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THREE NEW PROPOSITIONS THROUGH HARMONIC CONJUGATES

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Abstract. In this note, three new propositions are established, the first characterizes the harmonic conjugate of a point with respect to two others, the second has as a corollary the Butterfly Theorem, and the third proposition allows to build an uncountable infinity of butterflies.

1. Introduction

We prove the first proposition in section 2 of this note by using similarity of triangles. We prove the second proposition of the aforementioned section by using the first proposition we mention earlier, as well as the definition and the uniqueness of the conjugate harmonic of one point with respect to two others, so it is recommended to review Chapter 4 of [4], or Chapter VII of [1]; we also use the definition of power of a point with respect to a circle, so it is recommended to review chapter 6 of [4], or section E of chapter VIII of [1]; likewise, the theory of poles and polars will be use, so it is recommended to also see sections 8.1, 8.2 and 8.4 of chapter 8 of [4] and section C of chapter VIII of [1]. Then, the famous Butterfly Theorem (see [2] or [3]) is proved as a corollary of the second proposition we mentioned above. Finally, we include a third proposition that guarantees the existence of an uncountable infinity of butterflies and something more.

2. Three propositions of harmony

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Proposition 2.1. Let AB be the diameter of a circle $\mathcal{C}$ of center O, P a point between A and B, PR perpendicular to AB in P with R in $\mathcal{C}$, and Q a point at the exterior of $\mathcal{C}$ collinear with A and B. Then, RQ is tangent to $\mathcal{C}$ in R if and only if Q is the harmonic conjugate of P with respect to A and B.

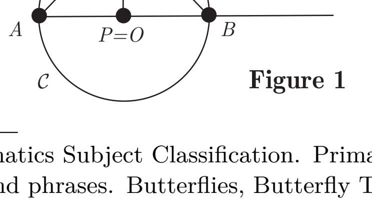

Figure 1

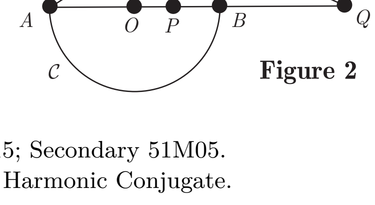

Figure 2

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Proof. If $P = O$, then the tangent t to $\mathcal{C}$ by R is perpendicular to $PR = OR$, so t is parallel to AB, as shown in Figure 1; also, since P is the midpoint of AB and considering Q as the point at infinity of the directed line AB, it is held that RQ is tangent to $\mathcal{C}$ in R if and only if Q is the harmonic conjugate of P with respect to A and B. Now, if $P \neq O$ as shown in figure 2, first suppose that RQ is tangent to $\mathcal{C}$ in R; then, when considering the chords AR and RB we stand that, by subtend the same arc $\angle BAR = \angle BRQ$; as in addition $\angle Q = \angle Q$; then, triangles ARQ and RBQ are similar, so

$$\left|\frac{AQ}{QR}\right| = \left|\frac{AR}{RB}\right| \Longrightarrow |AQ| = \frac{|AR| \cdot |QR|}{|RB|}, \text{ y}$$

$$\left|\frac{QB}{QR}\right| = \left|\frac{RB}{AR}\right| \Longrightarrow |QB| = \frac{|QR| \cdot |RB|}{|AR|,}$$

from where

$$\left|\frac{AQ}{QB}\right| = \left|\frac{AR}{RB}\right|^2, \tag{i}$$

furthermore, since $\angle APR = \angle ARB$ are right angles; as well as $\angle PAR = \angle PAR$; then, the triangles APR and ARB are similar, so

$$\left|\frac{AP}{AR}\right| = \left|\frac{PR}{RB}\right| \Longrightarrow |AP| = \frac{|AR| \cdot |PR|}{|RB|}, \tag{ii}$$

moreover, since $\angle RPB = \angle ARB$ are right angles, and $\angle RBP = \angle RBA$; then, the triangles RPB and ARB are similar, so

$$\left|\frac{PB}{RB}\right| = \left|\frac{PR}{AR}\right| \Longrightarrow |PB| = \frac{|PR| \cdot |RB|}{|AR|}, \tag{iii}$$

now, from (ii) and (iii) we have

$$\left|\frac{AP}{PB}\right| = \left|\frac{AR}{RB}\right|^2, \tag{iv}$$

and from (i) and (iv) it follows that $\left|\frac{AP}{PB}\right| = \left|\frac{AQ}{QB}\right|$, but $\frac{AP}{PB} > 0$ and $\frac{AQ}{QB} < 0$, so

$$\frac{AP}{PB} = -\frac{AQ}{QB};$$

that is to say, Q is the harmonic conjugate of P with respect to A and B.

Conversely, suppose that Q is the harmonic conjugate of P with respect to A and B; then, when considering the tangent t to $\mathcal{C}$ in R for the first part it follows that the intersection Q' of t with the line AB is the harmonic conjugate of P with respect to A and B; consequently, $Q' = Q$; thus, RQ is tangent to $\mathcal{C}$ in R. ■

Proposition 2.2. Let M be the midpoint of a chord PQ of a circle $\mathcal{C}$ of center O, and $ADBC$ a quadrilateral inscribed in $\mathcal{C}$ such that PQ intersects its opposite sides AD and BC in points X and Y, respectively, and one of their diagonals passes through M; then the diagonals AB and CD incise in M if and only if M is also the midpoint of XY.

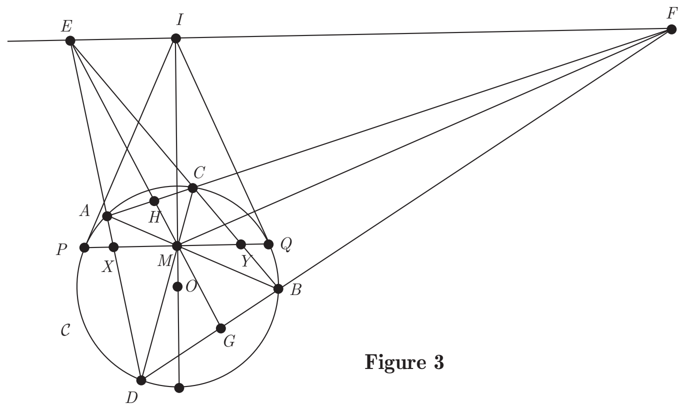

Figure 3

Proof. Suppose that PQ cuts opposite sides AD and CB in points X and Y, respectively, and the diagonals AB and CD of the quadrilateral $ADBC$ incise in M. It will be shown that M is the midpoint of XY. For this purpose, let be E the point of intersection of lines AD and BC; F the point of intersection of lines AC and BD; G the intersection of lines EM and BD; and H the intersection of lines EM and AC, as shown in figure 3; then, by the first theorem of section 4.14 of [4] it follows that G is the conjugate harmonic of F with respect to points D and B, and H is the harmonic conjugate of F with respect to points A and C; hence the $EHMG$ line, or briefly, line EM is the polar of F with respect to circle $\mathcal{C}$; likewise, the line FM is the polar of E with respect to circle $\mathcal{C}$; consequently, by the fundamental theorem of section 8.2 of [4] it follows that the polar of M with respect to circle $\mathcal{C}$ passes through points E and F. Also, considering the tangent PI to $\mathcal{C}$, where P is the point of tangency and I is a point on the line OM which is bisector of the chord PQ, by proposition 2.1 we have that I is the conjugate harmonic of M with respect to the intersections of OM with $\mathcal{C}$; therefore, PQ is the polar of I with respect to circle $\mathcal{C}$, and again, by the aforementioned fundamental theorem it follows that I is in the polar of M; consequently, I is in the line EF : Furthermore, since OM is the perpendicular bisector of PQ it follows that the triangles PMI and QMI are congruent, so $\measuredangle PIM = \measuredangle MIQ$; then by the Theorem from section 4.8 of [4] it follows that $I(PQMF)$ is a harmonic pencil, and since M is the midpoint of PQ, therefore

$$EIF = IF \parallel PQ; \qquad (i)$$

furthermore, since G is the harmonic conjugate of F with respect to the points D and B, then, by the theorem of section 4.6 of [4] it follows that $E(XYML)$ is a harmonic pencil, where L is the intersection of EIF with the line PQ; furthermore, by (i) it follows that L is the point at infinity of PQ; and by the corollary of the same section 4.6 it follows that $(XYML)$ is a harmonic row of points, where L is the point at infinity of the line $PQ = XY$, so M is the midpoint of XY.

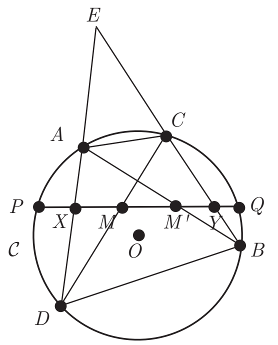

Figure 4

Conversely, suppose that M is the midpoint of XY and that one of its diagonals passes through M, as shown in figure 4. It will be proved that diagonals AB and CD are intersected in M. For this purpose, without loss of generality suppose that CD passes through M; then points D, M and C are collinear, so considering point E in the intersection of the lines AD and BC and applying Menelaus' Theorem to triangle XYE we obtain

$$\frac{YC}{CE} \cdot \frac{ED}{DX} \cdot \frac{XM}{MY} = -1, \tag{i}$$

but by hypothesis $XM = MY$; from this and (i) it follows that

$$\frac{CE}{YC} \cdot \frac{DX}{ED} = -1, \tag{ii}$$

Now if AB cuts XY in M', then using Menelaus' Theorem in triangle XYE it results

$$\frac{XA}{AE} \cdot \frac{EB}{BY} \cdot \frac{YM'}{M'X} = -1; \tag{iii}$$

Furthermore, by the definition of power of a point with respect to a circle, we have that $EA \cdot ED = EC \cdot EB$, from where

$$\frac{AE}{CE} \cdot \frac{ED}{EB} = 1; \tag{iv}$$

consequently, multiplying member by member (ii), (iii) and (iv) it is as follows

$$\frac{XA}{BY} \cdot \frac{DX}{YC} \cdot \frac{YM'}{M'X} = 1; \tag{v}$$

furthermore, by definition of power we have

$$\begin{aligned} XP \cdot XQ &= XA \cdot XD, \text{ y} \\ YP \cdot YQ &= YC \cdot YB, \end{aligned} \tag{vi}$$

but by hypothesis $PM = MQ$ and $XM = MY$; then, using directed segments, it results

$$XP = XM + MP = MY + QM = QY = -YQ, \quad \text{y}$$
$$XQ = XM + MQ = MY + PM = PY = -YP;$$

from this, and the relations given in (vi) it follows that

$$XA \cdot DX = YC \cdot BY, \tag{vii}$$

therefore, from (v) and (vii) we get $XM' = M'Y$, so $M' = M$. ∎

Butterfly Theorem. Let M be the midpoint of a chord PQ of a circle, AB and CD other chords through M such that C and D are on opposite sides of PQ. If AD and CB intersect PQ in X and Y, respectively, then M is also the midpoint of XY.

Proof. Follows immediately from proposition 2.2 ■

The following proposition guarantees the existence of an uncountable infinity of butterflies whose wings are cut by a chord PQ of a circle $\mathcal{C}$ in points X and Y symmetrical with respect to the midpoint M of the chord PQ.

Proposition 2.3. Let PQ be a chord of a circle $\mathcal{C}$ and M the midpoint of PQ, then:

(1) For every point X between P and M there is an uncountable infinity of quadrilaterals (butterflies) $ADBC$ inscribed in $\mathcal{C}$ such that PQ cuts their opposite sides AD and BC in X and a point Y, respectively, their diagonals AD and BC incise in M; and M is also the midpoint of XY.
(2) The intersections of the opposite sides of all quadrilaterals $ADBC$ from the previous part 1 are collinear.

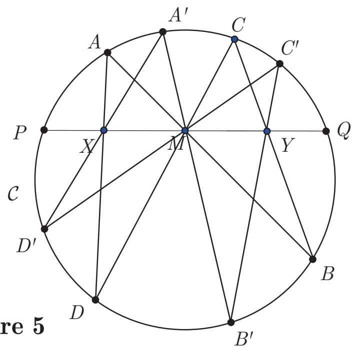

Figure 5

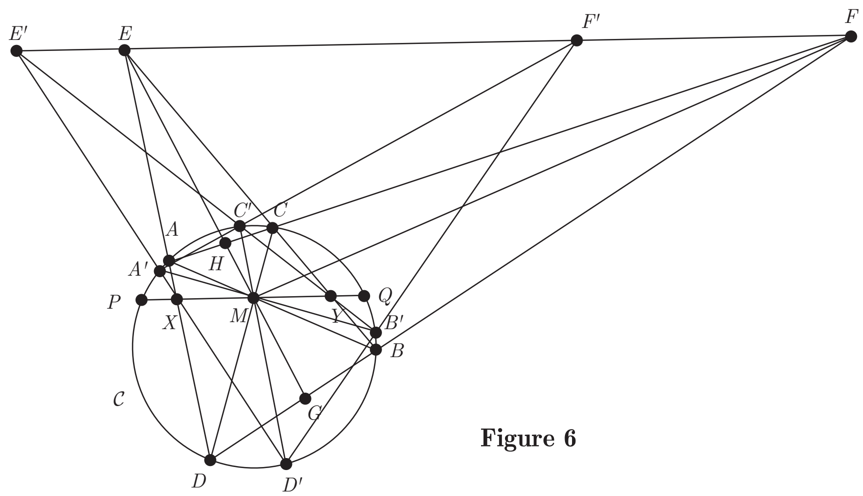

Figure 6

Proof.

(1) Let M be the midpoint of a chord PQ of a circle $\mathcal{C}$, X a point between P and M, as shown in figure 5. By taking Y in PQ as the symmetric of X with respect to M

we have that M is also the midpoint of XY; thus, starting from any point A in one of the two determined arcs by the chord PQ in $\mathcal{C}$, let's say the one above, we will obtain the vertices A, D, B and C, of one of the required $ADBC$ quadrilaterals, as shown in figure 5, and we will justify that its diagonals AB and CD incise in M. For this purpose, draw the chord AD that passes through X, then draw the DC chord that passes through M, then draw the CB chord that passes through Y, and join B with A; then, by proposition 2.2 the diagonals AB and CD incise M, and M is also the midpoint of XY. Repeating this procedure with each point A of one of the arcs determined by the chord PQ gives an infinity of the required quadrilaterals, and as each of these arcs has the same cardinality of the chord PQ which has the cardinality of the real numbers, it is concluded that there is an uncountable infinity of said quadrilaterals.

(2) Considering the complete quadrilateral $ADBC$ of Figure 6, by the Theorem from section 4.14 of [4] we have that point G is the harmonic conjugate of F with respect to points D and B; likewise, point H is the conjugate harmonic of F with respect to points A and C; consequently, the line $EHMG$, or briefly, the EM line is the polar of point F with respect to to circle $\mathcal{C}$; likewise, the line FM is the polar of point E with about circle $\mathcal{C}$; consequently, by the Fundamental Theorem of the section 8.2 of [4] the polar of M with respect to circle $\mathcal{C}$ passes through E and F. Similarly, we prove that the polar of M with respect to circle $\mathcal{C}$ passes through the intersections E' and F' of the opposite sides of any other quadrilateral $A'D'B'C'$. Thus, the intersections E and F of the respective opposite sides of quadrilaterals $ADBC$ from part 1 of the proposition are collinear. ■

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