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CONCURRENCE AND COLLINEARITY IN CONVEX QUADRILATERALS

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ABSTRACT. This paper highlights several concurrence and collinearity properties of convex quadrilaterals. Let w_a, w_b, w_c, w_d be the internal angle bisectors of the convex quadrilateral $ABCD$ and let $\{S\} = w_b \cap w_d$, $\{T\} = w_a \cap w_c$. The line ST intersects the sides CD, DA, AB, BC in X_a, X_b, X_c, X_d, respectively. Any of these points belongs to two more relevant lines (or three, if $ABCD$ is cyclic), as seen in Theorems 1, 2, 6 and 8. Similar results hold for external angle bisectors.

1. INTRODUCTION

Let $ABCD$ be a convex quadrilateral. We use the following notations (see Fig. 1a and 1b).

- A, B, C, D - vertices or measures of angles;
- w_a, w_b, w_c, w_d - internal bisectors of the angles $\angle A, \angle B, \angle C, \angle D$, respectively;
- w'_a, w'_b, w'_c, w'_d - external bisectors of the angles $\angle A, \angle B, \angle C, \angle D$, respectively;
- I - intersection of the diagonals;
- $\{M_a\} = w_a \cap BC$, $\{N_a\} = w_b \cap AD$, the points $M_b, N_b, M_c, N_c, M_d, N_d$ being similarly defined;
- $\{P_a\} = w_a \cap BD$, $\{Q_a\} = w_b \cap AC$ etc.;
- $\{M'_a\} = w'_a \cap BC$, $\{N'_a\} = w'_b \cap AD$ etc.;
- $\{P'_a\} = w'_a \cap BD$, $\{Q'_a\} = w'_b \cap AC$ etc.;
- $\{S\} = w_b \cap w_d$, $\{T\} = w_a \cap w_c$, and $\{S'\} = w'_b \cap w'_d$, $\{T'\} = w'_a \cap w'_c$.

By using subscripts a, b, c, d, we wish to highlight the side we are referring to. In doing so, certain points will end having two different notations. For instance, N_a and M_b or Q_a and P_b will represent the same points. Hopefully, this will not lead to any ambiguity.
Our purpose is to study the concurrence of the lines M_aN_a, P_aQ_a, CD and of their analogues. We shall show that the lines M_aN_a and P_aQ_a are either concurrent in a point X_a which belongs to CD or are parallel to CD, provided that they exist. We shall also prove that the points X_a, X_b, X_c, X_d are collinear, as they lie on the line ST. Similar results will be seen to hold for the external angle bisectors of the quadrilateral. A special attention will given to cyclic quadrilaterals. For basic properties of quadrilaterals, see [1], [2], [3].

2010 *Mathematics Subject Classification.* Quadrilateral, internal (external) angle bisector, collinear points, concurrent lines.
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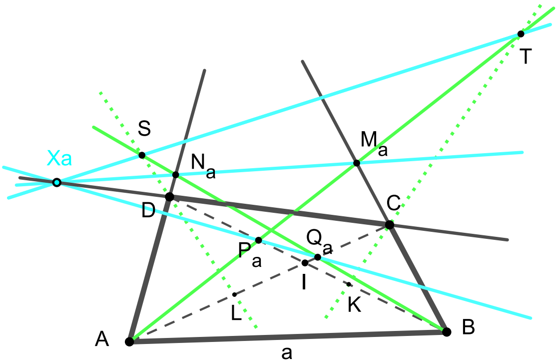

Fig. 1a

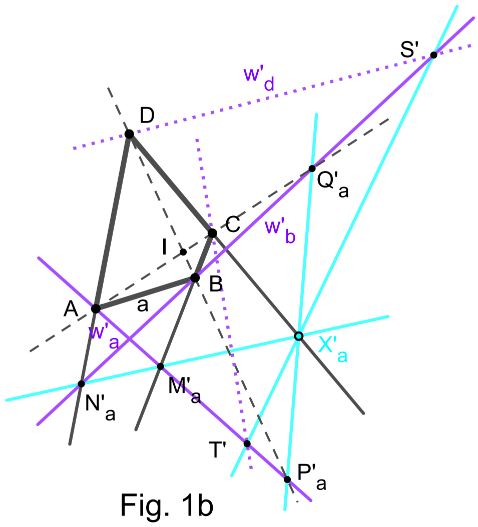

Fig. 1b

2. Convex quadrilateral, general case

First, note that the points defined above do not always exist. To discuss their existence, we must first observe the following basic properties of angle bisectors.

(1) The internal bisector w_a is parallel to the side BC if and only if $\dfrac{A}{2} + B = \pi$.

(2) The external bisector w'_a is parallel to the side BC if and only if $\dfrac{A}{2} + B = \dfrac{\pi}{2}$.

(3) The points P_a and Q_a coincide if and only if the bisectors w_a and w_b coincide with the diagonals AC and BD, respectively.

(4) The external bisector w'_a is parallel to the diagonal BD if and only if $a = d$.

(5) The internal (or external) bisectors of the opposite angles $\angle A$ and $\angle C$ are parallel if and only if $B = D$.

We now consider two cases.

2.1. **Internal angle bisectors.** Denote by Q_0 the set of convex quadrilaterals satisfying conditions (i), (ii) and (iii) indicated below.

(i) The points $M_a, N_a, M_b, N_b, M_c, N_c, M_d, N_d$ exist, i.e. the following conditions hold
  (i$_a$) $\dfrac{A}{2} + B \neq \pi$ and $A + \dfrac{B}{2} \neq \pi$ (M_a, N_a exist),
  (i$_b$) $\dfrac{B}{2} + C \neq \pi$ and $B + \dfrac{C}{2} \neq \pi$ (M_b, N_b exist),
  (i$_c$) $\dfrac{C}{2} + D \neq \pi$ and $C + \dfrac{D}{2} \neq \pi$ (M_c, N_c exist),
  (i$_d$) $\dfrac{D}{2} + A \neq \pi$ and $D + \dfrac{A}{2} \neq \pi$ (M_d, N_d exist).

(ii) $I \notin w_a$, $I \notin w_b$, $I \notin w_c$, $I \notin w_d$, i.e. the internal bisectors w_a, w_b, w_c, w_d are not diagonals).

(iii) $A \neq C$ and $B \neq D$, i.e. the points S and T exist.

The main results of this paper refer to quadrilaterals belonging to Q_0. However, many of them remain valid or can be adapted to quadrilaterals that satisfy only a few of these conditions.

From now on, we shall use an xy-Cartesian coordinate system in the plane of the quadrilateral $ABCD$, with origin A and x- and y- axes AB, AD, respectively, as seen in Fig. 2. The oblique Cartesian coordinates of the points A, B, D are then $(0,0), (a,0)$ and $(0,d)$, respectively. Moreover, we denote $C = C(\lambda, \mu)$ and by the sine formula applied to $\triangle CC'B$ with $CC' \parallel AD$ we have

$$\mu = \frac{b \sin B}{\sin A}, \quad \lambda = a - \frac{b \sin (A+B)}{\sin A},$$

regardless of the position of C' with respect to B. Hence

$$\lambda = \frac{c \sin D}{\sin A}, \quad \mu = \frac{b \sin B}{\sin A} \tag{2.1}$$

(for the formula $b(A+B) = a \sin A - c \sin D$ and its analogues, see [1, p.169]).
The sidelines BC, CD and the diagonals AC, BD have the equations:

$$\mu x - (\lambda - a)\, y = a\mu, \qquad (BC) \tag{2.2}$$
$$(d - \mu)\, x + \lambda y = d\lambda, \qquad (CD) \tag{2.3}$$
$$\mu x - \lambda y = 0, \qquad (AC) \tag{2.4}$$
$$dx + ay = ad. \qquad (BD) \tag{2.5}$$

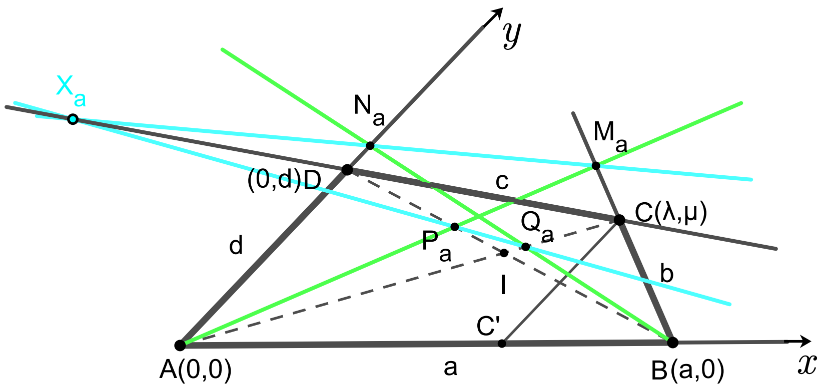

Fig. 2

Since, by the angle bisector theorem, P_a divides the segment BD in the ratio $\dfrac{\overline{P_aB}}{\overline{P_aD}} = -\dfrac{a}{d}$ and the coordinates of B and D are $(a,0)$ and $(0,d)$, we deduce that $P_a\left(\dfrac{ad}{a+d}, \dfrac{ad}{a+d}\right)$. Similarly, because $\dfrac{\overline{Q_aA}}{\overline{Q_aC}} = -\dfrac{a}{b}$, we obtain $Q_a\left(\dfrac{a\lambda}{a+b}, \dfrac{a\mu}{a+b}\right)$. Thus, we have

$$P_a\left(\frac{ad}{a+d}, \frac{ad}{a+d}\right), \quad Q_a\left(\frac{a\lambda}{a+b}, \frac{a\mu}{a+b}\right). \tag{2.6}$$

The equations of the angle bisectors w_a and w_b are

$$x = y, \qquad (w_a) \tag{2.7}$$

$$\mu x + (a + b - \lambda)\, y = a\mu \qquad (w_b) \tag{2.8}$$

(w_b passes through B and Q_a).
We are now ready to state and prove our first result. We fix our attention on the side AB of the quadrilateral. Similar results hold for the other sides.

Theorem 1. *Let $ABCD$ be a convex quadrilateral satisfying conditions (i_a) and $I \notin w_a$, $I \notin w_b$. The lines M_aN_a, P_aQ_a and CD either meet in a point X_a, or they are parallel (Fig. 2).*

Proof. From our hypotheses, we deduce that the points P_a and Q_a do not coincide. In view of (2.6), the equation of the line P_aQ_a is

$$[d\,(a + b) - \mu\,(a + d)]\, x + [\lambda\,(a + d) - d\,(a + b)]\, y + ad\,(\mu - \lambda) = 0. \quad (P_aQ_a) \tag{2.9}$$

The coordinates of M_a and N_a are obtained by solving the systems of equations (2.2) + (2.7) and ($x = 0$) + (2.8), respectively, under the hypothesis (i_a). We then obtain

$$M_a\left(\frac{a\mu}{a + \mu - \lambda}, \frac{a\mu}{a + \mu - \lambda}\right), \quad N_a\left(0, \frac{a\mu}{a + b - \lambda}\right) \tag{2.10}$$

(it is easy to verify the following equivalences: $a + \mu - \lambda = 0 \Longleftrightarrow \frac{A}{2} + B = \pi$ and $a + b - \lambda = 0 \Longleftrightarrow A + \frac{B}{2} = \pi$). Consequently, the equation of the line M_aN_a is

$$(b - \mu)\, x - (a + b - \lambda)\, y + a\mu = 0. \quad (M_aN_a) \tag{2.11}$$

By virtue of (2.3), (2.9) and (2.11), the condition ensuring the concurrence or collinearity of the lines CD, M_aN_a and P_aQ_a is

$$\begin{vmatrix} d - \mu & \lambda & -d\lambda \\ b - \mu & \lambda - a - b & a\mu \\ d\,(a + b) - \mu\,(a + d) & \lambda\,(a + d) - d\,(a + b) & ad\,(\mu - \lambda) \end{vmatrix} = 0.$$

Since the row R_3 of the determinant is equal to the linear combination $aR_1 + dR_2$, this equality is obvious.
Let Δ be defined by

$$\Delta = \begin{vmatrix} d - \mu & \lambda \\ b - \mu & \lambda - a - b \end{vmatrix} = (a + b)\,(\mu - d) + \lambda\,(d - b)\,. \tag{2.12}$$

If $\Delta = 0$, i.e.

$$(b - d)\,\lambda = (a + b)\,(\mu - d)\,, \tag{2.13}$$

then CD and M_aN_a are parallel and consequently CD, M_aN_a, P_aQ_a are also parallel. If $\Delta \neq 0$, then there exists a point X_a in which CD, M_aN_a, P_aQ_a intersect. The proof is now complete. □

Remark 1. Theorem 1 still holds if instead of conditions $I \notin w_a$ and $I \notin w_b$ we impose the less restrictive condition $I \notin w_a \cap w_b$. Indeed, the following hold true.

(1) If $I \in w_a$, and $I \notin w_b$, then X_a coincides with C.
(2) If $I \notin w_a$, and $I \in w_b$, then X_a coincides with D.
(3) If $I \notin w_a$, and $I \notin w_b$, then $X_a \in CD$ (at finite or infinite distance), $X_a \neq C$, and $X_a \neq D$.

Remark 2. Theorem 1 can be easily extended to cover the case in which only one of the conditions (i_a) is valid. Suppose that $\frac{A}{2} + B = \pi$ and $A + \frac{B}{2} \neq \pi$. Consequently, $w_a \| BC$ and only the point N_a exists. In this case, we may use the parallel to w_a through N_a in place of the line $M_a N_a$ (Fig. 3). We then obtain the desired result by adapting the proof of Theorem 1.

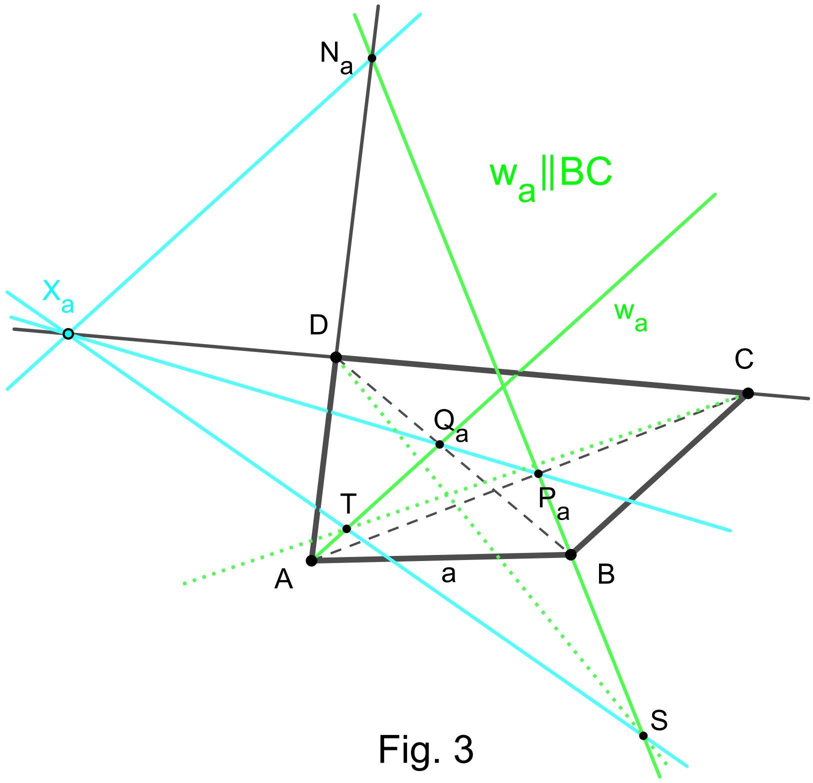

Fig. 3

Remark 3. If X_a exists, i.e. the equality (2.13) does not hold, we can easily find its coordinates. Indeed, the system (2.3)+(2.11), that is

$$\begin{cases} (d-\mu)\,x + \lambda y = d\lambda \\ (b-\mu)\,x + (\lambda - a - b)\,y + a\mu = 0, \end{cases}$$

has a unique solution, namely,

$$X_a\left(\frac{\lambda}{\Delta}\left[d\lambda + a\mu - d\,(a+b)\right], \frac{1}{\Delta}\left[\mu\,(d\lambda + a\mu) - d\,(b\lambda + a\mu)\right]\right). \qquad (2.14)$$

Remark 4. The condition $\Delta = 0$ may be reformulated in terms of the elements of the quadrilateral. From (2.1), we obtain

$$c\,(b-d)\sin D = (a+b)\,(b\sin B - d\sin A)$$

Since $b\sin B - d\sin A = c\sin(B+C)$ and $\sin(B+C) = -\sin(A+D)$, it follows that

$$(b-d)\sin D + (a+b)\sin(A+D) = 0.$$

Since $a\sin(A+D) = d\sin D - b\sin C$, one sees that

$$\sin(A+D) = \sin C - \sin D. \tag{2.15}$$

From here, using the same formula, we deduce that

$$(a+b)\sin C = (a+d)\sin D. \tag{2.16}$$

In particular, parallelograms and isosceles trapezoids satisfy the condition (2.15) or (2.16).

For the sake of simplicity, we shall consider that two parallel lines are concurrent at infinity. Thus, in the hypotheses of Theorem 1, the point X_a always exists and lies on CD at finite or infinite distance, depending upon the lines M_aN_a and P_aQ_a being concurrent or parallel. The next theorem completes the previous one.

Theorem 2. *Let $ABCD$ be a convex quadrilateral satisfying conditions (i_a), $I \notin w_a \cap w_b$, and (iii). Then the lines ST and CD intersect in X_a (Fig. 1a).*

Proof. We have seen that the equations of w_a and w_b are given by (2.7) and (2.8), respectively. Let us now denote $K = w_c \cap BD$ and $L = w_d \cap AC$. Applying the angle bisector theorem twice, we obtain:

$$\frac{\overline{KB}}{\overline{KD}} = -\frac{b}{c} \text{ and } \frac{\overline{LA}}{\overline{LC}} = -\frac{d}{c}.$$

Hence, the points K and L have the coordinates

$$K\left(\frac{ac}{b+c}, \frac{bd}{b+c}\right) \text{ and } L\left(\frac{d\lambda}{c+d}, \frac{d\mu}{c+d}\right).$$

The equations of w_c and w_d are then given by

$$[(b+c)\mu - bd]\,x - [(b+c)\lambda - ac]\,y + (bd\lambda - ac\mu) = 0, \quad (w_c) \tag{2.17}$$

$$(c+d-\mu)\,x + \lambda y - d\lambda = 0. \quad (w_d) \tag{2.18}$$

The hypothesis (iii) guarantees the existence of the points S and T. The coordinates (x_S, y_S) and (x_T, y_T) of these points are the solutions of the systems (2.8)+(2.18) and (2.7)+(2.17) respectively, i.e.

$$\begin{cases} \mu x + (a+b-\lambda)\,y = a\mu, \\ (c+d-\mu)\,x + \lambda y - d\lambda = 0 \end{cases}$$

and

$$\begin{cases} x = y, \\ [(b+c)\mu - bd]\,x - [(b+c)\lambda - ac]\,y + (bd\lambda - ac\mu) = 0. \end{cases}$$

We obtain

$$x_s = \frac{\lambda}{\Delta_1}\left[d\lambda + a\mu - d(a+b)\right], \quad y_s = \frac{\mu}{\Delta_1}\left[d\lambda + a\mu - a(c+d)\right], \tag{2.19}$$

where $\Delta_1 = \lambda(c+d) + \mu(a+b) - (a+b)(c+d)$ and

$$x_T = \frac{1}{\Delta_2}(ac\mu - bd\lambda), \quad y_T = \frac{1}{\Delta_2}(ac\mu - bd\lambda), \tag{2.20}$$

where $\Delta_2 = (b+c)(\mu-\lambda) + ac - bd$. Clearly, by (iii) we have $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$ (the reader can see by direct calculation that $A = C \Longleftrightarrow \Delta_1 = 0$ and $B = D \Longleftrightarrow \Delta_2 = 0$).
Case I. $\Delta \neq 0$, that is, X_a lies on CD at finite distance.
To prove that ST and CD intersect in X_a, it suffices to verify the collinearity of the points S, T and X_a, i.e. to show that

$$\begin{vmatrix} \lambda\left[d\lambda + a\mu - d(a+b)\right] & \mu\left[d\lambda + a\mu - a(c+d)\right] & \Delta_1 \\ ac\mu - bd\lambda & ac\mu - bd\lambda & \Delta_2 \\ \lambda\left[d\lambda + a\mu - d(a+b)\right] & \mu(d\lambda + a\mu) - d(b\lambda + a\mu) & \Delta \end{vmatrix} = 0.$$

This is a simple but tedious calculation (it is advisable first to obtain zero on positions a_{31} and a_{22}).
Case II. $\Delta = 0$. In this case, we claim that $ST \parallel CD$, that is, the slopes of the lines ST and CD are equal. Hence, we have to show that

$$\frac{y_S - y_T}{x_S - x_T} = \frac{\mu - d}{\lambda}$$

or

$$(\mu - d)(x_S - x_T) - \lambda(y_S - y_T) = 0. \tag{2.21}$$

By (2.19) and (2.20), we have:

$$\begin{aligned} x_S - x_T &= \frac{1}{\Delta_1\Delta_2}\left[\lambda\Delta_2\left(d\lambda + a\mu - d(a+b)\right) - \Delta_1\left(ac\mu - bd\lambda\right)\right] \\ &= \frac{k}{\Delta_1\Delta_2}\left[-(b+c)\lambda + c(a+b)\right], \\ y_S - y_T &= \frac{1}{\Delta_1\Delta_2}\left[\mu\Delta_2\left(d\lambda + a\mu - a(c+d)\right) - \Delta_1\left(ac\mu - bd\lambda\right)\right] \\ &= \frac{k}{\Delta_1\Delta_2}\left[-(b+c)\mu + b(c+d)\right], \end{aligned}$$

where $k = d\lambda^2 - a\mu^2 + (a-d)\lambda\mu - d(a+b)\lambda + a(c+d)\mu$. Consequently,

$$\begin{aligned} &(\mu - d)(x_S - x_T) - \lambda(y_S - y_T) \\ &= \frac{k}{\Delta_1\Delta_2}\left[(\mu - d)\left(-(b+c)\lambda + c(a+b)\right) - \lambda\left(-(b+c)\mu + b(c+d)\right)\right] \\ &= \frac{k}{\Delta_1\Delta_2}\left[(a+b)(\mu - d) + \lambda(d-b)\right] = \frac{k}{\Delta_1\Delta_2}\cdot\Delta = 0, \end{aligned}$$

that is, the equality (2.21) holds. The proof is now complete. □

Corollary 1. *If the convex quadrilateral ABCD satisfies conditions* (i_a)*,* $I \notin w_a \cap w_b$*, and (iii), then the lines* M_aN_a*,* P_aQ_a *and ST pass through a point* X_a *located on the sideline CD at finite or infinite distance (Fig. 1a and 4).*

Remark 5. Corollary 1 may be extended to the case in which only one of the points S and T exists. For instance, assume that $A = C$ and $B \neq D$, that is, $w_b \| w_d$ and $w_a \nparallel w_c$. Therefore, T exists, while S does not (Fig. 5). We may proceed as done above for the case of the pair (M_aN_a), for which the point M_a does not exist. In the role of the line ST in Corollary 1, we take the parallel to w_b through T (Fig. 5). Then M_aN_a, P_aQ_a and this parallel are concurrent in X_a. If neither of the points S and T exist, i.e. $A = C$ and $B = D$, then $ABCD$ is a parallelogram.

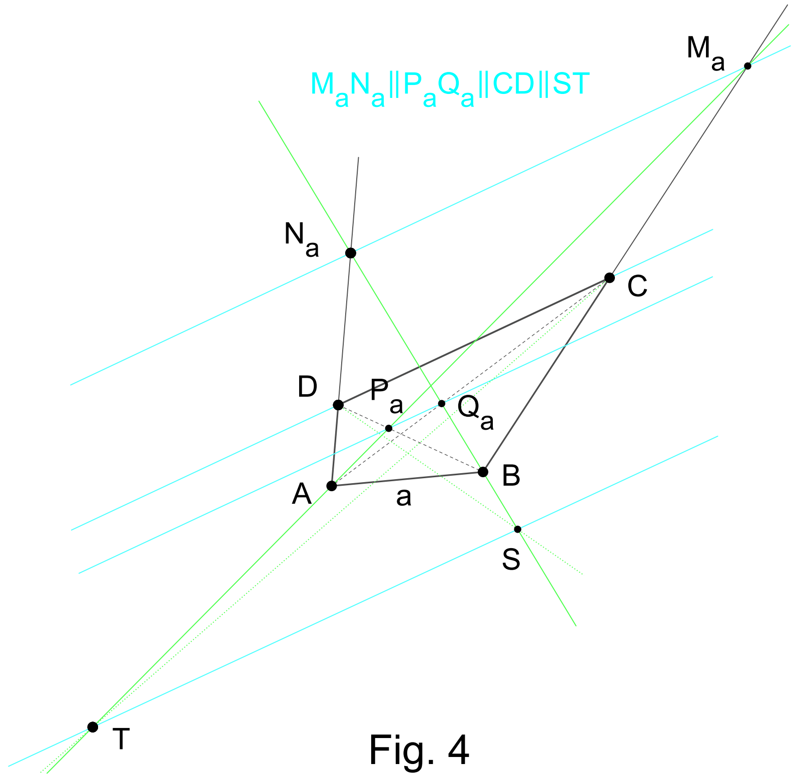

Fig. 4

The above considerations were made with regard to the side AB of the quadrilateral $ABCD$. Proceeding in the same way for the sides BC, CD, DA, we may obtain the points X_b, X_c, X_d, respectively.
We now turn our attention to the set Q_0 of convex quadrilaterals satisfying (i), (ii), (iii). By Theorem 1, for any quadrilateral belonging to Q_0 the points X_a, X_b, X_c, X_d associated with it always exist. These points have a beautiful property, as seen in the next assertion, a logical consequence of Theorems 1 and 2.

Theorem 3. *If $ABCD \in Q_0$, then the points X_a, X_b, X_c, X_d associated with it are collinear and lie on ST (Fig. 6).*

2.2. **External angle bisectors.** In this subsection, we shall see that the previous results remain also valid in the case of external angle bisectors.
The angle bisector w'_a has the following elementary properties:

- $w'_a \| BC$ if and only if $\dfrac{A}{2} + B = \dfrac{\pi}{2}$;
- $w'_a \| BD$ if and only if $a = d$;
- $w'_a \| w'_c$ if and only if $B = D$.

Similar properties hold for w'_b, w'_c, w'_d.
Denote by Q'_0 the set of convex quadrilaterals satisfying conditions:

(i′) the points M'_a, N'_a, M'_b, N'_b, M'_c, N'_c, M'_d, N'_d exist, i.e. we have:

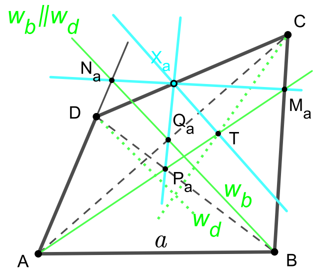

Fig. 5

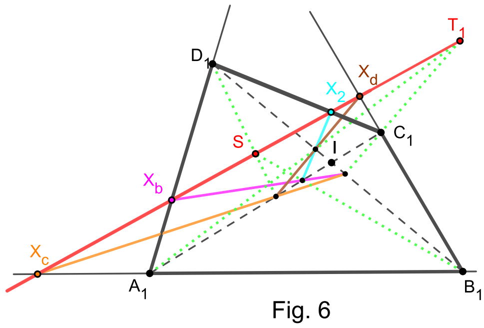

Fig. 6

(i$'_a$) $\dfrac{A}{2} + B \neq \dfrac{\pi}{2}$ and $A + \dfrac{B}{2} \neq \dfrac{\pi}{2}$ (M'_a, N'_a exist),
(i$'_b$) $\dfrac{B}{2} + C \neq \dfrac{\pi}{2}$ and $B + \dfrac{C}{2} \neq \dfrac{\pi}{2}$ (M'_b, N'_b exist),
(i$'_c$) $\dfrac{C}{2} + D \neq \dfrac{\pi}{2}$ and $C + \dfrac{D}{2} \neq \dfrac{\pi}{2}$ (M'_c, N'_c exist),
(i$'_d$) $\dfrac{D}{2} + A \neq \dfrac{\pi}{2}$ and $D + \dfrac{A}{2} \neq \dfrac{\pi}{2}$ (M'_d, N'_d exist);
(ii$'$) $a \neq b, b \neq c, c \neq d, d \neq a$ (the points $P'_a, Q'_a, P'_b, Q'_b, P'_c, Q'_c, P'_d, Q'_d$, exist);
(iii$'$) $A \neq C$ and $B \neq D$, i.e. the points S' and T' exist.

As done in the previous subsection, we fix our attention on the side AB. The sidelines BC, CD and the diagonals AC, BD have the equations (2.2)-(2.5).
Now, let us find the equations of the external angle bisectors w'_a, w'_b, w'_c, w'_d. If conditions (i$'_a$) and $a \neq b$, $a \neq d$ are verified, then the points M'_a, N'_a, P'_a, Q'_a exist. Since $\dfrac{\overline{P'_aB}}{\overline{P'_aD}} = \dfrac{a}{d}$ and $\dfrac{\overline{Q'_aA}}{\overline{Q'_aC}} = \dfrac{a}{b}$, we have:

$$P'_a\left(\frac{ad}{d-a}, -\frac{ad}{d-a}\right), \quad Q'_a\left(\frac{a\lambda}{a-b}, \frac{a\mu}{a-b}\right). \tag{2.22}$$

Therefore, the equations of the bisectors w'_a, w'_b are

$$x + y = 0, \quad (w'_a) \tag{2.23}$$

$$\mu x + (a - b - \lambda)\, y = a\mu \quad (w'_b)\,. \tag{2.24}$$

Solving the systems (2.2)+(2.23) and $(x = 0)$+(2.24), we find

$$M'_a\left(\frac{a\mu}{\lambda+\mu-a}, -\frac{a\mu}{\lambda+\mu-a}\right), \quad N'_a\left(0, \frac{a\mu}{a-b-\lambda}\right) \tag{2.25}$$

(it is left to the reader to verify that $\lambda + \mu - a \iff \frac{A}{2} + B = \frac{\pi}{2}$ and $a - b - \lambda \iff A + \frac{B}{2} = \frac{\pi}{2}$).

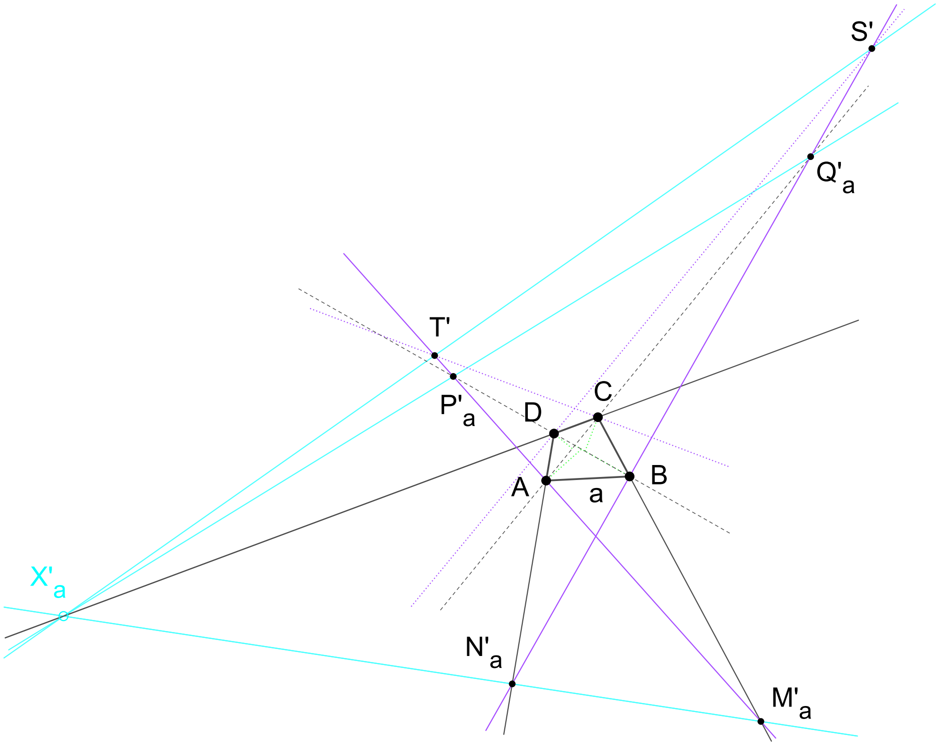

Fig. 7

Similar to how equations (2.17) and (2.18) were deduced, we obtain that the equations of the bisectors w_c', w_d' are given by

$$[(b-c)\,\mu - bd]\,x - [(b-c)\,\lambda + ac]\,y + (bd\lambda + ac\mu) = 0, \quad (w_c') \tag{2.26}$$

$$(\mu + c - d)\,x - \lambda y + d\lambda = 0. \quad (w_d') \tag{2.27}$$

Assume that condition (iii′) is satisfied. In this case, solving the linear systems

$$\begin{cases} \mu x + (a - b - \lambda)\,y = a\mu \\ (\mu + c - d)\,x - \lambda y = -d\lambda, \end{cases}$$

and

$$\begin{cases} x + y = 0 \\ [(b-c)\,\mu - bd]\,x - [(b-c)\,\lambda + ac]\,y = -\,(bd\lambda + ac\mu) \end{cases}$$

we find the coordinate of the points S' and T', respectively. Let Δ_1' and Δ_2' be the main determinants of these systems, i.e.

$$\Delta_1' = (d-c)\,(\lambda - a + b) + (a-b)\,\mu, \quad \Delta_2' = (c-b)\,(\lambda + \mu) + bd - ac.$$

By (iii′), we have $\Delta'_1 \neq 0$ and $\Delta'_2 \neq 0$ (the reader can see that $A = C \Longleftrightarrow \Delta'_1 = 0$ and $B = D \Longleftrightarrow \Delta'_2 = 0$). Consequently, we are able to determine that the coordinates of S' and T' are given by

$$S'\left(\frac{\lambda}{\Delta'_1}\left[d\lambda + a\mu - d\left(a-b\right)\right], \frac{\mu}{\Delta'_1}\left[d\lambda + a\mu - a\left(d-c\right)\right]\right), \tag{2.28}$$

$$T'\left(\frac{1}{\Delta'_2}\left(bd\lambda + ac\mu\right), -\frac{1}{\Delta'_2}\left(bd\lambda + ac\mu\right)\right). \tag{2.29}$$

We are now ready to state our results concerning the external angle bisectors of the quadrilateral.

Theorem 4. *Let $ABCD$ be a convex quadrilateral satisfying the conditions (i'_a), $a \neq b$, $a \neq d$ and (iii′). Then the lines $M'_aN'_a$, $P'_aQ'_a$ and $S'T'$ are concurrent in a point X'_a located on the sideline CD at finite or infinite distance (Fig. 7).*

Proof. Using (2.22) and (2.25), it is seen that the equations of the lines $M'_aN'_a$ and $P'_aQ'_a$ are given by

$$\left(b-\mu\right)x + \left(\lambda - a + b\right)y + a\mu = 0, \qquad \left(M'_aN'_a\right) \tag{2.30}$$

$$\left[\left(a-d\right)\mu - d\left(a-b\right)\right]x - \left[\left(a-d\right)\lambda + d\left(a-b\right)\right]y + ad\left(\lambda+\mu\right) = 0. \qquad \left(P'_aQ'_a\right) \tag{2.31}$$

The condition of concurrence or collinearity of the lines CD, $M'_aN'_a$ and $P'_aQ'_a$ is

$$\begin{vmatrix} d-\mu & \lambda & -d\lambda \\ b-\mu & \lambda-a+b & a\mu \\ \left(a-d\right)\mu - d\left(a-b\right) & -\left(a-d\right)\lambda - d\left(a-b\right) & ad\left(\lambda+\mu\right) \end{vmatrix} = 0.$$

Since this equality holds (the row R_3 is equal to the combination $-aR_1 + dR_2$), it follows that CD, $M'_aN'_a$, $P'_aQ'_a$ are concurrent or collinear. Note that they are concurrent provided that $\Delta' \neq 0$, where

$$\Delta' = \begin{vmatrix} d-\mu & \lambda \\ b-\mu & \lambda-a+b \end{vmatrix} = \left(d-b\right)\lambda + \left(a-b\right)\left(\mu-d\right). \tag{2.32}$$

In this case, solving the system (2.3)+(2.30), we find the coordinates of the intersection point X'_a

$$X'_a\left(\frac{\lambda}{\Delta'}\left[d\lambda + a\mu - d\left(a-b\right)\right], \frac{1}{\Delta'}\left[\mu\left(d\lambda + a\mu\right) - d\left(b\lambda + a\mu\right)\right]\right). \tag{2.33}$$

For the collinearity of points S', T' and X'_a, it suffices to show that

$$\begin{vmatrix} \lambda\left[d\lambda + a\mu - d\left(a-b\right)\right] & \mu\left[d\lambda + a\mu - a\left(d-c\right)\right] & \Delta'_1 \\ bd\lambda + ac\mu & -\left(bd\lambda + ac\mu\right) & \Delta'_2 \\ \lambda\left[d\lambda + a\mu - d\left(a-b\right)\right] & \mu\left(d\lambda + a\mu\right) - d\left(b\lambda + a\mu\right) & \Delta' \end{vmatrix} = 0.$$

We omit the routine computation.
If $\Delta' = 0$, we shall show that $S'T' \| CD$, i.e. the relation

$$\left(\mu - d\right)\left(x_{S'} - x_{T'}\right) - \lambda\left(y_{S'} - y_{T'}\right) = 0 \tag{2.34}$$

holds. Using (2.28) and (2.29), one sees that

$$x_{S'} - x_{T'} = \frac{k'}{\Delta_1' \Delta_2'} \left[(c-b)\,\lambda + c\,(b-a) \right],$$
$$y_{S'} - y_{T'} = \frac{k'}{\Delta_1' \Delta_2'} \left[(c-b)\,\mu + b\,(d-c) \right],$$

where $k' = d\lambda^2 + a\mu^2 + (a+d)\,\lambda\mu - d\,(a-b)\,\lambda + a\,(c-d)\,\mu$. Then, we have:

$$\begin{aligned}
&(\mu - d)\,(x_{S'} - x_{T'}) - \lambda\,(y_{S'} - y_{T'}) \\
&= \frac{k'}{\Delta_1' \Delta_2'} \left\{ (\mu - d) \left[(c-b)\,\lambda + c\,(b-a) \right] - \lambda \left[(c-b)\,\mu + b\,(d-c) \right] \right\} \\
&= \frac{k'}{\Delta_1' \Delta_2'} \cdot \left(-c\Delta' \right) = 0
\end{aligned}$$

Hence, (2.34) is valid. This concludes the proof. □

Remark 6. The previous result may be extended using considerations similar to those presented after the proof of Theorem 1. Also, using (2.1), we can reformulate the condition $\Delta' = 0$ in terms of the elements of quadrilateral. Corresponding to (2.15) and (2.16), we obtain in the same way the formulas:

$$\sin\,(A + D) = \sin D - \sin C$$

and

$$(a - b)\sin C = (a - d)\sin D.$$

The next result represents the analogue of Theorem 3.

Theorem 5. *If $ABCD \in Q_0'$, then the points X_a', X_b', X_c', X_d' are collinear, namely they are located on $S'T'$ (Fig. 8).*

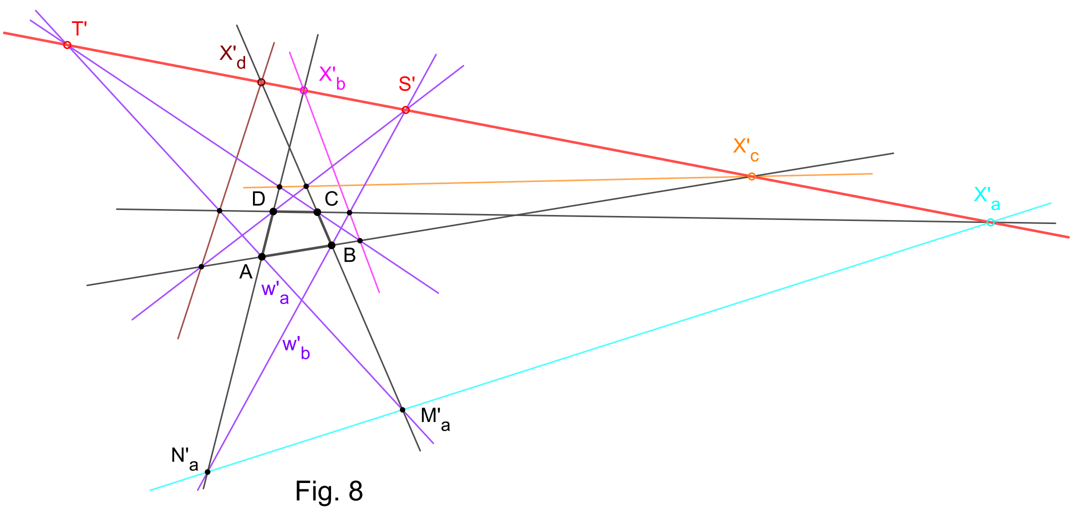

Fig. 8

3. Cyclic quadrilaterals

If $ABCD$ is a cyclic quadrilateral, then the coordinates λ and μ of the vertex C may be expressed in terms of the sidelenghts a, b, c, d. Indeed, by (2.1), and using the formulas:

$$\sin A = \frac{f}{2R}, \quad \sin B = \sin D = \frac{e}{2R}, \quad \frac{e}{f} = \frac{ad+bc}{ab+cd},$$

where e, f are the lengths of diagonals AC and BD, and R is the radius of circumcircle, we have:

$$\lambda = c\frac{ad+bc}{ab+cd}, \quad \mu = b\frac{ad+bc}{ab+cd}. \tag{3.1}$$

Consequently, the coordinates of the points, the equations of the lines, etc. from the previous section may be rewritten in terms of a, b, c, d. Thus, we have:

$$(ad+bc)\,x + \left(a^2+ab+cd-c^2\right) y = a\,(ad+bc)\,, \quad (w_b) \tag{3.2}$$

$$(ad+bc)\,x + \left(a^2-ab-cd-c^2\right) y = a\,(ad+bc)\,, \quad (w_b') \tag{3.3}$$

$$\Delta = \frac{bc\,(b-d)\,(a+b-c+d)}{ab+cd}, \quad \Delta' = \frac{bc\,(d-b)\,(-a+b+c+d)}{ab+cd}, \tag{3.4}$$

$$X_a\left(\frac{(c-d)\,(ad+bc)}{(b-d)\,(a+b-c+d)}, \frac{ad+bc}{a+b-c+d}\right), \tag{3.5}$$

$$X_a'\left(\frac{(c+d)\,(ad+bc)}{(d-b)\,(-a+b+c+d)}, -\frac{ad+bc}{-a+b+c+d}\right). \tag{3.6}$$

The coordinates of the points X_b, X_b', X_c, X_c', can also be expressed in similar terms.

3.1. **Internal angle bisectors.** Denote by $\mathcal{C}$ the circumcicle of the cyclic quadrilateral $ABCD$. Relative to the side BC, we denote the intersections of w_a and w_b with $\mathcal{C}$ by U_a, V_a, i.e. $U_a = w_a \cap \mathcal{C}$, $V_a = w_b \cap \mathcal{C}$.
In the xy-Cartesian coordinate system adopted above, the circle $\mathcal{C}$ has the equation

$$x^2 + y^2 + 2xy\cos A - ax - dy = 0, \tag{3.7}$$

as $A\,(0,0)\,, B\,(a,0)$ and $D\,(0,d)$ belong to $\mathcal{C}$. The pair of coordinates of U_a is the nonzero solution of the system

$$\begin{cases} x^2 + y^2 + 2xy\cos A - ax - dy = 0 \\ \hfill x = y. \end{cases}$$

We find

$$x = y = \frac{a+d}{2\,(1+\cos A)} = \frac{(a+d)\,(ad+bc)}{(a-b+c+d)\,(a+b-c+d)}$$

(see [3, p.152] for the formula $\cos A = \dfrac{a^2-b^2-c^2+d^2}{2\,(ad+bc)}$). Hence,

$$U_a\left(\frac{(a+d)\,(ad+bc)}{(a-b+c+d)\,(a+b-c+d)}, \frac{(a+d)\,(ad+bc)}{(a-b+c+d)\,(a+b-c+d)}\right). \tag{3.8}$$

With regard to point V_a, we need to find the solution different from $(a, 0)$ of the system (3.7)+(3.2), i.e.

$$\begin{cases} x^2 + y^2 + 2xy\cos A - ax - dy = 0 \\ (ad + bc)\, x + \left(a^2 + ab + cd - c^2\right) y = a\,(ad + bc)\,. \end{cases}$$

From the second equation, we have:

$$x = a - \frac{a^2 + ab + cd - c^2}{ad + bc}\, y.$$

Substituting this expression of x in the equation (3.7) and removing the factor y, we obtain:

$$\left(a - \frac{a^2 + ab + cd - c^2}{ad + bc} y\right)\left(-\frac{a^2 + ab + cd - c^2}{ad + bc} + 2\cos A\right) + y - d = 0$$

or

$$\left(a - \frac{a^2 + ab + cd - c^2}{ad + bc} y\right) \cdot \frac{d^2 - b^2 - ab - cd}{ad + bc} + y - d = 0.$$

This equation can be written as

$$\left[1 - \frac{\left(a^2 + ab + cd - c^2\right)\left(d^2 - b^2 - ab - cd\right)}{(ad + bc)^2}\right] y = d - a\frac{d^2 - b^2 - ab - cd}{ad + bc}.$$

Consequently,

$$y = \frac{(a + b)\,(ad + bc)}{(a + b - c + d)\,(a + b + c - d)}.$$

Finally, we obtain

$$V_a\left(\frac{(c - d)\,(ad + bc)}{(a + b - c + d)\,(a + b + c - d)}, \frac{(a + b)\,(ad + bc)}{(a + b - c + d)\,(a + b + c - d)}\right). \tag{3.9}$$

Theorem 6. *Let $ABCD$ be a cyclic quadrilateral.*

(1) If $b \neq d$, then U_aV_a and CD intersect in X_a (Fig. 9).
(2) If $b = d$, then U_aV_a and CD are parallel.

Proof. According to (3.4), $\Delta = 0 \iff b = d$. Hence, X_a given by (3.5) is at finite or infinite distance, depending upon whether $b \neq d$ or $b = d$.
1. If $b \neq d$, we need to prove the collinearity of X_a, U_a, and V_a (Fig. 9). By (3.5), (3.8) and (3.9), we have to show that

$$\begin{vmatrix} \dfrac{a - d}{b - d} & 1 & 1 \\ \dfrac{a + d}{a - b + c + d} & \dfrac{a + d}{a - b + c + d} & 1 \\ \dfrac{c - d}{a + b + c - d} & \dfrac{a + b}{a + b + c - d} & 1 \end{vmatrix} = 0.$$

It is easy to see that

$$\begin{vmatrix} c - d & b - d & b - d \\ a + d & a + d & a - b + c + d \\ c - d & a + b & a + b + c - d \end{vmatrix} = \begin{vmatrix} c - d & b - c & 0 \\ a + d & 0 & c - b \\ 0 & a + d & c - d \end{vmatrix} = 0,$$

i.e. the first assertion is true.
2. If $b = d$, then the involved points are

$$U_a\left(\frac{b(a+b)}{a+2b-c}, \frac{b(a+b)}{a+2b-c}\right), V_a\left(\frac{b(c-b)}{a+2b-c}, \frac{b(a+b)}{a+2b-c}\right), C(c,b), D(0,b).$$

It is obvious that CD and U_aV_a are parallel to AB. In this case $ABCD$ is nothing other than a square or an isosceles trapezoid with nonparallel sides equal to a base. □

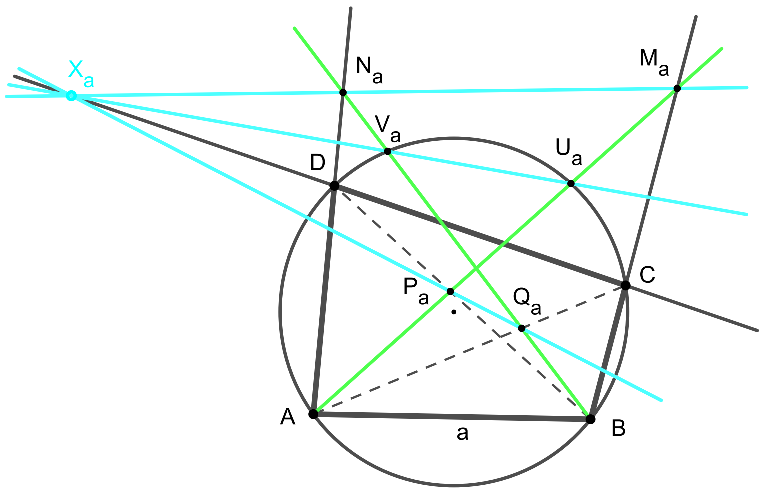

Fig. 9

3.2. **External angle bisectors.** Let $U'_a = w'_a \cap \mathcal{C}$ and $V'_a = w'_b \cap \mathcal{C}$ (Fig. 10). The coordinates of U'_a can be determined by solving the system

$$\begin{cases} x^2 + y^2 + 2xy\cos A - ax - dy = 0 \\ x + y = 0. \end{cases}$$

The nonzero solution is given by

$$x = -y = \frac{a-d}{2(1-\cos A)} = \frac{(a-d)(ad+bc)}{(-a+b+c+d)(a+b+c-d)}.$$

Hence,

$$U'_a\left(\frac{(a-d)(ad+bc)}{(-a+b+c+d)(a+b+c-d)}, -\frac{(a-d)(ad+bc)}{(-a+b+c+d)(a+b+c-d)}\right). \quad (3.10)$$

To find the coordinates of V'_a, we solve the system

$$\begin{cases} x^2 + y^2 + 2xy\cos A - ax - dy = 0 \\ (ad+bc)x + \left(a^2 - ab - cd - c^2\right)y = a(ad+bc). \end{cases}$$

Substituting

$$x = a - \frac{a^2 - ab - cd - c^2}{ad+bc}y$$

in (3.7), we obtain:

$$\left(a-\frac{a^2-ab-cd-c^2}{ad+bc}y\right)\left(-\frac{a^2-ab-cd-c^2}{ad+bc}+2\cos A\right)+y-d=0$$

or

$$\left(a-\frac{a^2-ab-cd-c^2}{ad+bc}y\right)\cdot\frac{d^2-b^2+ab+cd}{ad+bc}+y-d=0,$$

which leads to

$$y=-\frac{(a-b)(ad+bc)}{(-a+b+c+d)(a-b+c+d)}.$$

Consequently, we have

$$V_a'\left(-\frac{(c+d)(ad+bc)}{(-a+b+c+d)(a-b+c+d)},-\frac{(a-b)(ad+bc)}{(-a+b+c+d)(a-b+c+d)}\right). \tag{3.11}$$

The following theorem is analogous to Theorem 6, its proof being left to the reader.

Theorem 7. *Let $ABCD$ be a cyclic quadrilateral.*

(1) If $b\neq d$, then $U_a'V_a'$ intersect CD in X_a' (Fig. 10).
(2) If $b=d$, then $U_a'V_a'$ and CD are parallel.

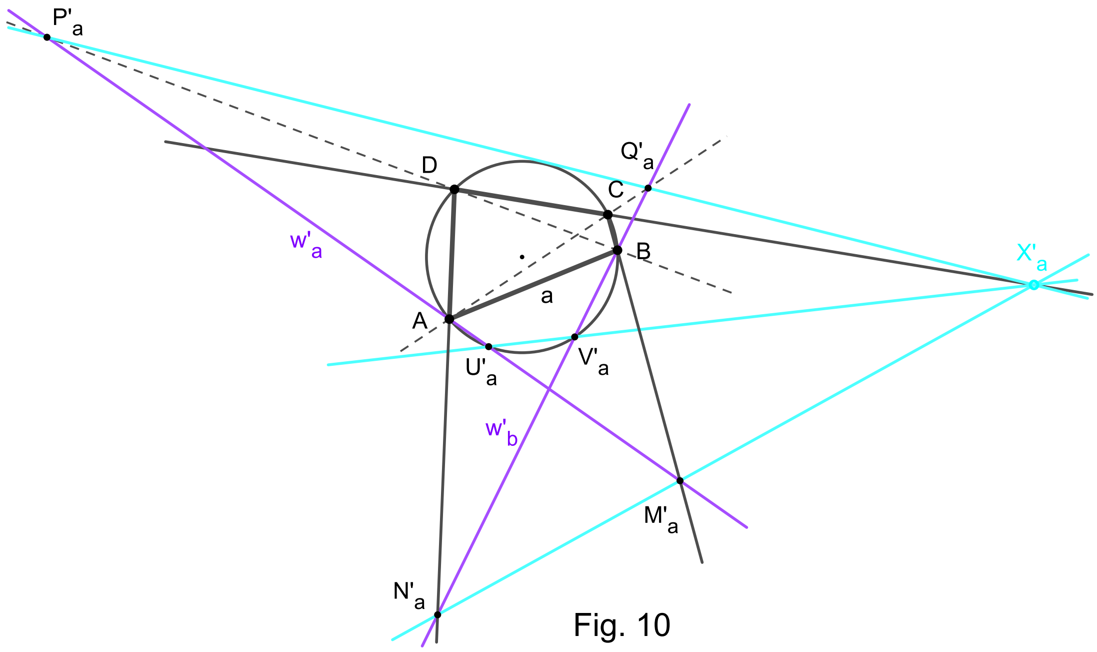

Fig. 10

Let us connect these results to those of Section 2. First, we shall observe that the conditions to be imposed on cyclic quadrilaterals can be relaxed.

Lemma 1. *If the quadrilateral $ABCD$ is cyclic, then the systems of conditions (i)-(iii) and (i′)-(iii′) are equivalent.*

Proof. (i)$\Longleftrightarrow$(i′). Indeed, $\frac{A}{2}+B=\pi \Longleftrightarrow \frac{\pi-C}{2}+(\pi-D)=\pi \Longleftrightarrow \frac{C}{2}+D=\frac{\pi}{2}$. Hence, $\frac{A}{2}+B\neq\pi \Longleftrightarrow \frac{C}{2}+D\neq\frac{\pi}{2}$, and so on.

(ii)$\Longleftrightarrow$(ii′). Indeed, $I\in w_a \Longleftrightarrow \overset{\frown}{BC}=\overset{\frown}{CD} \Longleftrightarrow b=c$. Hence, $I\notin w_a \Longleftrightarrow b\neq c$, and so on. We conclude by noting that (iii) is in fact the same condition as (iii′). ☐

As a consequence, one may obtain the following result, which is valid for cyclic quadrilaterals.

Theorem 8. *Let $ABCD$ be a cyclic quadrilateral satisfying the conditions (i)-(iii). The following assertions are valid.*

(1) The lines M_aN_a, P_aQ_a, U_aV_a are concurrent in a point X_a located on CD.
(2) The lines $M'_aN'_a$, $P'_aQ'_a$, $U'_aV'_a$are concurrent in a point X'_a located on CD.
(3) The points X_a, X_b, X_c, X_d are collinear and lie on the line ST.
(4) The points X'_a, X'_b,X'_c, X'_d are collinear and lie on the line $S'T'$.

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