem 3.1.** A 3-dimensional pseudoprojectively flat $f$-Kenmotsu manifold with respect to semi-symmetric metric connection is an $\eta$–Einstein manifold of the form (3.6).

4. **$\phi$-PROJECTIVELY FLAT 3-DIMENSIONAL $f$-KENMOTSU MANIFOLD WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION**

**Definition 4.1.** An $f$-Kenmotsu manifold is said to be $\phi$-projectively flat with respect to the semi-symmetric metric connection if
\[
g(\mathcal{R}(\phi X, \phi Y)\phi Z, \phi W) = 0
\] (4.1)
for all $X, Y, Z, W \in \chi(M)$.

Let $M$ be a 3-dimensional $\phi$-projectively flat $f$-Kenmotsu manifold with respect to the semi-symmetric metric connection. Then from (3.2) and (4.1), it follows that
\[
g(\mathcal{R}(\phi X, \phi Y)\phi Z, \phi W)) = \frac{1}{2}\{\mathcal{S}(\phi Y, \phi Z)g(\phi X, \phi W) - \mathcal{S}(\phi X, \phi Z)g(\phi Y, \phi W)\}.
\] (4.2)

Let $\{e_1, e_2, e_3 = \xi\}$ be a local orthonormal basis of the vector fields in $M$. Then $\{\phi e_1, \phi e_2, \xi\}$ is also local orthonormal basis of the vector fields in this manifold. Putting $Y = Z = e_i$ in (4.2) and taking summation from $i = 1$ to $i = 2$, we get
\[
\mathcal{S}(\phi X, \phi W) = \frac{7}{3}g(\phi X, \phi W)
\] (4.3)
which in view of (2.3), (2.22) and (2.23) becomes
\[
S(\phi X, \phi W) = \frac{1}{3} (r + f + 1) [g(X, W) - \eta(X)\eta(W)]. \tag{4.4}
\]
Now replacing \( X \) by \( \phi X \), \( W \) by \( \phi W \) in (4.4) and using (2.1), (2.2), we get
\[
S(X, W) - \eta(W)S(X, \xi) - \eta(X)S(W, \xi) - 2(f^2 + f')\eta(X)\eta(W) = \frac{1}{3} (r + f + 1) g(\phi X, \phi W)
\]
which by using (2.3) and (2.12) turns to
\[
S(X, W) = \left\{ \frac{(r + f + 1)}{3} \right\} g(X, W) \tag{4.5}
\]
\[
- \left\{ \frac{(r + f + 1)}{3} + 2(f^2 + f') \right\} \eta(X)\eta(W).
\]
Next, by comparing (4.5) and (2.8), we get \( r = 2(f + 1) - 6(f^2 + f') \). Thus we have the following:

**Theorem 4.1.** A 3-dimensional \( \phi \)-projectively flat \( f \)-Kenmotsu manifold with respect to the semi-symmetric metric connection is an \( \eta \)-Einstein manifold of the form (4.5) with the scalar curvature \( 2(f + 1) - 6(f^2 + f') \).

5. 3-DIMENSIONAL \( f \)-KENMOTSU MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION ADMITTING CYCLIC PARALLEL RICCI TENSOR

**Definition 5.1.** An \( f \)-Kenmotsu manifold with respect to the semi-symmetric metric connection is said to have cyclic parallel Ricci tensor if its Ricci tensor \( S \) of type \((0,2)\) is non-zero and satisfies the following condition
\[
(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0 \tag{5.1}
\]
for all \( X, Y, Z \in \chi(M) \).

Let a 3-dimensional \( f \)-Kenmotsu manifold with respect to the semi-symmetric metric connection admits cyclic parallel Ricci tensor, then (5.1) holds. From (2.8) and (2.22), we have
\[
\bar{S}(Y, Z) = \left( \frac{r}{2} + f^2 + f' - 3f - 1 \right) g(Y, Z) - \left( \frac{r}{2} + 3f^2 + 3f' - f - 1 \right) \eta(Y)\eta(Z). \tag{5.2}
\]
Taking covariant derivative of (5.2) with respect to \( X \), we have
\[
(\nabla_X \bar{S})(Y, Z) = \left[ \frac{dr(X)}{2} + (2f - 3)(Xf) + (Xf') \right] g(Y, Z) \tag{5.3}
\]
\[
- \left[ \frac{dr(X)}{2} + (6f - 1)(Xf) + 3(Xf') \right] \eta(Y)\eta(Z)
\]
\[
- (1 + f) \left( \frac{r}{2} + 3f^2 + 3f' - f - 1 \right) [g(\phi X, \phi Y)\eta(Z) + g(\phi X, \phi Z)\eta(Y)].
\]
Similarly, we can find
\[ (\nabla_Y \bar{S})(Z, X) = \left[ \frac{dr(Y)}{2} + (2f - 3)(Yf) + (Yf') \right] g(Z, X) \] (5.4)
\[ -\left[ \frac{dr(Y)}{2} + (6f - 1)(Yf) + 3(Yf') \right] \eta(Z) \eta(X) \]
\[ -(1 + f) \left[ \frac{r}{2} + 3f^2 + 3f' - f - 1 \right] (g(\phi Y, \phi Z) \eta(X) + g(\phi Y, \phi X) \eta(Z)) \]
and
\[ (\nabla_Z \bar{S})(X, Y) = \left[ \frac{dr(Z)}{2} + (2f - 3)(Zf) + (Zf') \right] g(X, Y) \] (5.5)
\[ -\left[ \frac{dr(Z)}{2} + (6f - 1)(Zf) + 3(Zf') \right] \eta(X) \eta(Y) \]
\[ -(1 + f) \left[ \frac{r}{2} + 3f^2 + 3f' - f - 1 \right] (g(\phi Z, \phi X) \eta(Y) + g(\phi Z, \phi Y) \eta(X)). \]

Adding the equations (5.3)-(5.5), we find \((\nabla_X \bar{S})(Y, Z) + (\nabla_Y \bar{S})(Z, X) + (\nabla_Z \bar{S})(X, Y) = 0\), if \(r = -6f^2 + 2f + 2\), \(f\) being constant. Thus we have the following:

**Theorem 5.1.** A 3-dimensional \(f\)-Kenmotsu manifold with respect to the semi-symmetric metric connection admits cyclic parallel Ricci tensor if the scalar curvature given by \(-6f^2 + 2f + 2\) is constant, provided \(f\) constant.

6. **\(\phi\)-RICCI SYMMETRIC 3-DIMENSIONAL \(f\)-KENMOTSU MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION**

Let \(M\) be a \(\phi\)-Ricci symmetric 3-dimensional \(f\)-Kenmotsu manifold with respect to the semi-symmetric metric connection, i.e., \(\phi^2(\nabla_X \bar{Q}) Y = 0\), which by virtue of (2.1) turns to
\[ -(\nabla_X \bar{Q}) Y + \eta(\nabla_X \bar{Q}) Y \xi = 0. \] (6.1)

Taking the inner product of (6.1) with \(Z\) and using (2.2), we find
\[ g((\nabla_X \bar{Q}) Y, Z) - \eta((\nabla_X \bar{Q}) Y) \eta(Z) = 0 \]
from which it follows that
\[ g(\nabla_X \bar{Q} Y, Z) - S(\nabla_X Y, Z) - \eta((\nabla_X \bar{Q}) Y) \eta(Z) = 0. \] (6.2)

Now putting \(Y = \xi\) in (6.2) then using (2.25) and (2.26), we obtain
\[ -2[\{ (2f + 1)(Xf) + (Xf') \} \eta(Z) + (f^2 + f' + f)(1 + f) g(X, Z)] \]
\[ -(1 + f) \bar{S}(X, Z) - \eta((\nabla_X \bar{Q}) \xi) \eta(Z) = 0. \] (6.3)
Replacing \(X\) by \(\phi X\) and \(Z\) by \(\phi Z\) in (6.3) yields
\[ \bar{S}(\phi X, \phi Z) + 2(f^2 + f' + f) g(\phi X, \phi Z) = 0, \quad (1 + f) \neq 0. \] (6.4)

In view of (2.3) and (2.28), (6.4) turns to
\[ S(X, Z) = -(2f^2 + 2f' - f - 1) g(X, Z) - (f + 1) \eta(X) \eta(Z). \] (6.5)
Contracting (6.5) over \(X\) and \(Z\), we get \(r = -6(f^2 + f') + 2(f + 1)\). Thus we can state the following:
Theorem 6.1. A φ-Ricci symmetric 3-dimensional f-Kenmotsu manifold with respect to the semi-symmetric metric connection is an η-Einstein manifold of the form (6.5) with the scalar curvature \(-6(f^2 + f') + 2(f + 1)\).

Next, from (2.8) and (2.22), we have
\[
\overline{Q}Y = QY + (f + 1)\eta(Y)\xi - (3f + 1)Y,
\]
\[
QY = (\frac{r}{2} + f^2 + f')Y - (\frac{r}{2} + 3f^2 + 3f')\eta(Y)\xi,
\]
respectively. By using (6.7) in (6.6), we have
\[
\overline{Q}Y = (\frac{r}{2} + f^2 + f')Y - (\frac{r}{2} + 3f^2 + 3f' - f - 1)\eta(Y)\xi.
\]

By covariant differentiation of (6.8) with respect to X and using (2.26), (2.27), we find
\[
(\nabla_X \overline{Q})Y = \left[\frac{dr(X)}{2} + (2f - 3)(Xf) + (Xf')\right]Y - (\frac{r}{2} + 3f^2 + 3f' - f - 1)(1 + f)[g(X, Y - \eta(X)\eta(Y))\xi + \eta(Y)X - \eta(X)\eta(Y)\xi].
\]

Operating \(\phi^2\) to both sides of (6.9) and using (2.1), we have
\[
\phi^2(\nabla_X \overline{Q})Y = \left[\frac{dr(X)}{2} + (2f - 3)(Xf) + (Xf')\right](-Y + \eta(Y)\xi) + (\frac{r}{2} + 3f^2 + 3f' - f - 1)(1 + f)\eta(Y)\phi^2 X.
\]

Suppose that Y is orthogonal to \(\xi\), then from the last equation it follows that
\[
\phi^2(\nabla_X \overline{Q})Y = -\left[\frac{dr(X)}{2} + (2f - 3)(Xf) + (Xf')\right]Y.
\]

In particular, if f is constant, then from (6.11), we have
\[
\phi^2(\nabla_X \overline{Q})Y = -\frac{dr(X)}{2}.
\]

Thus we can state the following:

Theorem 6.2. A 3-dimensional f-Kenmotsu manifold with respect to the semi-symmetric metric connection is locally φ-Ricci symmetric if and only if the scalar curvature \(r\) is constant, provided \(f\) is a constant.

A 3-dimensional f-Kenmotsu manifold with respect to the semi-symmetric metric connection is said to have \(\eta\)–parallel Ricci tensor if \((\nabla_X \overline{S})(\phi Y, \phi Z) = 0\) for any \(X, Y, Z\) on \(M\) [18].

Replacing \(Y\) by \(\phi Y\) and \(Z\) by \(\phi Z\) in (5.3), we have
\[
(\nabla_X \overline{S})(\phi Y, \phi Z) = \left[\frac{dr(X)}{2} + (2f - 3)(Xf) + (Xf')\right]g(\phi Y, \phi Z).
\]

In particular, if \(f\) is constant, then from (6.13), we have
\[
(\nabla_X \overline{S})(\phi Y, \phi Z) = \frac{dr(X)}{2}.
\]

Thus we can state the following:
**Theorem 6.3.** In a 3-dimensional $f$-Kenmotsu manifold with respect to the semi-symmetric metric connection, the Ricci tensor is $\eta$-parallel if and only if the scalar curvature is constant, provided $f$ is a constant.

From the Theorems 6.2 and 6.3, we can state the following:

**Theorem 6.4.** In a 3-dimensional $f$-Kenmotsu manifold with respect to the semi-symmetric metric connection, the Ricci tensor is $\eta$-parallel if and only if it is locally $\phi$-Ricci symmetric, provided $f$ is a constant.

**Example.** We consider the 3-dimensional manifold $\tilde{M} = \{(x, y, z) \in \mathbb{R}^3 : y \neq 0\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. Define the vector fields

\[ e_1 = x^y \frac{\partial}{\partial x}, \quad e_2 = x^y \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y} = \xi, \]

which are linearly independent at each point of $\tilde{M}$ and form basis of tangent space at each point of $\tilde{M}$. Let $g$ be the Riemannian metric defined by

\[ g_{ij} = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}. \]

Let $\eta$ be the 1-form defined by $\eta(X) = g(X, e_3)$ for any $X \in \chi(\tilde{M})$. Let $\phi$ be the $(1, 1)$-tensor field defined by

\[ \phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0. \]

Then from the linearity of $\phi$ and $g$, we have

\[ \eta(e_3) = 1, \quad \phi^2(X) = -X + \eta(X)e_3, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \]

for any $X, Y \in \chi(\tilde{M})$. Thus for $e_3 = \xi$, $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $\tilde{M}$. Now by computation, we obtain

\[ [e_1, e_2] = 0, \quad [e_1, e_3] = -e_1, \quad [e_3, e_2] = e_2. \]

The Levi-Civita connection $\nabla$ of the metric tensor $g$ is given by

\[ 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [X, Z]) + g(Z, [X, Y]). \]

From the above formula, we get

\[ \nabla_{e_1} e_1 = e_3, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_1 = 0, \]

\[ \nabla_{e_3} e_3 = -e_1, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_3} e_3 = -e_2, \]

\[ \nabla_{e_3} e_2 = 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_3} e_2 = e_3. \]

It can be easily verified that the manifold satisfies $\nabla_X \xi = f[X - \eta(X)\xi]$ for $\xi = e_3$, where $f = -1$. Hence we conclude that $M$ is an $f$-Kenmotsu manifold. Also $f^2 + f' \neq 0$. Hence $M$ is a regular $f$-Kenmotsu manifold. It known that

\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \]

With the help of the above formula it can be easily obtain

\[ R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1, \]

\[ R(e_1, e_2)e_2 = -e_1, \quad R(e_3, e_2)e_2 = -e_3, \quad R(e_1, e_3)e_2 = 0, \]

\[ R(e_2, e_1)e_1 = -e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_3, e_1)e_1 = -e_3. \]
From the above expressions of the curvature tensors, we obtain

\[ S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = -2. \]

Thus, we have \( S(X, Y) = -2g(X, Y) \). Hence we get \( r = -6 \). Now, we consider a linear connection \( \nabla \) such that

\[ \nabla e_i e_j = \nabla e_i e_j + \eta(e_j)e_i - g(e_i, e_j)e_3 \]

for all \( i, j = 1, 2, 3 \).

It can be easily seen that \( \nabla e_i e_j = 0 \) \((1 \leq i, j \leq 3)\) from which it follows that \( R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z = 0 \) \((1 \leq i, j \leq 3)\). Thus the manifold is a flat with respect to the semi-symmetric metric connection.

Hence, this is an example 3-dimensional \( f \)-Kenmotsu manifold which is an Einstein manifold with respect to the Levi-Civita connection and is flat with respect to the semi-symmetric metric connection. For \( f = -1 \), it can be seen that the given example satisfies all the theorems of the paper.

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