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GENERALIZATIONS OF THE LESTER CIRCLE

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ABSTRACT. Using barycentric coordinates, Cartesian coordinates with some help from SageMath − a free and open-source mathematical software system, we prove some properties of the $t-$pedal triangles and two generalizations of the Lester circle.

1. INTRODUCTION

In 1997, June Lester published an article [8]. In which she proved that two Fermat points, nine-point center, and circumcenter of any scalene triangle are concyclic. The circle through these four points were later called *the Lester circle*.
There have been at least three generalizations of the Lester circle so far. Bernard Gibert found the first generalization [6][12], in which this circle is a property of the Kiepert hyperbola.
Later, Dao Thanh Oai gave another generalization and his proof [1][2]. In this generalization, the circle is a property of rectangular hyperbola.

Theorem 1.1 (Dao Thanh Oai, [2]). *Let H and G be two points on one branch of a rectangular hyperbola, and*

(i) F_+ and F_- are antipodal points on the hyperbola, the tangents at F_+, F_- are parallel to the line HG,
(ii) K_+ and K_- are two points on the hyperbola, the tangents at K_+, K_- intersect at a point E on the line HG,

If the line K_+K_- intersects HG at D, and the perpendicular bisector of DE intersects the hyperbola at G_+ and G_-, then the six points F_+, F_-, D, E, G_+, G_- lie on a circle.

Not so long after that, Dao Thanh Oai conjectured the third generalization. This generalization was confirmed to be correct by Cezar Lozada [9][7]:

Theorem 1.2 (Dao Thanh Oai, [3]). *Given a triangle ABC and a point P lies on the Neuberg cubic of ABC.*
Let P_a, P_b, P_c be the reflections of P in BC, CA, AB, respectively.
AP_a, BP_b, CP_c are concurrent at a point, let it be Q_P.
Then P, Q_P and two Fermat points of triangle ABC are concyclic.

In the third generalization, according to the properties of the Neuberg cubic [5], P, Q_P, the isogonal conjugate P^* of P are collinear, and PQ_P is parallel to the Euler line.
Furthermore, the Euler line is parallel to the tangent lines of the Kiepert hyperbola at two Fermat points. So I came up with the idea of proving Theorem 1.2 by using Theorem 1.1. But in Theorem 1.1, the line DE has common points with the rectangular hyperbola. So it is not applicable for Theorem 1.2, since PQ_P and the Kiepert hyperbola do not necessarily have common

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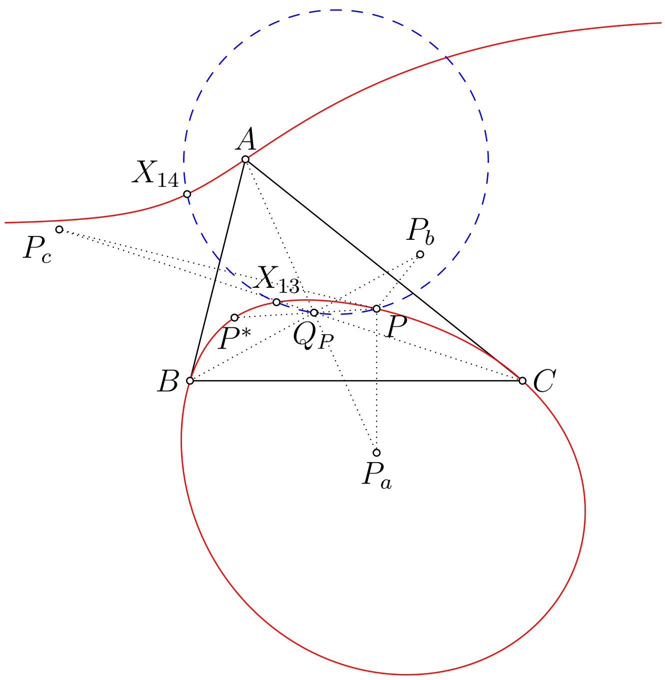

Figure 1. P, Q_P and two Fermat points are concyclic

points. To resolve this limitation of Theorem 1.1 and prove Theorem 1.2, I suggest an alternative for Theorem 1.1 as follow:

Theorem 1.3. *Let F_+, F_- be the antipodal points on a rectangular hyperbola $(\mathcal{H})$. The tangent lines at F_+, F_- of $(\mathcal{H})$ are τ_+, τ_-.*
P, Q are two points on $(\mathcal{H})$ such that PQ is perpendicular to τ_+.
M, N are conjugate points with respect to $(\mathcal{H})$, the line MN is parallel or coincident to τ_+, τ_- and two points M, N are not coincident.
The perpendicular bisector of MN intersects $(\mathcal{H})$ at G_+, G_-.

(i) The circle with diameter PQ passes through F_+, F_-.
(ii) F_+, F_-, G_+, G_-, M, N are concyclic.

In Theorem 1.3, part (i) can be considered as a degenerate case of part (ii), when MN degenerates to a point.
Later in this article, we will prove Theorem 1.3 and then use it to prove Theorem 1.2. But first, to provide rigorous proof for the third generalization, we need to prepare with some properties of the t–pedal triangles.

2. Preliminaries

Throughout this article, we will assume that triangle ABC is scalene and adopt the following notations from [11]:

Notations	Meanings
ABC	Reference triangle
a, b, c	Lengths of BC, CA, AB
S, S_A, S_B, S_C	Conway triangle notations
O, H	Circumcenter and orthocenter of triangle ABC
I, I_a, I_b, I_c	Incenter and excenters of triangle ABC

And we imply that P, P^* are isogonal conjugate points with respect to triangle ABC.

2.1. **Collinear points.**

Theorem 2.1 ([11]). *Given three points P_1, P_2, P_3, which have barycentric coordinates:*

$$P_1 = (x_1 : y_1 : z_1),$$
$$P_2 = (x_2 : y_2 : z_2),$$
$$P_3 = (x_3 : y_3 : z_3).$$

These three points are collinear if and only if:

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

2.2. **Conics and conjugation.** The results in this section follow [4].
Let's consider a conic (Γ). The general equation of (Γ) in barycentric coordinates is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} a_{xx} & a_{xy} & a_{zx} \\ a_{xy} & a_{yy} & a_{yz} \\ a_{zx} & a_{yz} & a_{zz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,$$

and the general equation of (Γ) in Cartesian coordinates is:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} a_{xx} & a_{xy} & a_{x} \\ a_{xy} & a_{yy} & a_{y} \\ a_{x} & a_{y} & a \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.$$

The square matrices in two above equations are called *the symmetric matrices of* (Γ).

Theorem 2.2. *The polar of a point:*

(i) In barycentric coordinates, let $P = (\alpha_0 : \beta_0 : \gamma_0)$, then the polar of P with respect to (Γ) has equation:

$$\begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} a_{xx} & a_{xy} & a_{zx} \\ a_{xy} & a_{yy} & a_{yz} \\ a_{zx} & a_{yz} & a_{zz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

(ii) In Cartesian coordinates, let $P = (x_0, y_0)$, then the polar of P with respect to (Γ) has equation:

$$\begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} a_{xx} & a_{xy} & a_{x} \\ a_{xy} & a_{yy} & a_{y} \\ a_{x} & a_{y} & a \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.$$

Two points are conjugate if and only if each lies on the polar line of the other. Thus, we obtain the following corollaries:

Theorem 2.3. *Conjugate points:*

(i) In barycentric coordinates, two points $P_1 = (x_1 : y_1 : z_1)$ and $P_2 = (x_2 : y_2 : z_2)$ are conjugate with respect to (Γ) if and only if:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} a_{xx} & a_{xy} & a_{zx} \\ a_{xy} & a_{yy} & a_{yz} \\ a_{zx} & a_{yz} & a_{zz} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = 0.$$

(ii) In Cartesian coordinates, two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are conjugate with respect to (Γ) if and only if:

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} a_{xx} & a_{xy} & a_x \\ a_{xy} & a_{yy} & a_y \\ a_x & a_y & a \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = 0.$$

3. $t-$PEDAL TRIANGLE

In this section, we study the t–pedal triangles, which play an important role in the proof of Theorem 1.2.
The term t–pedal triangle first appeared in an article by Pinkernell [10].
Given a triangle ABC and a point P. $P_AP_BP_C$ is the pedal triangle of P with respect to triangle ABC. $P_A(t), P_B(t), P_C(t)$ lies on PP_A, PP_B, PP_C such that:

$$\overline{PP_A(t)} = t \cdot \overline{PP_A}, \qquad \overline{PP_B(t)} = t \cdot \overline{PP_B}, \qquad \overline{PP_C(t)} = t \cdot \overline{PP_C}.$$

then $P_A(t)P_B(t)P_C(t)$ is called the t–pedal triangle of P with respect to triangle ABC, where t is a real number.
However, this definition does not include the case $t = \infty$. So let's convention that if P does not coincide with any vertices of triangle ABC, $P_A(\infty)$, $P_B(\infty)$, $P_C(\infty)$ will be the points at infinity of the corresponding altitudes of triangle ABC. Otherwise, if P coincides with A, B or C, then:

$$\begin{cases} A_B(\infty) \equiv A_C(\infty) \equiv A, \\ B_C(\infty) \equiv B_A(\infty) \equiv B, \\ C_A(\infty) \equiv C_B(\infty) \equiv C. \end{cases}$$

These conventions ensure the compatibility with later results.

In barycentric coordinates, let $P = (\alpha : \beta : \gamma)$, then the vertices of the pedal triangle of P have coordinates:

$$\begin{aligned} P_A &= \left(0 : a^2\beta + S_C\alpha : a^2\gamma + S_B\alpha\right), \\ P_B &= \left(b^2\alpha + S_C\beta : 0 : b^2\gamma + S_A\beta\right), \\ P_C &= \left(c^2\alpha + S_B\gamma : c^2\beta + S_A\gamma : 0\right). \end{aligned} \tag{3.1}$$

As a consequence, the vertices of the t–pedal triangle have coordinates:

$$\begin{aligned} P_A(t) &= \left((1-t)a^2\alpha : a^2\beta + tS_C\alpha : a^2\gamma + tS_B\alpha\right), \\ P_B(t) &= \left(b^2\alpha + tS_C\beta : (1-t)b^2\beta : b^2\gamma + tS_A\beta\right), \\ P_C(t) &= \left(c^2\alpha + tS_B\gamma : c^2\beta + tS_A\gamma : (1-t)c^2\gamma\right). \end{aligned} \tag{3.2}$$

The above formulas do not include the case $t = \infty$. However, we can replace t by two real numbers p, q which are not simultaneously zero such that:

$$\begin{cases} q = t \cdot p & (\text{if } t \neq \infty), \\ q = q_0 \neq 0,\ p = 0 & (\text{if } t = \infty). \end{cases}$$

By using these numbers, the coordinates of the vertices of the t–pedal triangle become:

$$\begin{aligned} P_A(t) &= \Big((p-q)a^2\alpha : pa^2\beta + qS_C\alpha : pa^2\gamma + qS_B\alpha\Big)\,, \\ P_B(t) &= \Big(pb^2\alpha + qS_C\beta : (p-q)b^2\beta : pb^2\gamma + qS_A\beta\Big)\,, \\ P_C(t) &= \Big(pc^2\alpha + qS_B\gamma : pc^2\beta + qS_A\gamma : (p-q)c^2\gamma\Big)\,. \end{aligned} \tag{3.3}$$

Now these have included the case $t = \infty$.

Proposition 3.1 ([10]). *Given a point P and its isogonal conjugate point P^* with respect to triangle ABC, such that P does not coincide with A, B, C, H, O, I, I_a, I_b, I_c.*
PP^* intersects OH at T (which might be point at infinity).
Let $P_A(2t)P_B(2t)P_C(2t)$ be the $2t$–pedal triangle of P, where $t = \frac{\overline{TO}}{\overline{TH}}$.
Then $AP_A(2t), BP_B(2t), CP_C(2t), PP^*$ are concurrent.

Proof. To avoid division by zero, let p, q be two real numbers which are not simultaneously zero and $p \cdot \overline{TO} = q \cdot \overline{TH}$. Together with the coordinates of O, H:

$$O = (a^2S_A : b^2S_B : c^2S_C), \qquad H = (2S_BS_C : 2S_CS_A : 2S_AS_B).$$

we obtain the coordinates of T as follow:

$$T = (pa^2S_A - 2qS_BS_C : pb^2S_B - 2qS_CS_A : pc^2S_C - 2qS_AS_B).$$

T, P, P^* are collinear if and only if:

$$\begin{vmatrix} pa^2S_A - 2qS_BS_C & pb^2S_B - 2qS_CS_A & pc^2S_C - 2qS_AS_B \\ \alpha & \beta & \gamma \\ a^2\beta\gamma & b^2\gamma\alpha & c^2\alpha\beta \end{vmatrix} = 0$$

$$\Leftrightarrow \underbrace{2q \sum_{cyclic} S_BS_C\alpha(c^2\beta^2 - b^2\gamma^2)}_{\text{Orthocubic } \mathbf{K006}} = \underbrace{p \sum_{cyclic} a^2S_A\alpha(c^2\beta^2 - b^2\gamma^2)}_{\text{McCay cubic } \mathbf{K003}}\,.$$

We can choose

$$p = \sum_{cyclic} S_BS_C\alpha(c^2\beta^2 - b^2\gamma^2) \quad \text{and} \quad q = \frac{1}{2}\sum_{cyclic} a^2S_A\alpha(c^2\beta^2 - b^2\gamma^2).$$

This selection satisfies p, q are not simultaneously zero since P is not the common point of the Orthocubic and the McCay cubic (there are nine common points, including A, B, C, H, O, I, I_a, I_b, I_c).
According to (3.3), the coordinates of $P_A(2t)$, $P_B(2t)$, $P_C(2t)$ can be written as follow:

$$\begin{aligned} P_A(2t) &= \Big(a^2\alpha(p-2q) : pa^2\beta + 2q\alpha S_C : pa^2\gamma + 2q\alpha S_B\Big)\,, \\ P_B(2t) &= \Big(pb^2\alpha + 2q\beta S_C : b^2\beta(p-2q) : pb^2\gamma + 2q\beta S_A\Big)\,, \\ P_C(2t) &= \Big(pc^2\alpha + 2q\gamma S_B : pc^2\beta + 2q\gamma S_A : c^2\gamma(p-2q)\Big)\,. \end{aligned}$$

From here, we will factorize $pa^2\beta + 2q\alpha S_C$ and $pa^2\gamma + 2q\alpha S_B$.

$$
\begin{aligned}
& pa^2\beta + 2q\alpha S_C \\
= \quad & \gamma(b^2\alpha^2 - a^2\beta^2)(a^2 S_A S_B \beta + c^2 S_C S_C \alpha) \\
& + \alpha(c^2\beta^2 - b^2\gamma^2)(a^2 S_B S_C \beta + a^2 S_A S_C \alpha) \\
& + \beta(a^2\gamma^2 - c^2\alpha^2)(a^2 S_C S_A \beta + b^2 S_B S_C \alpha) \\
= \quad & (b^2\alpha^2 - a^2\beta^2)(a^2 S_A S_B \beta\gamma + c^2 S_C S_C \alpha\gamma) \\
& + \alpha(c^2 a^2 S_B S_C \beta^3 - a^2 b^2 S_B S_C \beta\gamma^2 + a^2 c^2 S_A S_C \alpha\beta^2 - a^2 b^2 S_A S_C \alpha\gamma^2) \\
& + \beta(a^4 S_A S_C \beta\gamma^2 - a^2 c^2 S_C S_A \alpha^2 \beta + a^2 b^2 S_B S_C \alpha\gamma^2 - b^2 c^2 S_B S_C \alpha^3) \\
= \quad & (b^2\alpha^2 - a^2\beta^2)(a^2 S_A S_B \beta\gamma + c^2 S_C S_C \alpha\gamma) \\
& + (c^2 a^2 S_B S_C \alpha\beta^3 - b^2 c^2 S_B S_C \alpha^3 \beta + a^4 S_A S_C \beta^2\gamma^2 - a^2 b^2 S_A S_C \alpha^2\gamma^2) \\
= \quad & (b^2\alpha^2 - a^2\beta^2)(a^2 S_A S_B \beta\gamma + c^2 S_C S_C \alpha\gamma) \\
& + c^2 S_B S_C \alpha\beta(a^2\beta^2 - b^2\alpha^2) + a^2 S_A S_C \gamma^2 (a^2\beta^2 - b^2\alpha^2) \\
= \quad & (b^2\alpha^2 - a^2\beta^2)(a^2 S_A S_B \beta\gamma - a^2 S_A S_C \gamma^2 + c^2 S_C S_C \alpha\gamma - c^2 S_B S_C \alpha\beta) \\
= \quad & (b^2\alpha^2 - a^2\beta^2)(a^2 S_A \gamma - c^2 S_C \alpha)(S_B \beta - S_C \gamma).
\end{aligned}
$$

Analogously, we have:

$$pa^2\gamma + 2q\alpha S_B = (a^2\gamma^2 - c^2\alpha^2)(b^2 S_B \alpha - a^2 S_A \beta)(S_B\beta - S_C\gamma).$$

Therefore, $AP_A(2t)$ passes through:

$$
\begin{aligned}
Q_P = ((c^2 S_C \beta - b^2 S_B \gamma)(a^2\gamma^2 - c^2\alpha^2)(b^2\alpha^2 - a^2\beta^2) : \\
(a^2 S_A \gamma - c^2 S_C \alpha)(b^2\alpha^2 - a^2\beta^2)(c^2\beta^2 - b^2\gamma^2) : \\
(b^2 S_S \alpha - a^2 S_A \beta)(c^2\beta^2 - b^2\gamma^2)(a^2\gamma^2 - c^2\alpha^2)).
\end{aligned}
$$

Cyclically, $BP_B(2t), CP_C(2t)$ also pass through Q_P.

Q_P, P, P^* are collinear since:

$$
\begin{aligned}
& \begin{vmatrix}
(c^2 S_C \beta - b^2 S_B \gamma)(a^2\gamma^2 - c^2\alpha^2)(b^2\alpha^2 - a^2\beta^2) & \alpha & a^2\beta\gamma \\
(a^2 S_A \gamma - c^2 S_C \alpha)(b^2\alpha^2 - a^2\beta^2)(c^2\beta^2 - b^2\gamma^2) & \beta & b^2\gamma\alpha \\
(b^2 S_S \alpha - a^2 S_A \beta)(c^2\beta^2 - b^2\gamma^2)(a^2\gamma^2 - c^2\alpha^2) & \gamma & c^2\alpha\beta
\end{vmatrix} \\
= \quad & (c^2\beta^2 - b^2\gamma^2)(a^2\gamma^2 - c^2\alpha^2)(b^2\alpha^2 - a^2\beta^2) \sum_{cyclic} (c^2 S_C \alpha\beta - b^2 S_B \gamma\alpha) \\
= \quad & 0.
\end{aligned}
$$

In conclusion, $AP_A(2t), BP_B(2t), CP_C(2t), PP^*$ are concurrent. □

The point of concurrence is called *the Euler-Pinkernell perspector of P with respect to triangle ABC*, or just *the Euler-Pinkernell perspector of P*.
The following gives some properties of this perspector.

Proposition 3.2. *Let $P = (\alpha : \beta : \gamma)$ be a point other than A, B, C, H, O, I, I_a, I_b, I_c.*
Let Q_P be the Euler-Pinkernell perspector of P.

(i) Q_P, P and P^* are collinear.

(ii) Q_P *has coordinates:*

$$((c^2S_C\beta - b^2S_B\gamma)(a^2\gamma^2 - c^2\alpha^2)(b^2\alpha^2 - a^2\beta^2) : \\ (a^2S_A\gamma - c^2S_C\alpha)(b^2\alpha^2 - a^2\beta^2)(c^2\beta^2 - b^2\gamma^2) : \\ (b^2S_S\alpha - a^2S_A\beta)(c^2\beta^2 - b^2\gamma^2)(a^2\gamma^2 - c^2\alpha^2)).$$

(iii) Q_P *coincides with* P *if and only if* P *lies on the McCay cubic.*
(iv) Q_P *coincides with* H *if and only if* P *lies on the Orthocubic.*

In general, the locus of P such that $\frac{\overline{TO}}{\overline{TH}} =$ const is a pivotal isogonal cubic with pivot T. The cubics, of which pivots lie on the Euler line, belong to a pencil of cubics called *the Euler pencil*. In the Euler pencil, every cubic contains nine points $A, B, C, H, O, I, I_a, I_b, I_c$. Many other properties of the Euler pencil and its members are summarized at Bernard Gibert's website [5].
Nine points $A, B, C, H, O, I, I_a, I_b, I_c$ are excluded from Proposition 3.1 and 3.2 since for all t, their $t-$pedal triangles are perspective with triangle ABC. The coordinates and some properties of the perspectors of these nine points are summarized in the following:

Proposition 3.3. *Let $A(t)$, $B(t)$, $C(t)$, $H(t)$, $O(t)$, $I(t)$, $I_a(t)$, $I_b(t)$, $I_c(t)$ be the perspectors of triangle ABC with the $t-$pedal triangles of A, B, C, H, O, I, I_a, I_b, I_c respectively.*
Let T be a point on OH such that $\frac{\overline{TO}}{\overline{TH}} = t$.

(i) $A(t) \equiv A$, $B(t) \equiv B$, $C(t) \equiv C$, $H(t) \equiv H$.
(ii) If $t \neq \infty$:

$$\begin{aligned} O(t) &= (a^2S_A + tS_BS_C : b^2S_B + tS_CS_A : c^2S_C + tS_AS_B), \\ I(t) &= ((ca + tS_B)(ab + tS_C) : (ab + tS_C)(bc + tS_A) : (bc + tS_A)(ca + tS_B)), \\ I_a(t) &= ((ca - tS_B)(ab - tS_C) : -(ab - tS_C)(bc + tS_A) : -(ca - tS_B)(bc + tS_A)), \\ I_b(t) &= (-(ab - tS_C)(ca + tS_B) : (ab - tS_C)(bc - tS_A) : -(bc - tS_A)(ca + tS_B)), \\ I_c(t) &= (-(ca - tS_B)(ab + tS_C) : -(bc - tS_A)(ab + tS_C) : (ca - tS_B)(bc - tS_A)). \end{aligned} \tag{3.4}$$

(iii) $O(\infty)$, $I(\infty)$, $I_a(\infty)$, $I_b(\infty)$, $I_c(\infty)$ *coincide with* H.
(iv) The following triplets are collinear:

$$(A, A(2t), T), (B, B(2t), T), (C, C(2t), T), (H, H(2t), T), (O, O(2t), T), \\ (I, I(2t), T), (I_a, I_a(2t), T), (I_b, I_b(2t), T), (I_c, I_c(2t), T).$$

(v) The following pairs are conjugate with respect to the Kiepert hyperbola:

$$(A, A(t)), (B, B(t)), (C, C(t)), (H, H(t)), (O, O(2)), \\ (I, I(2)), (I_a, I_a(2)), (I_b, I_b(2)), (I_c, I_c(2)).$$

Proof. Part (i). A, B, C coincide with $A(t)$, $B(t)$, $C(t)$, this follows the formulas given in (3.2) (when t is a real number) and the convention (when $t = \infty$).
If ABC is not a right triangle, then the pedal triangle of H does not degenerate and the vertices of its $t-$pedal triangle lie on the corresponding altitudes of triangle ABC, so $H(t)$ coincides with H. Otherwise, ABC is a right triangle, H coincides with a vertice of triangle ABC, the result still holds.

Part (ii). The coordinates of $O(t)$, $I(t)$, $I_a(t)$, $I_b(t)$, $I_c(t)$ are calculated from the coordinates of the vertices of the t–pedal triangles of O, I, I_a, I_b, I_c, which follow the formulas given in (3.2).

Part (iii). Since the circumcenter, incenter and excenters do not coincide with any vertice of triangle ABC, then the result follows the convention.

Part (iv) and (v). The pairs that contain A, B, C, H are obvious, since A, B, C, H coincide with $A(t)$, $B(t)$, $C(t)$, $H(t)$, respectively and they lie on the Kiepert hyperbola.
If t is a real number, the coordinates of $O(2t)$ indicate that $O(2t)$ lies on the Euler line, so O, $O(2t)$, T are collinear. Otherwise, $t = \infty$, $O(\infty)$ and T coincide with H, the result still holds.
$O(2)$ is the nine-point center N of triangle ABC. N and O are conjugate with respect to the Kiepert hyperbola, which follows the results in [12].

However, proof for the pairs that contain the incenter or excenters is not so straightforward. But it suffices to prove for the incenter, since the barycentric coordinates of the incenter and excenters have the form of $(\pm a : \pm b : \pm c)$.
If $t = \infty$, $I(\infty)$, $I_a(\infty)$, $I_b(\infty)$, $I_c(\infty)$ and T coincide with H. Otherwise, when t is a real number, I introduce a proof using SageMath.
Here is the SageMath script that I have used to prove I, $I(2t)$, T are collinear and I, $I(2)$ are conjugate with respect to the Kiepert hyperbola. These scripts use the formulas in Theorem 2.1 and 2.3.

```
a, b, c, t = var("a, b, c, t")
# a, b, c are the sidelengths of BC, CA, AB
# t = TO / TH
# Conway triangle notations
sa = (-a*a + b*b + c*c)/2
sb = (a*a - b*b + c*c)/2
sc = (a*a + b*b - c*c)/2

# coordinates of T
T = [
    a*a*sa-2*t*sb*sc,
    b*b*sb-2*t*sc*sa,
    c*c*sc-2*t*sa*sb
]
# coordinates of the incenter
I = [a, b, c]
# coordinates of the perspector
P = [
    (a*c+2*t*sb)*(a*b+2*t*sc),
    (a*b+2*t*sc)*(b*c+2*t*sa),
    (b*c+2*t*sa)*(c*a+2*t*sb)
]
# coordinates of I(2)
I2 = [
    (a*c+2*sb)*(a*b+2*sc),
    (a*b+2*sc)*(b*c+2*sa),
    (b*c+2*sa)*(c*a+2*sb)
]
# symmetric matrix of the Kiepert hyperbola
K = matrix([
    [0, a*a-b*b, c*c-a*a],
    [a*a-b*b, 0, b*b-c*c],
    [c*c-a*a, b*b-c*c, 0]
])
```

```
# I, I(2t), T are collinear
print(matrix([T, I, P]).det().expand()) # equals zero

# I, I(2) are conjugate with respect to the Kiepert hyperbola
d = vector(I2) * K * vector(I).column()
print(d[0].expand()) # equals zero
```

□

Proposition 3.4. *A point P (other than A, B, C, H, O, I, I_a, I_b, I_c) and its Euler-Pinkernell perspector are conjugate with respect to the Kiepert hyperbola if and only if P lies on the union of the Neuberg cubic and the Grebe cubic.*

Proof. In barycentric coordinates, the Kiepert hyperbola has equation:

$$(b^2-c^2)yz+(c^2-a^2)zx+(a^2-b^2)xy=0.$$

According to Theorem 2.3 and 3.2, $P=(\alpha:\beta:\gamma)$ and Q_P are conjugate with respect to the Kiepert hyperbola if and only if:

$$\begin{pmatrix}(c^2S_C\beta-b^2S_B\gamma)(a^2\gamma^2-c^2\alpha^2)(b^2\alpha^2-a^2\beta^2)\\(a^2S_A\gamma-c^2S_C\alpha)(b^2\alpha^2-a^2\beta^2)(c^2\beta^2-b^2\gamma^2)\\(b^2S_S\alpha-a^2S_A\beta)(c^2\beta^2-b^2\gamma^2)(a^2\gamma^2-c^2\alpha^2)\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}0&a^2-b^2&c^2-a^2\\a^2-b^2&0&b^2-c^2\\c^2-a^2&b^2-c^2&0\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=0.$$

Here is the SageMath script that I have used to expand and factorize the left-hand side of the above equation:

```
a, b, c, x, y, z = var("a, b, c, x, y, z")

# a = S_A, b = S_B, c = S_C (Conway triangle notations)
# x = alpha, y = beta, z = gamma

A = (a + b) * y^2 - (c + a) * z^2
B = (b + c) * z^2 - (a + b) * x^2
C = (c + a) * x^2 - (b + c) * y^2
AA = c * (a + b) * y - b * (c + a) * z
BB = a * (b + c) * z - c * (a + b) * x
CC = b * (c + a) * x - a * (b + c) * y

P = [x, y, z]

# Euler-Pinkernell perspector
Q = [AA * B * C, A * BB * C, A * B * CC]

# symmetric matrix of the Kiepert hyperbola
K = matrix([
  [0, b - a, a - c],
  [b - a, 0, c - b],
  [a - c, c - b, 0]
])
```

```
product = vector(Q) * K * vector(P).column()

print(factor(product[0]))
```

After factorization, the condition equation becomes:

$$\underbrace{\sum_{cyclic}(a^2S_A - 2S_BS_C)\alpha(c^2\beta^2 - b^2\gamma^2)}_{\text{Neuberg cubic } \mathbf{K001}} \times \underbrace{\sum_{cyclic} a^2\alpha(c^2\beta^2 - b^2\gamma^2)}_{\text{Grebe cubic } \mathbf{K102}} = 0.$$

and the result follows. □

4. Proofs of two generalizations

4.1. **Proof of Theorem 1.3.** Put $(\mathcal{H})$, F_+, F_-, P, Q, M, N, G_+, G_- into the Cartesian plane. Since all rectangular hyperbolas are similar, it suffices to prove in case the equation of $(\mathcal{H})$ is $xy = 1$.
Proof of part (i).
Let the coordinates of F_+ and F_- be:

$$F_+ = \left(a, \frac{1}{a}\right) \qquad F_- = \left(-a, -\frac{1}{a}\right).$$

where a is a real number. The tangents at F_+ and F_- of $(\mathcal{H})$ have equations:

$$\tau_+ : \quad x + a^2y = 2a,$$
$$\tau_- : \quad x + a^2y = -2a.$$

PQ is perpendicular to τ_+, τ_-, then PQ has equation $a^2x - y = b$, where b is a real number.
The coordinates of P, Q (intersections of $(\mathcal{H})$ and PQ) are the solutions of the system:

$$\begin{cases} a^2x - y = b, \\ xy = 1. \end{cases}$$

By substitution, we obtain that the $x-$coordinate x_1, x_2 of P, Q are the roots of $a^2x^2 - bx - 1 = 0$. The discriminant of this equation is positive, which implies that any line perpendicular to τ_+ would meet $(\mathcal{H})$ at two distinct points. According to Vieta's formula, $x_1x_2 = -\frac{1}{a^2}$.

$$\begin{aligned}
\overrightarrow{F_+P} \cdot \overrightarrow{F_+Q} &= (x_1 - a)(x_2 - a) + \left(\frac{1}{x_1} - \frac{1}{a}\right)\left(\frac{1}{x_2} - \frac{1}{a}\right) \\
&= (x_1 - a)(x_2 - a)\left(1 + \frac{1}{a^2x_1x_2}\right) = 0, \\
\overrightarrow{F_-P} \cdot \overrightarrow{F_-Q} &= (x_1 + a)(x_2 + a) + \left(\frac{1}{x_1} + \frac{1}{a}\right)\left(\frac{1}{x_2} + \frac{1}{a}\right) \\
&= (x_1 + a)(x_2 + a)\left(1 + \frac{1}{a^2x_1x_2}\right) = 0.
\end{aligned}$$

Hence, F_+, F_- lie on the circle of which diameter is PQ.

Proof of part (ii).

In two points M, N:

- If one point lies on τ_+ (or τ_-), then the other coincides with F_+ (or F_-).
- If one point coincides with F_+ (or F_-), then the other is an arbitrary point on τ_+ (or τ_-) other than F_+ and F_- (since M and N are not coincident).

Boths cases lead to M, N, F_+, F_- are concyclic.
If M, N do not lie on τ_+ nor τ_-, we will use the converse of the intersecting chords (secants) theorem. Let the coordinates of M be (x_0, y_0).
MN has equations:

$$MN: \quad x + a^2 y = x_0 + a^2 y_0.$$

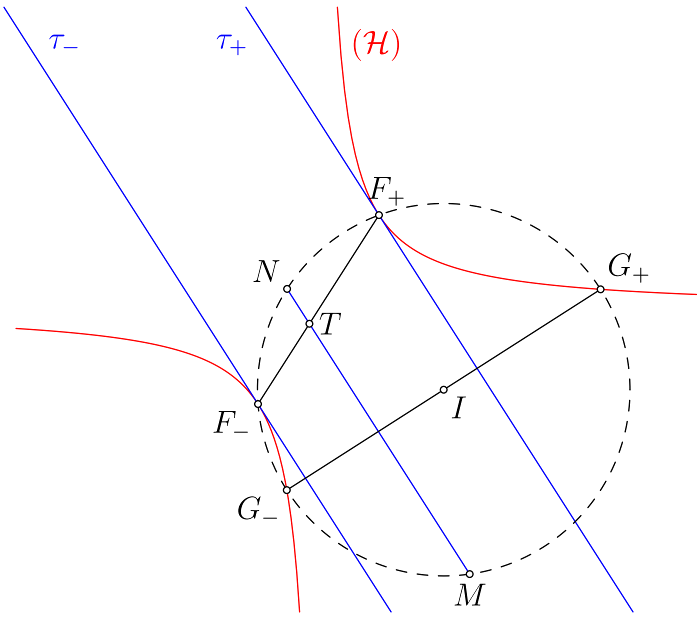

Figure 2. Theorem 1.3

The line MN has direction vector $(a^2, -1)$. Let $N = (x_0 + ka^2, y_0 - k)$.
According to Theorem 2.3, M, N are conjugate with respect to $(\mathcal{H})$ if and only if $x_0(y_0 - k) + y_0(x_0 + ka^2) - 2 = 0$.
Solving for k, we obtain that

$$N = \left(x_0 + \frac{2a^2(x_0 y_0 - 1)}{x_0 - a^2 y_0}, y_0 - \frac{2(x_0 y_0 - 1)}{x_0 - a^2 y_0} \right).$$

The intersection T of MN and F_+F_- has coordinates:

$$T = \left(\frac{1}{2}(x_0 + a^2 y_0), \frac{1}{2a^2}(x_0 + a^2 y_0) \right).$$

$$\begin{aligned}
\overline{TF_+} \cdot \overline{TF_-} &= \overrightarrow{TF_+} \cdot \overrightarrow{TF_-} \\
&= \left(\frac{1}{2}(x_0 + a^2y_0) - a\right)\left(\frac{1}{2}(x_0 + a^2y_0) + a\right) \\
&+ \left(\frac{1}{2a^2}(x_0 + a^2y_0) - \frac{1}{a}\right)\left(\frac{1}{2a^2}(x_0 + a^2y_0) + \frac{1}{a}\right) \\
&= \left(\frac{x_0 + a^2y_0}{2}\right)^2 - a^2 + \left(\frac{x_0 + a^2y_0}{2a^2}\right)^2 - \frac{1}{a^2} \\
&= \left(\frac{x_0 + a^2y_0}{2}\right)^2 \cdot \left(1 + \frac{1}{a^4}\right) - a^2 - \frac{1}{a^2}.
\end{aligned}$$

$$\begin{aligned}
\overline{TM} \cdot \overline{TN} &= \overrightarrow{TM} \cdot \overrightarrow{TN} \\
&= \frac{1}{2}(x_0 - a^2y_0)\left(\frac{1}{2}(x_0 - a^2y_0) + \frac{2a^2(x_0y_0 - 1)}{x_0 - a^2y_0}\right) \\
&+ \frac{1}{2a^2}(a^2y_0 - x_0)\left(\frac{1}{2a^2}(a^2y_0 - x_0) - \frac{2(x_0y_0 - 1)}{x_0 - a^2y_0}\right) \\
&= \left(\frac{x_0 - a^2y_0}{2}\right)^2 \cdot \left(1 + \frac{1}{a^4}\right) + \left(a^2 + \frac{1}{a^2}\right)(x_0y_0 - 1) \\
&= \left(\frac{x_0 + a^2y_0}{2}\right)^2 \cdot \left(1 + \frac{1}{a^4}\right) - a^2x_0y_0\left(1 + \frac{1}{a^4}\right) + \left(a^2 + \frac{1}{a^2}\right)(x_0y_0 - 1) \\
&= \left(\frac{x_0 + a^2y_0}{2}\right)^2 \cdot \left(1 + \frac{1}{a^4}\right) - a^2 - \frac{1}{a^2}.
\end{aligned}$$

Therefore, $\overline{TF_+} \cdot \overline{TF_-} = \overline{TM} \cdot \overline{TN}$, then M, N, F_+, F_- are concyclic.
G_+G_- is perpendicular to τ_+ so G_+, G_-, F_+, F_- lie on a circle of which diameter is G_+G_- (this is part (i)).
Let I be the midpoint of G_+G_-. Since F_+, F_- lie on a circle with diameter G_+G_-, then I is the center of this circle.
G_+G_- is the perpendicular bisector of MN and F_+, F_-, M, N are concyclic, then I is also the center of the circle through F_+, F_-, M, N.
Thus, six points F_+, F_-, M, N, G_+, G_- are concyclic. □

4.2. **Proof of Theorem 1.2.** According to Proposition 3.3 and 3.4, P and Q_P are conjugate with respect to the Kiepert hyperbola.
P, P^*, Q_P are collinear PP^* is parallel to the Euler line due to Proposition 3.1 and 3.2. The tangent lines of the Kiepert hyperbola at two Fermat points are parallel to the Euler line.
Hence, by Theorem 1.3, P, Q_P and two Fermat points are concyclic. □

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