THREE NEW PROPOSITIONS THROUGH HARMONIC CONJUGATES

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Abstract. In this note, three new propositions are established, the first characterizes the harmonic conjugate of a point with respect to two others, the second has as a corollary the Butterfly Theorem, and the third proposition allows to build an uncountable infinity of butterflies.

1. Introduction

We prove the first proposition in section 2 of this note by using similarity of triangles. We prove the second proposition of the aforementioned section by using the first proposition we mention earlier, as well as the definition and the uniqueness of the conjugate harmonic of one point with respect to two others, so it is recommended to review Chapter 4 of [4], or Chapter VII of [1]; we also use the definition of power of a point with respect to a circle, so it is recommended to review chapter 6 of [4], or section E of chapter VIII of [1]; likewise, the theory of poles and polars will be used, so it is recommended to also see sections 8.1, 8.2 and 8.4 of chapter 8 of [4] and section C of chapter VIII of [1]. Then, the famous Butterfly Theorem (see [2] or [3]) is proved as a corollary of the second proposition we mentioned above. Finally, we include a third proposition that guarantees the existence of an uncountable infinity of butterflies and something more.

2. Three propositions of harmony

Proposition 2.1. Let $AB$ be the diameter of a circle $C$ of center $O$, $P$ a point between $A$ and $B$, $PR$ perpendicular to $AB$ in $P$ with $R$ in $C$, and $Q$ a point at the exterior of $C$ collinear with $A$ and $B$. Then, $RQ$ is tangent to $C$ in $R$ if and only if $Q$ is the harmonic conjugate of $P$ with respect to $A$ and $B$.

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Proof. If \( P = O \), then the tangent \( t \) to \( C \) by \( R \) is perpendicular to \( PR = OR \), so \( t \) is parallel to \( AB \), as shown in Figure 1; also, since \( P \) is the midpoint of \( AB \) and considering \( Q \) as the point at infinity of the directed line \( AB \), it is held that \( RQ \) is tangent to \( C \) in \( R \) if and only if \( Q \) is the harmonic conjugate of \( P \) with respect to \( A \) and \( B \). Now, if \( P \neq O \) as shown in figure 2, first suppose that \( RQ \) is tangent to \( C \) in \( R \); then, when considering the chords \( AR \) and \( RB \) we stand that, by subtend the same arc \( \angle BAR = \angle BRQ \); as in addition \( \angle Q = \angle Q' \); then, triangles \( ARQ \) and \( RBQ \) are similar, so

\[
\frac{|AQ|}{|QB|} = \frac{|AR|}{|RB|} \implies |AQ| = \frac{|AR| \cdot |QR|}{|RB|}, \quad y
\]

\[
\frac{|QB|}{|QR|} = \frac{|RB|}{|AR|} \implies |QB| = \frac{|QR| \cdot |RB|}{|AR|},
\]

from where

\[
\frac{|AQ|}{|QB|} = \left(\frac{|AR|}{|RB|}\right)^2, \quad (i)
\]

furthermore, since \( \angle APR = \angle ARB \) are right angles; as well as \( \angle PAR = \angle PAR \); then, the triangles \( APR \) and \( ARB \) are similar, so

\[
\frac{|AP|}{|AR|} = \frac{|PR|}{|RB|} \implies |AP| = \frac{|AR| \cdot |PR|}{|RB|}, \quad (ii)
\]

moreover, since \( \angle RPB = \angle ARB \) are right angles, and \( \angle RBP = \angle RBA \); then, the triangles \( RPB \) and \( ARB \) are similar, so

\[
\frac{|PB|}{|RB|} = \frac{|PR|}{|AR|} \implies |PB| = \frac{|PR| \cdot |RB|}{|AR|}, \quad (iii)
\]

now, from \((ii)\) and \((iii)\) we have

\[
\frac{|AP|}{|PB|} = \left(\frac{|AR|}{|RB|}\right)^2, \quad (iv)
\]

and from \((i)\) and \((iv)\) it follows that \( \frac{|AP|}{|PB|} = \frac{|AQ|}{|QB|} \), but \( \frac{AP}{PB} > 0 \) and \( \frac{AQ}{QB} < 0 \), so

\[
\frac{AP}{PB} = -\frac{AQ}{QB},
\]

that is to say, \( Q \) is the harmonic conjugate of \( P \) with respect to \( A \) and \( B \).

Conversely, suppose that \( Q \) is the harmonic conjugate of \( P \) with respect to \( A \) and \( B \); then, when considering the tangent \( t \) to \( C \) in \( R \) for the first part it follows that the intersection \( Q' \) of \( t \) with the line \( AB \) is the harmonic conjugate of \( P \) with respect to \( A \) and \( B \); consequently, \( Q' = Q \); thus, \( RQ \) is tangent to \( C \) in \( R \).

Proposition 2.2. Let \( M \) be the midpoint of a chord \( PQ \) of a circle \( C \) of center \( O \), and \( ADBC \) a quadrilateral inscribed in \( C \) such that \( PQ \) intersects its opposite sides \( AD \) and \( BC \) in points \( X \) and \( Y \), respectively, and one of their diagonals passes through \( M \); then the diagonals \( AB \) and \( CD \) incise in \( M \) if and only if \( M \) is also the midpoint of \( XY \).
Proof. Suppose that $PQ$ cuts opposite sides $AD$ and $CB$ in points $X$ and $Y$, respectively, and the diagonals $AB$ and $CD$ of the quadrilateral $ADBC$ incise in $M$. It will be shown that $M$ is the midpoint of $XY$. For this purpose, let be $E$ the point of intersection of lines $AD$ and $BC$; $F$ the point of intersection of lines $AC$ and $BD$; $G$ the intersection of lines $EM$ and $AC$, as shown in figure 3; then, by the first theorem of section 4.14 of [4] it follows that $G$ is the conjugate harmonic of $F$ with respect to points $D$ and $B$, and $H$ is the harmonic conjugate of $F$ with respect to points $A$ and $C$; hence the $EHMG$ line, or briefly, line $EM$ is the polar of $F$ with respect to circle $C$; likewise, the line $FM$ is the polar of $E$ with respect to circle $C$; consequently, by the fundamental theorem of section 8.2 of [4] it follows that the polar of $M$ with respect to circle $C$ passes through points $E$ and $F$. Also, considering the tangent $PI$ to $C$, where $P$ is the point of tangency and $I$ is a point on the line $OM$ which is bisector of the chord $PQ$, by proposition 2.1 we have that $I$ is the conjugate harmonic of $M$ with respect to the intersections of $OM$ with $C$; therefore, $PQ$ is the polar of $I$ with respect to circle $C$, and again, by the aforementioned fundamental theorem it follows that $I$ is in the polar of $M$; consequently, $I$ is in the line $EF$: Furthermore, since $OM$ is the perpendicular bisector of $PQ$ it follows that the triangles $PMI$ and $QMI$ are congruent, so $\angle PIM = \angle MIQ$; then by the Theorem from section 4.8 of [4] it follows that $I(PQMF)$ is a harmonic pencil, and since $M$ is the midpoint of $PQ$, therefore

$$EIF = IF \parallel PQ;$$  \hspace{1cm} (i)

furthermore, since $G$ is the harmonic conjugate of $F$ with respect to the points $D$ and $B$, then, by the theorem of section 4.6 of [4] it follows that $E(XYML)$ is a harmonic pencil, where $L$ is the intersection of $EIF$ with the line $PQ$; furthermore, by (i) it follows that $L$ is the point at infinity of $PQ$; and by the corollary of the same section 4.6 it follows that $(XYML)$ is a harmonic row of points, where $L$ is the point at infinity of the line $PQ = XY$, so $M$ is the midpoint of $XY$. 

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Conversely, suppose that $M$ is the midpoint of $XY$ and that one of its diagonals passes through $M$, as shown in figure 4. It will be proved that diagonals $AB$ and $CD$ are intersected in $M$. For this purpose, without loss of generality suppose that $CD$ passes through $M$; then points $D$, $M$ and $C$ are collinear, so considering point $E$ in the intersection of the lines $AD$ and $BC$ and applying Menelaus’ Theorem to triangle $XYE$ we obtain

$$\frac{YC}{CE} \cdot \frac{ED}{DX} \cdot \frac{XM}{MY} = -1,$$

but by hypothesis $XM = MY$; from this and (i) it follows that

$$\frac{CE}{YC} \cdot \frac{DX}{ED} = -1,$$  \hspace{1cm} (ii)

Now if $AB$ cuts $XY$ in $M'$, then using Menelaus’ Theorem in triangle $XYE$ it results

$$\frac{XA}{AE} \cdot \frac{EB}{BY} \cdot \frac{YM'}{M'X} = -1;$$  \hspace{1cm} (iii)

Furthermore, by the definition of power of a point with respect to a circle, we have that $EA \cdot ED = EC \cdot EB$, from where

$$\frac{AE}{EC} \cdot \frac{ED}{EB} = 1;$$  \hspace{1cm} (iv)

consequently, multiplying member by member (ii), (iii) and (iv) it is as follows

$$\frac{XA}{BY} \cdot \frac{DX}{YC} \cdot \frac{YM'}{M'X} = 1;$$  \hspace{1cm} (v)

furthermore, by definition of power we have

$$XP \cdot XQ =XA \cdot XD,$$  \hspace{1cm} (vi)

$$YP \cdot YQ =YC \cdot YB,$$

but by hypothesis $PM = MQ$ and $XM = MY$; then, using directed segments, it results

$$XP = XM + MP = MY + QM = QY = -YQ,$$

$$XQ = XM + MQ = MY + PM = PY = -YP;$$

from this, and the relations given in (vi) it follows that

$$XA \cdot DX = YC \cdot BY;$$  \hspace{1cm} (vii)

therefore, from (v) and (vii) we get $XM' = M'Y$, so $M' = M$. 

\[\blacksquare\]
Butterfly Theorem. Let $M$ be the midpoint of a chord $PQ$ of a circle, $AB$ and $CD$ other chords through $M$ such that $C$ and $D$ are on opposite sides of $PQ$. If $AD$ and $CB$ intersect $PQ$ in $X$ and $Y$, respectively, then $M$ is also the midpoint of $XY$.

Proof. Follows immediately from proposition 2.2

The following proposition guarantees the existence of an uncountable infinity of butterflies whose wings are cut by a chord $PQ$ of a circle $C$ in points $X$ and $Y$ symmetrical with respect to the midpoint $M$ of the chord $PQ$.

Proposition 2.3. Let $PQ$ be a chord of a circle $C$ and $M$ the midpoint of $PQ$, then:

1. For every point $X$ between $P$ and $M$ there is an uncountable infinity of quadrilaterals (butterflies) $ADBC$ inscribed in $C$ such that $PQ$ cuts their opposite sides $AD$ and $BC$ in $X$ and a point $Y$, respectively, their diagonals $AD$ and $BC$ incise in $M$; and $M$ is also the midpoint of $XY$.
2. The intersections of the opposite sides of all quadrilaterals $ADBC$ from the previous part 1 are collinear.

Proof.

1. Let $M$ be the midpoint of a chord $PQ$ of a circle $C$, $X$ a point between $P$ and $M$, as shown in figure 5. By taking $Y$ in $PQ$ as the symmetric of $X$ with respect to $M$
we have that $M$ is also the midpoint of $XY$; thus, starting from any point $A$ in one of the two determined arcs by the chord $PQ$ in $C$, let’s say the one above, we will obtain the vertices $A$, $D$, $B$ and $C$, of one of the required $ADBC$ quadrilaterals, as shown in figure 5, and we will justify that its diagonals $AB$ and $CD$ incise in $M$. For this purpose, draw the chord $AD$ that passes through $X$, then draw the $DC$ chord that passes through $M$, then draw the $CB$ chord that passes through $Y$, and join $B$ with $A$; then, by proposition 2.2 the diagonals $AB$ and $CD$ incise $M$, and $M$ is also the midpoint of $XY$. Repeating this procedure with each point $A$ of one of the arcs determined by the chord $PQ$ gives an infinity of the required quadrilaterals, and as each of these arcs has the same cardinality of the chord $PQ$ which has the cardinality of the real numbers, it is concluded that there is an uncountable infinity of said quadrilaterals.

(2) Considering the complete quadrilateral $ADBC$ of Figure 6, by the Theorem from section 4.14 of [4] we have that point $G$ is the harmonic conjugate of $F$ with respect to points $D$ and $B$; likewise, point $H$ is the conjugate harmonic of $F$ with respect to points $A$ and $C$; consequently, the line $EHMG$, or briefly, the $EM$ line is the polar of point $F$ with respect to circle $C$; likewise, the line $FM$ is the polar of point $E$ with about circle $C$; consequently, by the Fundamental Theorem of the section 8.2 of [4] the polar of $M$ with respect to circle $C$ passes through $E$ and $F$. Similarly, we prove that the polar of $M$ with respect to circle $C$ passes through the intersections $E'$ and $F'$ of the opposite sides of any other quadrilateral $A'D'B'C'$. Thus, the intersections $E$ and $F$ of the respective opposite sides of quadrilaterals $ADBC$ from part 1 of the proposition are collinear.

References


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