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ON THE ISOTOMIC INSCRIBED QUADRILATERALS

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ABSTRACT. In this paper we examine the conditions for which isotomic inscribed quadrilaterals have equal areas. In premier part of article we get an approach syntetic of this question, in the seconnd part we discuss the problem analytic, in oblique coordinates.

1. INTRODUCTION

In [1] the author prove that the isotomic inscribed triangles have equal areas and investigate some interesting properties of these triangles. For example, the centroids of isotomic inscribed triangles are symmetric with respect to the centroid of the reference triangle ABC (Proposition 2, p. 126). In this paper we extend the investigation for the quadrilaterals. Let $ABCD$ an arbitrary quadrilateral, $PQRS$ its Varignon parallelogram ($P \in AB$, $Q \in BC$, $R \in CD$, $S \in DA$). Denote with $TUVW$ a quadrilateral inscribed in the quadrilateral $ABCD$ ($T \in AB$, $U \in BC$, $V \in CD$, $W \in DA$). Consider the symmetric of points T, U, V, W with respect to the midpoints P, Q, R, S, respectively (Figure 1). Denote these points with the symbols T', U', V', W'.

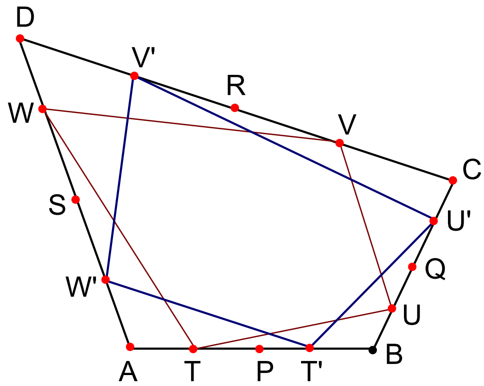

Figure 1

The quadrilaterals $TUVW$ and $T'U'V'W'$ we shall nom *isotomic inscribed quadrilaterals*. In this paper we analize the conditions for which the isotomic inscribed quadrilaterals have equal areas. In general they have not equal areas.

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2. Synthetic approach

Theorem 2.1. *If $ABCD$ is parallelogram, then the isotomic inscribed quadrilaterals $TUVW$ and $T'U'V'W'$ have equal areas.*

Proof. We execute the demonstration when T, U, V, W are interior of the sides AB, BC, CD, DA, respectively (Figure 2).

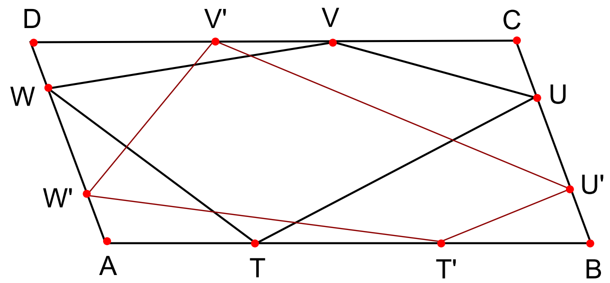

Figure 2

$$\begin{aligned}TUVW &= ABCD - WAT - TBU - UCV - VDW\\
&= ABCD - \frac{\sin A}{2}(WA \cdot AT + TB \cdot BU + UC \cdot CV + VD \cdot DW)\\
&= ABCD - \frac{\sin A}{2}[(AD - WD)(AB - TB) + (AB - AT) \cdot (AD - UC)\\
&\quad +(AD - BU)(AB - VD) + (AB - CV)(AD - AW)]\\
&= ABCD - \frac{\sin A}{2}[(AD - AW')(AB - AT') + (AB - BT')(AD - BU')\\
&\quad +(AD - CU')(AB - CV') + (AB - DV')(AD - DW')]\\
&= ABCD - \frac{\sin A}{2}[4AB \cdot AD - AB(AW' + BU' + CU' + DW')\\
&\quad -AD(AT'+BT'+CV'+DV')+W'A \cdot AT'+T'B \cdot BU'+U'C \cdot CV'+V'D \cdot DW']\\
&= ABCD - \frac{\sin A}{2}(4AB \cdot AD - AB \cdot 2AD - AD \cdot 2AB\\
&\quad +W'A \cdot AT' + T'B \cdot BU' + U'C \cdot CV' + V'D \cdot DW')\\
&= ABCD - \frac{\sin A}{2}(W'A \cdot AT' + T'B \cdot BU' + U'C \cdot CV' + V'D \cdot DW')\\
&= ABCD - W'AT' - T'BU' - U'CV' - V'DW' = T'U'V'W'.\end{aligned}$$

The above property is available for all point T, U, V, W on the line AB, BC, CD, DA, respectively. □

Theorem 2.2. *On the side AB of the quadrilateral $ABCD$ consider an arbitrary point T. If $TU \| AC$, $UV \| BD$, $VW \| AC$ and $U \in BC$, $V \in CD$, $W \in DA$ then $TUVW$ is parallelogram.*

Proof. We have:

$$TU \| AC \Leftrightarrow \frac{BT}{AB} = \frac{BU}{BC} = \frac{TU}{AC}, \quad UV \| BD \Leftrightarrow \frac{CU}{BC} = \frac{CV}{CD} = \frac{UV}{BD},$$

$$VW \| AC \Leftrightarrow \frac{DV}{CD} = \frac{DW}{DA} = \frac{VW}{AC}.$$

Then again:

$$\frac{CU}{BC} = \frac{CV}{CD} \Leftrightarrow 1 - \frac{CU}{BC} = 1 - \frac{CV}{CD} \Leftrightarrow \frac{BU}{BC} = \frac{DV}{CD} \Leftrightarrow \frac{TU}{AC} = \frac{VW}{AC}.$$

From this we obtain that $TU \equiv VW$ (Figure 3).

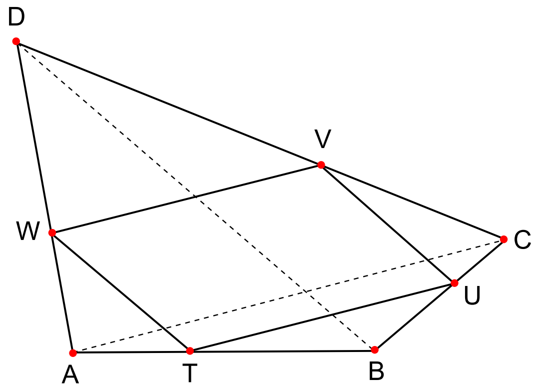

Figure 3

Since $TU \| AC \| VW$ and $TU \equiv VW$, then the segment TW is parallel with UV, consequently with BD, too. Therefore $TUVW$ is parallelogram. □

The sides of inscribed parallelogram $TUVW$ are parallel with the diagonal AC, respectively BD, of the reference quadrilateral $ABCD$. Call a such parallelogram *inscribed diagonal parallelogram*.

Theorem 2.3. *The isotomic quadrilateral of an inscribed diagonal parallelogram is an inscribed diagonal parallelogram, too.*

Proof. We will prove that if $TUVW$ is an inscribed diagonal parallelogram, then the conditions $TU \| AC$ and $T'U' \| AC$ are equivalents (Figure 4).

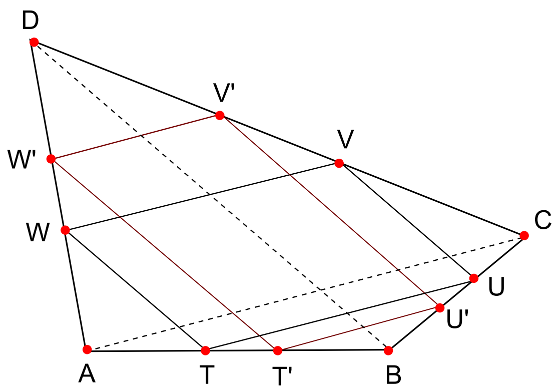

Figure 4

Indeed

$$TU \| AC \Leftrightarrow \frac{AB}{BT} = \frac{BC}{BU} \Leftrightarrow \frac{AT' + BT'}{AT'} = \frac{BU' + CU'}{CU'} \Leftrightarrow$$
$$\Leftrightarrow \frac{BT'}{AT'} = \frac{BU'}{CU'} \Leftrightarrow T'U' \| AC.$$

Similarly we can demonstrate that $U'V' \| BD$, $V'W' \| AC$ and $W'T' \| BT$.
Therefore $T'U'V'W'$ is an inscribed diagonal parallelogram. □

Theorem 2.4. *The isotomic inscribed diagonal parallelograms have equal areas.*

Proof. Denote with φ the angle formed by the diagonals AC and BD of quadrilaterals $ABCD$. All arbitrary point T of line AB determine two isotomic inscribed diagonal parallelograms $TUVW$ and $T'U'V'W'$ (Figure 4):

$$TUVW = TW \cdot TU \sin\varphi = \frac{BD \cdot AT}{AB} \cdot \frac{AC \cdot BT}{AB} \sin\varphi = \frac{2AT \cdot BT}{AB^2} ABCD,$$
$$T'U'V'W' = T'W' \cdot T'U' \sin\varphi = \frac{BD \cdot AC}{AB^2} AT' \cdot BT' \sin\varphi$$
$$= \frac{BD \cdot AC}{AB^2} AT \cdot BT \sin\varphi = \frac{2AT \cdot BT}{AB^2} ABCD = TUVW.$$

□

Let the point M on the line AB between A and B. This order we will denote with (A, M, B).

Theorem 2.5. (a) *If* (A, T, T', B) *and* (D, W, W', A) *or* (T, A, B, T') *and* (W, D, A, W'), *then*

$$ATW = AT'W' \Leftrightarrow \frac{AT}{AB} = \frac{DW}{DA}.$$

(b) *If* (B, U, U', C) *and* (A, T, T', B) *or* (U, B, C, U') *and* (T, A, B, T'), *then*

$$BUT = BU'T' \Leftrightarrow \frac{BU}{BC} = \frac{AT}{AB}.$$

(c) *If* (C, V, V', D) *and* (B, U, U', C) *or* (V, C, D, V') *and* (U, B, C, U'), *then*

$$CVU = CV'U' \Leftrightarrow \frac{CV}{CD} = \frac{BU}{BC}.$$

(d) *If* (D, W, W', A) *and* (C, V, V', D) *or* (W, D, A, W') *and* (V, C, D, V'), *then*

$$DWV = DW'V' = \frac{DW}{DA} = \frac{CV}{CD}.$$

Proof. (a) Let us assume the order (A, T, T', B) and (D, W, W', A). Therefore:

$$ATW = AT'W \Leftrightarrow AT \cdot AW = AT' \cdot AW' \Leftrightarrow AT(AD - DW)$$
$$= (AB - AT)DW \Leftrightarrow AT \cdot AD = AB \cdot DW \Leftrightarrow \frac{AT}{AB} = \frac{DW}{DA}.$$

The demonstration is similar in case of orders (T, A, B, T') and (W, D, A, W') (Fig. 5 – 6).

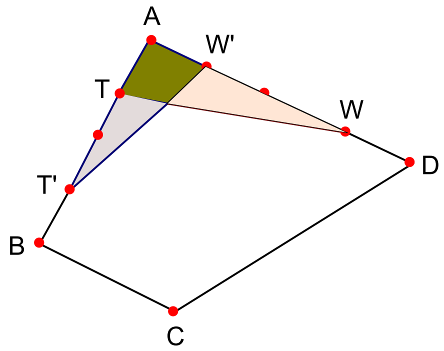

Figure 5

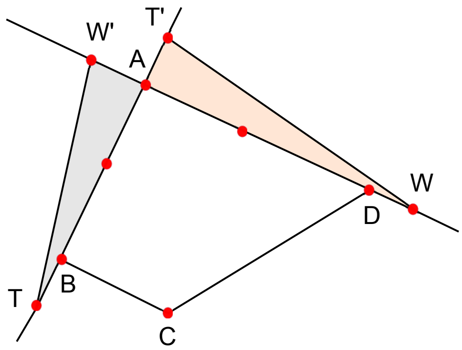

Figure 6

□

Definition 2.1. Let X a point of the line EF. We shall say that the segments EX and EF have same orientation if X is on semi-straight line $(EF$. Otherwise we say that the segments EX and EF have opposite orientation.

Theorem 2.6. *If the pairs of segments (AT, AB), (BU, BC), (CV, CD), (DW, DA) have same orientation and $\frac{AT}{AB} = \frac{BU}{BC} = \frac{CV}{CD} = \frac{DW}{DA}$, then the isotomic inscribed quadrilaterals $TUVW$ and $T'U'V'W'$ have equal areas.*

Proof. In this case the orders of the vertices of quadrilaterals $TUVW$ and $T'U'V'W'$ on the side AB, BC, CD, DA must be (A, T, T', B), (B, U, U', C), (C, V, V', D), (D, W, W', A), respectively (Figure 1).
The areas of quadrilaterals $TUVW$ and $T'U'V'W'$ can be represent in the forms

$$TUVW = ABCD - ATW - BUT - CVU - DWV,$$
$$T'U'V'W' = ABCD - AT'W' - BU'T' - CV'U' - DW'V'.$$

But $ATW = AT'W'$, $BUT = BU'T'$, $CVU = CV'U'$, $DWV = DW'V'$, consequently $TUVW$ and $T'U'V'W'$ have equal areas. □

Theorem 2.7. *If the pairs of segments (AT, AB), (BU, BC), (CV, CD), (DW, DA) have opposite orientation and $\frac{AT}{AB} = \frac{BU}{BC} = \frac{CV}{CD} = \frac{DW}{DA}$, then the isotomic inscribed quadrilaterals $TUVW$ and $T'U'V'W'$ have equals areas.*

Proof. In this case the orders of the vertices of quadrilaterals $TUVW$ and $T'U'V'W'$ on the sides AB, BC, CD, DA are (T, A, B, T'), (U, B, C, U'), (V, C, D, V'), (W, D, A, W'), respectively (Figure 7).

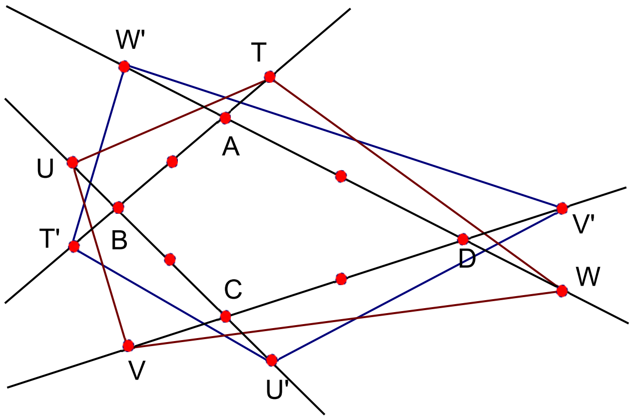

Figure 7

The areas of quadrilaterals $TUVW$ and $T'U'V'W'$ are

$$TUVW = ABCD + ATW + BUT + CVU + DWV,$$
$$T'U'V'W' = ABCD + AT'W' + BU'T' + CV'U' + DW'V'.$$

But $ATW = AT'W'$, $BUT = BU'T'$, $CVU = CV'U'$, $DWV = DW'V'$, consequently $TUVW$ and $T'U'V'W'$ have equal areas. □

Theorem 2.8. *If the pairs of segments* (AT, AB) *and* (CV, CD) *have simultaneously same* (*or opposite*) *orientation,* $\frac{AT}{AB} = \frac{CV}{CD}$ *and* $\frac{BU}{BC} = \frac{DW}{DA}$ *, then* $ATW + CVU = BU'T' + DW'V'$.

Proof. Let us assume that the orders of vertices of quadrilaterals $TUVW$ and $T'U'V'W'$ on the sides AB and CD are (A, T, T', B), (C, V, V', D), respectively (Figure 8).

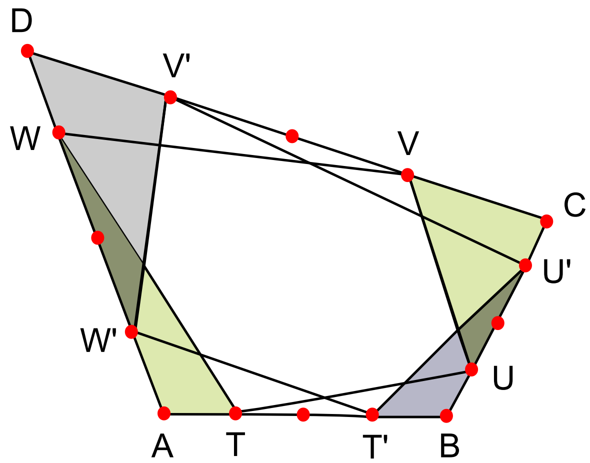

Figure 8

We have:

$$ATW + CVU = BU'T' + DW'V'$$
$$\Leftrightarrow AT \cdot AW \sin A + CV \cdot CU \sin C = BU' \cdot BT' \sin B + DW' \cdot DV' \sin D$$
$$\Leftrightarrow AT \cdot AW \sin A + CV \cdot CU \sin C = BU' \cdot AT \sin B + DW' \cdot CV \sin D$$

$$\Leftrightarrow \frac{AT}{AB \cdot CD} AW \sin A + \frac{CV}{AB \cdot CD} CU \sin C = BU' \cdot \frac{AT}{AB \cdot CD} \sin B + DW' \frac{CV}{AB \cdot CD} \sin D$$
$$\Leftrightarrow \frac{1}{CD} AW \sin A + \frac{1}{AB} CU \sin C = BU' \frac{1}{CD} \sin B + DW' \frac{1}{AB} \sin D$$
$$\Leftrightarrow AB \cdot AW \sin A + CD \cdot CU \sin C = CU \cdot AB \sin B + AW \cdot CD \sin D$$
$$\Leftrightarrow AB(DA - DW) \sin A + CD(BC - BU) \sin C$$
$$= (BC - BU) AB \sin B + (DA - DW) CD \sin D$$
$$\Leftrightarrow 2ABD - AB \cdot DW \sin A + 2BCD - CD \cdot BU \sin C$$
$$= 2ABC - BU \cdot AB \sin B + 2DAC - DW \cdot CD \sin D$$
$$\Leftrightarrow AB \cdot DW \sin A + CD \cdot BU \sin C = BU \cdot AB \sin B + DW \cdot CD \sin D$$
$$\Leftrightarrow AB \frac{DW}{BC \cdot DA} \sin A + CD \frac{BU}{BC \cdot DA} \sin C$$
$$= \frac{BU}{BC \cdot DA} AB \sin B + \frac{DW}{BC \cdot DA} CD \sin D$$
$$\Leftrightarrow AB \frac{1}{BC} \sin A + CD \frac{1}{DA} \sin C = \frac{1}{DA} AB \sin B + \frac{1}{BC} CD \sin D$$
$$\Leftrightarrow AB \cdot DA \sin A + CD \cdot BC \sin C = BC \cdot AB \sin B + DA \cdot CD \sin D$$
$$\Leftrightarrow 2ABD + 2BCD = 2ABC + 2DAC \Leftrightarrow ABCD = ABCD.$$

□

Theorem 2.9. *If the pairs of segments* (BU, BC), (DW, DA) *have simultaneously same* (*or opposite*) *orientation,* $\frac{AT}{AB} = \frac{CV}{CD}$ *and* $\frac{BU}{BC} = \frac{DW}{DA}$, *then* $BUT + DWV = AT'W' + CV'U'$.

Proof. Let us assume that the orders of vertices of quadrilaterals $TUVW$ and $T'U'V'W'$ on the sides BC and DA are (U, B, C, U'), (W, D, A, W'), respectively (Figure 9).

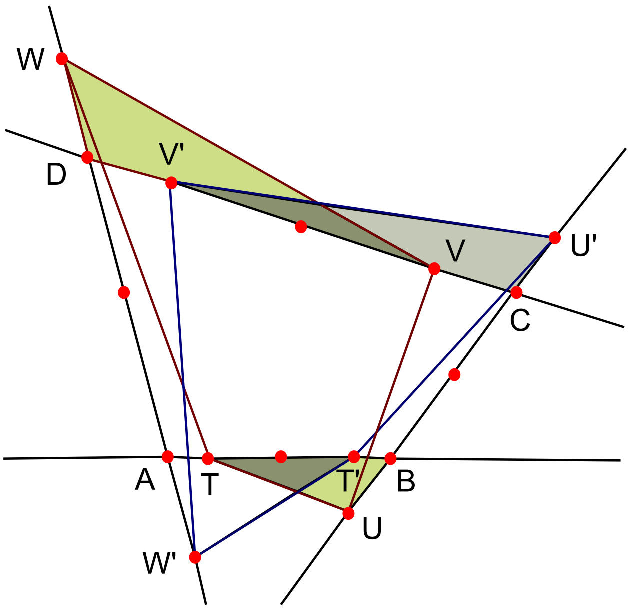

Figure 9

We have:

$$BUT + DWV = AT'W' + CV'U'$$
$$\Leftrightarrow BT \cdot BU \sin B + DW \cdot DV \sin D = AT' \cdot AW' \sin A + CV' \sin A + CV' \cdot CU' \sin C$$
$$\Leftrightarrow BT \cdot BU \sin B + DW \cdot DV \sin D = AT' \cdot DW \sin A + CV' \cdot BU \sin C$$
$$\Leftrightarrow BT \frac{BU}{BC \cdot DA} \sin B + \frac{DW}{BC \cdot DA} DV \sin D$$
$$= AT' \frac{DW}{BC \cdot DA} \sin A + CV' \frac{BU}{BC \cdot DA} \sin C$$
$$\Leftrightarrow BT \frac{1}{DA} \sin B + \frac{1}{BC} DV \sin D = AT' \frac{1}{BC} \sin A + CV' \frac{1}{DA} \sin C$$
$$\Leftrightarrow BT \cdot BC \sin B + DA \cdot DV \sin D = AT' \cdot DA \sin A + CV' \cdot BC \sin C$$
$$\Leftrightarrow (AB - AT) BC \sin B + DA(CD - CV) \sin D$$
$$= (AB - AT) DA \sin A + (CD - CV) BC \sin C$$
$$\Leftrightarrow 2ABC - AT \cdot BC \sin B + 2DAC - DA \cdot CV \sin D$$
$$= 2ABD - AT \cdot DA \sin A + 2BCD - CV \cdot BC \sin C$$
$$\Leftrightarrow \frac{AT}{AB \cdot CD} BC \sin B + DA \frac{CV}{AB \cdot CD} \sin D$$
$$= \frac{AT}{AB \cdot CD} DA \sin A + \frac{CV}{AB \cdot CD} BC \sin C$$
$$\Leftrightarrow \frac{1}{CD} BC \sin B + DA \frac{1}{AB} \sin D = \frac{1}{CD} DA \sin A + \frac{1}{AB} BC \sin C$$
$$\Leftrightarrow AB \cdot BC \sin B + DA \cdot CD \sin D = AB \cdot DA \sin A + CD \cdot BC \sin C$$
$$\Leftrightarrow 2ABD + 2BCD = 2ABC + 2DAC \Leftrightarrow ABCD = ABCD.$$

□

Theorem 2.10. *If the pairs of segments (AT, AB), (CV, CD), respectively (BU, BC), (DW, DA) have simultaneously same (or opposite) orientation, $\frac{AT}{AB} = \frac{CV}{CD}$ and $\frac{BU}{BC} = \frac{DW}{DA}$, then the isotomic inscribed quadrilaterals $TUVW$ and $T'U'V'W'$ have equal areas.*

Proof. We will consider the situation of Figure 9:

$$TUVW = ABCD - ATW + BUT - CVU + DWV$$
$$= ABCD + AT'W' - BU'T' + CV'U' - DW'V' = T'U'V'W'.$$

□

3. Analytic discussion

Henceforth we will work in oblique coordinates system [2]. We consider a parallelogram $PQRS$ and we attach to this parallelogram a coordinate system. Let be the x-axis the line determined by the midpoints of segment (PQ) and (RS), the y-axis the line determined by the midpoints of segment (PS) and (QR). A quadrilateral $ABCD$ we can generated in the following mode: take a point M in the plan of $PQRS$ no situated on its sides. Construct the anticomplementary triangle ABC of the triangle PQM. Let P, Q, M be the midpoints of segments (AB), (BC), (CA). The vertex D will be the symmetric of C with respect to R (Figure 10).

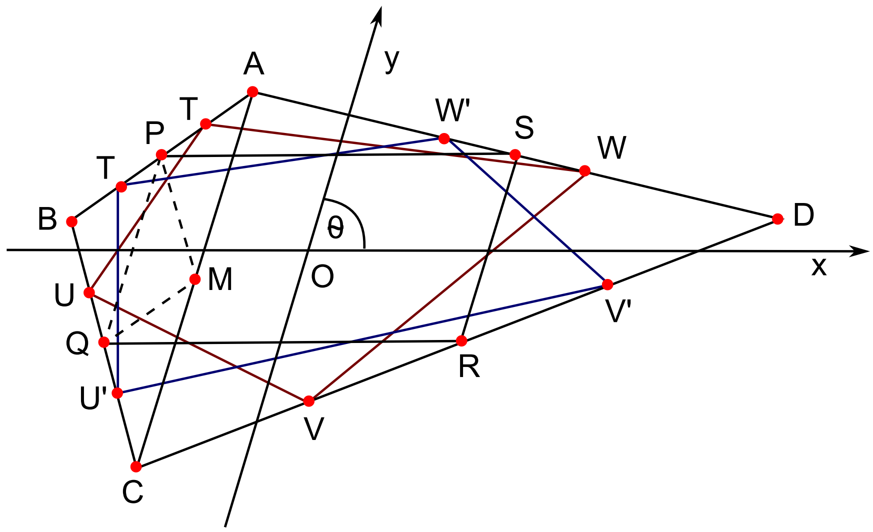

Figure 10

We introduce the following notations:

$$PS = QR = 2p,\ PQ = RS = 2q,\ \text{where}\ \ p > 0, q > 0.$$

Consequently $P = (-p, q)$, $Q = (-p, -q)$, $R = (p, -q)$, $S = (p, q)$.
If $M = (\alpha, \beta)$, then
$A = (\alpha, \beta + 2q)$, $B = (-\alpha - 2p, -\beta)$, $C = (\alpha, \beta - 2q)$, $D = (-\alpha + 2p, -\beta)$.
Let $TUVW$ a quadrilateral inscribed in $ABCD$ ($T \in AB$, $U \in BC$, $V \in CD$, $W \in DA$). We will determine the coordinates of the points T, U, V, W. The equation of the line AB is

$$\begin{vmatrix} x & y & 1 \\ \alpha & \beta + 2q & 1 \\ -\alpha - 2p & -\beta & 1 \end{vmatrix} = 0$$
$$\Leftrightarrow (\beta + q)x - (\alpha + p)y + p(\beta + q) + q(\alpha + p) = 0.$$

We can chose the coordinates of T as $T = [(\alpha + p)t - p, (\beta + q)t + q]$, where t is a real number. Similarly we can obtain that $U = [(\alpha + p)u - p, (\beta - q)u - q]$, $V = [-(\alpha - p)v + p, -(\beta - q)v - q]$, $W = [-(\alpha - p)w + p, -(\beta + q)w + q]$, where u, v, w are real numbers.
Let $T'U'V'W'$ the isotomic inscribed quadrilaterals of $TUVW$. We have:

$T' = [-(\alpha + p)t - p, -(\beta + q)t + q]$, $U' = [-(\alpha + p)u - p, -(\beta - q)u - q]$.
$V' = [(\alpha - p)v + p, (\beta - q)v - q]$, $W' = [(\alpha - p)w + p, (\beta + q)w + q]$.

Now we will calculate the difference of the area of isotomic inscribed quadrilaterals $TUVW$ and $T'U'V'W'$. In oblique coordinates the signed area of the triangle determined by the points $M_1 = (x_1, y_1)$, $M_2 = (x_2, y_2)$, $M_3 = (x_3, y_3)$ is

$M_1M_2M_3 = \dfrac{\sin\theta}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$, where θ denote the angle of coordinate axes x and y

$(0 < \theta < \pi)$ (Figure 10). We have:

$$TUVW = ABCD - ATW - BUT - CVU - DWV,$$
$$T'U'V'W' = ABCD - AT'W' - BU'T' - CV'U' - DW'V',$$
$$TUVW - T'U'V'W' = AT'W' - ATW + BU'T' - BUT + CV'U' - CVU + DW'V' - DWV,$$

$$AT'W' = \frac{\sin\theta}{2}\begin{vmatrix} \alpha & \beta+2q & 1 \\ -(\alpha+p)t-p & -(\beta+q)t+q & 1 \\ (\alpha-p)w+p & (\beta+q)w+q & 1 \end{vmatrix}$$
$$= -p(\beta+q)(t+1)(w-1)\sin\theta,$$

$$ATW = \frac{\sin\theta}{2}\begin{vmatrix} \alpha & \beta+2q & 1 \\ (\alpha+p)t-p & (\beta+q)t+q & 1 \\ -(\alpha-p)w+p & -(\beta+q)w+q & 1 \end{vmatrix}$$
$$= -p(\beta+q)(t-1)(w+1)\sin\theta,$$

$$BU'T' = \frac{\sin\theta}{2}\begin{vmatrix} -\alpha-2p & -\beta & 1 \\ -(\alpha+p)u-p & -(\beta-q)u-q & 1 \\ -(\alpha+p)t-p & -(\beta+q)t+q & 1 \end{vmatrix}$$
$$= (\alpha+p)q(u-1)(t-1)\sin\theta,$$

$$BUT = \frac{\sin\theta}{2}\begin{vmatrix} -\alpha-2p & -\beta & 1 \\ (\alpha+p)u-p & (\beta-q)u-q & 1 \\ (\alpha+p)t-p & (\beta+q)t+q & 1 \end{vmatrix}$$
$$= (\alpha+p)q(u+1)(t+1)\sin\theta,$$

$$CV'U' = \frac{\sin\theta}{2}\begin{vmatrix} \alpha & \beta-2q & 1 \\ (\alpha-p)v+p & (\beta-q)v-q & 1 \\ -(\alpha+p)u-p & -(\beta-q)u-q & 1 \end{vmatrix}$$
$$= p(\beta-q)(v-1)(u+1)\sin\theta,$$

$$CVU = \frac{\sin\theta}{2}\begin{vmatrix} \alpha & \beta-2q & 1 \\ -(\alpha-p)v+p & -(\beta-q)v-q & 1 \\ (\alpha+p)u-p & (\beta-q)u-q & 1 \end{vmatrix}$$
$$= p(\beta-q)(v+1)(u-1)\sin\theta,$$

$$DW'V' = \frac{\sin\theta}{2}\begin{vmatrix} -\alpha+2p & -\beta & 1 \\ (\alpha-p)w+p & (\beta+q)w+q & 1 \\ (\alpha-p)v+p & (\beta-q)v-q & 1 \end{vmatrix}$$
$$= -(\alpha-p)q(w+1)(v+1)\sin\theta,$$

$$DWV = \frac{\sin\theta}{2}\begin{vmatrix} -\alpha-2p & -\beta & 1 \\ -(\alpha-p)w+p & -(\beta+q)w+q & 1 \\ -(\alpha-p)v+p & -(\beta-q)v-q & 1 \end{vmatrix}$$
$$= -(\alpha-p)q(w-1)(v-1)\sin\theta,$$

$$\begin{aligned}AT'W' - ATW &= 2p(\beta+q)(t-w)\sin\theta,\\ BU'T' - BUT &= -2(\alpha+p)q(u+t)\sin\theta,\\ CV'U' - CVU &= 2p(\beta-q)(v-u)\sin\theta,\\ DW'V' - DWV &= -2(\alpha-p)q(w+v)\sin\theta,\end{aligned}$$

$$\begin{aligned}TUVW - T'U'V'W' &= 2[p(\beta+q)(t-w) - (\alpha+p)q(u+t)\\ &\quad + p(\beta-q)(v-u) - (\alpha-p)q(w+v)]\sin\theta\\ &= 2[-q(t+u+v+w)\alpha + p(t-u+v-w)\beta]\sin\theta.\end{aligned}$$

Our first observation is that if $\alpha = 0 = \beta$, then $TUVW - T'U'V'W' = 0$, i.e. the isotomic inscribed quadrilaterals $TUVW$ and $T'U'V'W'$ have equal areas. In this case $A = (0, 2q)$, $B = (-2p, 0)$, $C = (0, -2q)$, $D = (2p, 0)$, in other word the quadrilateral $ABCD$ is parallelogram (see Theorem 1.1).

Theorem 3.1. *In all quadrilateral $ABCD$ and for all points T, U, V, W on the sides AB, BC, CD, DA respectively, we have:*

$$(a)\quad \frac{AT}{AB} = \frac{1}{2}|t-1|, \frac{BU}{BC} = \frac{1}{2}|u+1|, \frac{CV}{CD} = \frac{1}{2}|v+1|, \frac{DW}{DA} = \frac{1}{2}|w-1|.$$

$$(b)\quad \frac{AT'}{AB} = \frac{1}{2}|t+1|, \frac{BU'}{BC} = \frac{1}{2}|u-1|, \frac{CV'}{CD} = \frac{1}{2}|v-1|, \frac{DW'}{DA} = \frac{1}{2}|w+1|.$$

Proof. In oblique coordinates the length of segment determined by two points $M_1 = (x_1, y_1)$ and $M_2 = (x_2, y_2)$ is

$$(M_1M_2)^2 = (x_1-x_2)^2 + (y_1-y_2)^2 + 2(x_1-x_2)(y_1-y_2)\cos\theta.$$

We will prove two equalities:

$$\begin{aligned}AB^2 &= 4[(\alpha+p)^2 + (\beta+q)^2 + 2(\alpha+p)(\beta+q)\cos\theta],\\ AT^2 &= (t-1)^2[(\alpha+p)^2 + (\beta+q)^2 + 2(\alpha+p)(\beta+q)\cos\theta] = (t-1)^2\frac{AB^2}{4}\\ BC^2 &= 4[(\alpha+p)^2 + (\beta-q)^2 + 2(\alpha+p)(\beta-q)\cos\theta],\\ BU'^2 &= (u-1)^2[(\alpha+p)^2 + (\beta-q)^2 + 2(\alpha+p)(\beta-q)\cos\theta] = (u-1)^2\frac{BC^2}{4}.\end{aligned}$$

If $t+u+v+w = 0$ and $t-u+v-w = 0$, i.e. if $t = -v$ and $u = -w$, then $TUVW - T'U'V'W' = 0$. In this case

$$\frac{AT}{AB} = \frac{1}{2}|t-1| = \frac{1}{2}|-v-1| = \frac{1}{2}|v+1| = \frac{CV}{CD}$$

and

$$\frac{BU}{BC} = \frac{1}{2}|u+1| = \frac{1}{2}|-w+1| = \frac{1}{2}|w-1| = \frac{DW}{DA}$$

(see Theorem 2.10). □

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