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INTEGRAL FORMULAS FOR A RIEMANNIAN MANIFOLD WITH SEVERAL ORTHOGONAL COMPLEMENTARY DISTRIBUTIONS

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ABSTRACT. A Riemannian manifold endowed with $k > 2$ orthogonal complementary distributions (called a Riemannian almost multi-product manifold) appears in such topics as multiply warped products, the webs composed of several foliations, and proper Dupin hypersurfaces of real space-forms. In the paper we introduce the curvature invariant (called the mixed scalar curvature) of a Riemannian almost multi-product manifold, prove a novel integral formula with this curvature, generalizing well-known formula for $k = 2$, and give applications to splitting and isometric immersions of Riemannian manifolds, in particular, multiply warped products, and to hypersurfaces with $k > 2$ distinct principal curvatures of constant multiplicities.

INTRODUCTION

Distributions on a manifold (i.e., subbundles of the tangent bundle) are used to build up notions of integrability, and specifically of a foliated manifold. Distributions and foliations (that correspond to involutive distributions) on Riemannian manifolds appear in various situations, e.g., [3, 8, 12], among them twisted (and warped) products are fruitful generalizations of the direct product playing important role in mathematics and physics, e.g., [7]. Integral formulas for distributions and foliations are usually obtained by applying the Divergence Theorem to suitable vector fields. Integral formulae are useful for many problems in the differential geometry of foliations, e.g., [2, 12, 17]:

- characterizing of foliations, whose leaves have a given geometric property;
- prescribing the higher mean curvatures of the leaves of a foliation;
- minimizing functionals like volume defined for tensor fields on a foliated manifold.

The first known integral formula for a closed Riemannian manifold endowed with a codimension one foliation tells us that the integral mean curvature of the leaves vanishes, see [11]. The second formula in the series of total σ_k's – elementary symmetric functions of principal curvatures of the leaves – says that for a codimension one foliation with a unit normal N to the leaves the total σ_2 is a half of the total Ricci curvature in the N-direction, e.g., [2]:

$$\int_M (2\,\sigma_2 - \mathrm{Ric}_{N,N})\,\mathrm{d}\,\mathrm{vol} = 0, \tag{0.1}$$

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which is a consequence of Stokes' theorem applied to $\nabla_N N + \sigma_1 N$ (where ∇ is the Levi-Civita connection). One can see directly that (0.1) implies nonexistence of totally umbilical foliations on a closed manifold of negative curvature: if the integrand is strictly positive, so is the value of the integral. The *mixed scalar curvature* $\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp}$ is one of the simplest curvature invariants of a Riemannian almost-product manifold $(M, g; \mathcal{D}, \mathcal{D}^\perp)$, which can be defined as an averaged sum of sectional curvatures of planes that nontrivially intersect with each of complementary orthogonal distributions, that is

$$\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} = \sum_{1\le a\le \dim\mathcal{D},\, \dim\mathcal{D}<b\le\dim M} K(E_a, E_b).$$

Here $\{E_i\}$ is a local orthonormal frame on M such that $E_a \in \mathcal{D}$ for $1 \le a \le \dim\mathcal{D}$, and $K(E_a, E_b) = \langle R(E_a, E_b)\, E_b,\, E_a\rangle$ is the (mixed) sectional curvature of the plane $E_a \wedge E_b$. Recall that $R(X,Y)\,Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ is the Riemannian curvature tensor. The following integral formula for a closed Riemannian manifold endowed with $\mathcal{D}$ and $\mathcal{D}^\perp$, see [17] (and [10] for foliations), generalizes (0.1) and has many interesting global corollaries (e.g., decomposition criteria using the sign $\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp}$, see [13, 14]):

$$\int_M \big(\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} + \|h_1\|^2 + \|h_2\|^2 - \|H_1\|^2 - \|H_2\|^2 - \|T_1\|^2 - \|T_2\|^2\big)\,\mathrm{d\,vol}_g = 0. \tag{0.2}$$

Here h_i, $H_i = \mathrm{Tr}_g\, h_i$ and T_i are the second fundamental form, the mean curvature vector field and the integrability tensor of distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on (M, g). The formula (0.2) was obtained by calculation of the divergence of the vector field $H_1 + H_2$,

$$\operatorname{div}(H_1 + H_2) = \mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} + \|h_1\|^2 + \|h_2\|^2 - \|H_1\|^2 - \|H_2\|^2 - \|T_1\|^2 - \|T_2\|^2, \tag{0.3}$$

and then applying Stokes' theorem.

The construction of twisted (and warped) products was recently extended to multiply twisted (and multiply warped) products, e.g., [7]. The multiply twisted product, in turn, is a special case of a Riemannian almost multi-product manifold with $k \in \{2, \dots, n\}$ foliations. This structure can be also viewed in the theory of webs composed of several foliations, see [1], and on proper Dupin hypersurfaces of real space-forms, i.e., the number k of distinct principal curvatures is constant and each principal curvature function is constant along its corresponding surface of curvature, see [6]. Thus, the problem of generalizing (0.2) to the case of $k > 2$ distributions is actual. In Section 1, we solve this problem for arbitrary $k > 2$. In Sections 2 and 3 we give applications to splitting and isometric immersions of manifolds, in particular, multiply twisted (or warped) products, and to hypersurfaces with $k > 2$ distinct principal curvatures of constant multiplicities.

1. New integral formulas

Let an n-dimensional manifold M with a Riemannian metric $g = \langle\cdot,\cdot\rangle$ and the Levi-Civita connection ∇ be endowed with $k > 2$ pairwise orthogonal n_i-dimensional distributions $\mathcal{D}_i$ $(1 \le i \le k)$ with $\sum n_i = \dim M$. Such $(M, g; \mathcal{D}_1, \dots, \mathcal{D}_k)$ is called a *Riemannian almost multi-product manifold*, see [8] for $k = 2$. A plane in TM spanned by two vectors belonging to different distributions, say, $\mathcal{D}_i$ and $\mathcal{D}_j$ is called *mixed*.

There exists on $(M, g; \mathcal{D}_1, \dots, \mathcal{D}_k)$ a local adapted orthonormal frame $\{E_1, \dots, E_n\}$, where

$$\{E_1, \dots, E_{n_1}\} \subset \mathcal{D}_1, \quad \{E_{n_{i-1}+1}, \dots, E_{n_i}\} \subset \mathcal{D}_i, \quad 2 \le i \le k.$$

We will generalize the well-known concept of almost-product manifolds.

Definition 1.1. The function on a Riemannian almost multi-product manifold

$$\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \sum\nolimits_{i<j} \mathrm{S}_{\mathcal{D}_i,\mathcal{D}_j} \tag{1.1}$$

will be called the *mixed scalar curvature* of $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$, where

$$\mathrm{S}_{\mathcal{D}_i,\mathcal{D}_j} = \sum_{n_{i-1}<a\leq n_i,\ n_{j-1}<b\leq n_j} K(E_a, E_b), \quad i \neq j.$$

Note that $\mathrm{S}_{\mathcal{D}_i,\mathcal{D}_j}$ is the mixed scalar curvature of the pair $\{\mathcal{D}_i, \mathcal{D}_j\}$ of distributions. Let $P_i : TM \to \mathcal{D}_i$ be the orthoprojector, and $P_i^\perp$ – the orthoprojector onto $\mathcal{D}_i^\perp = \bigoplus_{j\neq i}\mathcal{D}_j$. The second fundamental form $h_i : \mathcal{D}_i \times \mathcal{D}_i \to \mathcal{D}_i^\perp$ (symmetric) and the integrability tensor $T_i : \mathcal{D}_i \times \mathcal{D}_i \to \mathcal{D}_i^\perp$ (skew-symmetric) of $\mathcal{D}_i$ are defined by

$$2\,h_i(X,Y) = P_i^\perp(\nabla_X Y + \nabla_Y X), \quad 2\,T_i(X,Y) = P_i^\perp(\nabla_X Y - \nabla_Y X) = P_i^\perp\,[X,Y].$$

Similarly, $h_i^\perp$, $H_i^\perp = \mathrm{Tr}_g\, h_i^\perp$, $T_i^\perp$ are the second fundamental forms, mean curvature vector fields and the integrability tensors of $\mathcal{D}_i^\perp$, and h_{ij}, $H_{ij} = \mathrm{Tr}_g\, h_{ij}$, T_{ij} be the second fundamental forms, mean curvature vector fields and the integrability tensors of distributions $\mathcal{D}_{i,j} = \mathcal{D}_i \oplus \mathcal{D}_j$ on M, and $P_{ij} : TM \to \mathcal{D}_{i,j}$ be orthoprojectors, etc. Note that $H_i = \sum_{j\neq i} P_j H_i$, etc. Recall that a distribution $\mathcal{D}_i$ is integrable if $T_i = 0$, and $\mathcal{D}_i$ is totally umbilical, harmonic, or totally geodesic, if $h_i = (H_i/n_i)\,g$, $H_i = 0$, or $h_i = 0$, respectively. For the scalar curvature $\mathrm{S} : M \to \mathbb{R}$ of (M,g) we have $\mathrm{S} = 2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} + \sum_{i=1}^k \mathrm{S}(\mathcal{D}_i)$, where $\mathrm{S}(\mathcal{D}_i)$ are scalar curvatures of suitable distributions (functions on M).

Lemma 1.1. *We have the following decomposition formula of the mixed scalar curvature:*

$$2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \sum\nolimits_i \mathrm{S}_{\mathcal{D}_i,\mathcal{D}_i^\perp}.$$

Proof. This directly follows from definition (1.1). □

Let $S(r,k)$ be the set of all r-combinations (i.e., subsets of r distinct elements) of $\{1,\ldots,k\}$. For example, $S(k-1,k)$ contains k elements. For any $q \in S(r,k)$ we may assume $q_1 < \ldots < q_r$. Set $h_q = h_{q_1,\ldots,q_r}$ and $T_q = T_{q_1,\ldots,q_r}$ for the distribution $\mathcal{D}_{q_1,\ldots,q_r} = \mathcal{D}_{q_1} \oplus \ldots \oplus \mathcal{D}_{q_r}$. Our main goal are the following formulas, which for $k = 2$ coincide with (0.3) and (0.2).

Theorem 1.1. *For a Riemannian almost multi-product manifold $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ we have*

$$\operatorname{div} X = 2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} + \sum\nolimits_{r\in\{1,k-1\}} \sum\nolimits_{q\in S(r,k)} \big(\langle h_q, h_q\rangle - \langle H_q, H_q\rangle - \langle T_q, T_q\rangle\big), \tag{1.2}$$

where $X = \sum_{r\in\{1,k-1\}} \sum_{q\in S(r,k)} H_q$. On a closed manifold M we have the integral formula

$$\int_M \big(2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} + \sum\nolimits_{r\in\{1,k-1\}} \sum\nolimits_{q\in S(r,k)} (\|h_q\|^2 - \|H_q\|^2 - \|T_q\|^2)\big)\,\mathrm{d}\,\mathrm{vol}_g = 0. \tag{1.3}$$

Proof. For $k = 2$ we have (0.3). To illustrate the proof for $k > 2$, first consider the case of $k = 3$. Using (0.3) for the distributions $\mathcal{D}_1$ and $\mathcal{D}_1^\perp = \mathcal{D}_2 \oplus \mathcal{D}_3$, we get

$$\operatorname{div}(H_1 + H_1^\perp) = 2\,\mathrm{S}_{\mathcal{D}_1,\mathcal{D}_1^\perp} + (\|h_1\|^2 - \|H_1\|^2 - \|T_1\|^2) + (\|h_1^\perp\|^2 - \|H_1^\perp\|^2 - \|T_1^\perp\|^2), \tag{1.4}$$

and similarly for $(\mathcal{D}_i, \mathcal{D}_i^\perp)$ $(i = 2,3)$. Summing 3 copies of (1.4), we get (1.2) for $k = 3$:

$$\begin{aligned}\operatorname{div}\sum\nolimits_i \big(H_i + H_i^\perp\big) &= 2\,\mathrm{S}_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3} \\ &+ \sum\nolimits_i (\|h_i\|^2 - \|H_i\|^2 - \|T_i\|^2 + \|h_i^\perp\|^2 - \|H_i^\perp\|^2 - \|T_i^\perp\|^2).\end{aligned} \tag{1.5}$$

Next, for arbitrary $k > 2$, we apply (0.3) for pairs of distributions $(\mathcal{D}_i, \mathcal{D}_i^\perp)$ and get k equalities. Summing these equations and using Lemma 1.1, we get (1.2). Using Stokes' theorem for (1.2) on a closed manifold M yields (1.3). □

Remark 1.2. Just the particular case of (1.2) for $k = 3$, see (1.5), can find many geometrical applications, because an almost 3-product structure appears naturally in several topics:
1) almost para-f-manifolds, e.g., [15].
2) orientable 3-manifolds, since they admits 3 linearly independent vector fields ($n_i = 1$).
3) the theory of 3-webs composed of 3 generic foliations of different dimensions, see [16].
4) minimal hypersurfaces in the sphere with 3 distinct principal curvatures, see [9].
5) tubes over standard embeddings of a projective plane FP^2, for $F = \mathbb{R}; \mathbb{C}; \mathbb{H}$ or $\mathbb{O}$ (Cayley numbers), in $S^4; S^7; S^{13}$, and S^{25}, respectively, see [6].

Remark 1.3. Formula (1.2) can be equivalently written as

$$\operatorname{div} \sum\nolimits_i \big(H_i + H_i^\perp\big) = 2\,\mathrm{S}_{\mathcal{D}_1,\dots,\mathcal{D}_k} + \sum\nolimits_i (\|h_i\|^2 - \|H_i\|^2 - \|T_i\|^2 + \|h_i^\perp\|^2 - \|H_i^\perp\|^2 - \|T_i^\perp\|^2).$$

Accordingly, the integral formula (1.3) can be written as

$$\int_M \big(2\,\mathrm{S}_{\mathcal{D}_1,\dots,\mathcal{D}_k} + \sum\nolimits_i (\|h_i\|^2 - \|H_i\|^2 - \|T_i\|^2 + \|h_i^\perp\|^2 - \|H_i^\perp\|^2 - \|T_i^\perp\|^2)\big)\,\mathrm{d}\,\mathrm{vol}_g = 0.$$

Example 1.1. We apply the method of Theorem 1.1 to a simpler case. Let (M^n, g) admits n pairwise orthogonal codimension-one foliations $\mathcal{F}_i$, and let N_i be unit vector fields orthogonal to $\mathcal{F}_i$. Writing down (0.1) for each N_i, summing for $i = 1, \dots, n$, and using the equality $\mathrm{S} = \sum_i \mathrm{Ric}_{N_i, N_i}$, yields the integral formula with the scalar curvature S of (M, g),

$$\int_M \big(2 \sum\nolimits_i \sigma_2(\mathcal{F}_i) - \mathrm{S}\big)\,\mathrm{d}\,\mathrm{vol}_g = 0. \tag{1.6}$$

Two immediately consequences of (1.6):
- if $\mathrm{S} < 0$ then each foliation $\mathcal{F}_i$ cannot be totally umbilical;
- if $\mathrm{S} > 0$ then each foliation $\mathcal{F}_i$ cannot be harmonic.

In order to simplify the LHS of (1.2) to the shorter view $\operatorname{div} \sum_i H_i$, we reorganize the terms $\operatorname{div} \sum_{q \in S(k-1,k)} H_q$ for $r > 1$ and obtain new integral formulae.

Theorem 1.4. *For $(M, g; \mathcal{D}_1, \dots, \mathcal{D}_k)$ and any $r \in \{2, \dots, k-1\}$ we have*

$$\operatorname{div} X = \sum\nolimits_{q \in S(r,k)} \big(\|H_q\|^2 + \langle \sum\nolimits_{i=1}^r H_{q_i} - r\,H_q\,, \sum\nolimits_{j \notin q} H_j\rangle\big), \tag{1.7}$$

where $X = \sum_{q \in S(r,k)} H_q - C_{k-2}^{r-1} \sum_i H_i$. The following integral formulae for $r \in \{2, \dots, k-1\}$ take place on a closed Riemannian almost multi-product manifold:

$$\int_M \sum\nolimits_{q \in S(r,k)} \big(\|H_q\|^2 + \langle \sum\nolimits_{i=1}^r H_{q_i} - r\,H_q\,, \sum\nolimits_{j \notin q} H_j\rangle\big)\,\mathrm{d}\,\mathrm{vol}_g = 0. \tag{1.8}$$

Proof. Using equality $H_{1\dots r} = P_{r+1\dots k}(H_1 + \ldots + H_r)$ for $q = \{1, \dots, r\}$, we find

$$\begin{aligned}\operatorname{div} H_{1\dots r} &= \operatorname{div}_{r+1\dots k} H_{1\dots r} - \|H_{1\dots r}\|^2 \\ &= \operatorname{div}_{r+1\dots k}(H_1 + \ldots + H_r) + \langle H_1 + \ldots + H_r,\ H_{r+1\dots k}\rangle - \|H_{1\dots r}\|^2,\end{aligned}$$

and similarly for all C_k^r cases of $q \in S(r, k)$. Summing the above, we use equalities

$$\operatorname{div}_{r+1\dots k} H_1 = \sum\nolimits_{j>r} \operatorname{div}_j H_1, \quad \operatorname{div}_{2\dots k} H_1 = \operatorname{div} H_1 + \|H_1\|^2, \quad \text{etc.}$$

Using induction, we write (1.7) for $k-1$ distributions $\mathcal{D}_1,\ldots,\mathcal{D}_{k-2}$ and $\mathcal{D}_{k-1}\oplus\mathcal{D}_k$, and then write similar formulas for other choices of a pair of distributions. Summing these C^2_k equations, we get equation of the form (1.7) with k, comparing coefficients yields the first claim. Using Stokes' theorem for (1.7) on a closed manifold M yields (1.8). $\square$

Example 1.2. For $k=3$ and $r=2$, the formula (1.8) reads as

$$\int_M \big(\sum\nolimits_i \|H_i\|^2 + \sum\nolimits_{i<j}\big(2\langle H_i, H_j\rangle - \|H_{ij}\|^2\big)\big)\,\mathrm{d}\,\mathrm{vol}_g = 0,$$

and for $k=4$ and $r=2$, the formula (1.8) reads as

$$\int_M \big(2\sum\nolimits_i \|H_i\|^2 + \sum\nolimits_{i,j<s}\langle H_i,\, H_{js}\rangle - \sum\nolimits_{i<j}\|H_{ij}\|^2\big)\,\mathrm{d}\,\mathrm{vol}_g = 0.$$

One can easily find the integral formula corresponding to (1.8) with $k=4$ and $r=3$.

From Theorems 1.1 and 1.4 for $r=k-1$, we obtain the following companion of (1.2).

Theorem 1.5. *For a Riemannian almost multi-product manifold $(M,g;\mathcal{D}_1,\ldots,\mathcal{D}_k)$ we have*

$$\operatorname{div}\sum\nolimits_i H_i = \mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} + \frac12\sum\nolimits_{r\in\{1,k-1\}}\sum\nolimits_{q\in S(r,k)}\big(\|h_q\|^2 - \|H_q\|^2 - \|T_q\|^2\big)$$
$$-\frac12\sum\nolimits_{q\in S(k-1,k)}\big(\|H_q\|^2 + \big\langle \sum\nolimits_{i=1}^{k-1} H_{q_i} - (k-1)\,H_q\,, \sum\nolimits_{j\notin q} H_j\,\big\rangle\big).$$

Corollary 1.1. *The following integral formula takes place on a closed $(M,g;\mathcal{D}_1,\ldots,\mathcal{D}_k)$:*

$$\int_M \big(2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} + \sum\nolimits_{r\in\{1,k-1\}}\sum\nolimits_{q\in S(k-1,k)}(\|h_q\|^2 - \|H_q\|^2 - \|T_q\|^2)$$
$$-\sum\nolimits_{q\in S(k-1,k)}(\|H_q\|^2 + \big\langle \sum\nolimits_{i=1}^{k-1} H_{q_i} - (k-1)\,H_q\,, \sum\nolimits_{j\notin q} H_j\,\big\rangle)\big)\,\mathrm{d}\,\mathrm{vol}_g = 0.$$

Remark 1.6. For particular case of $k=3$, the integral formula of Corollary 1.1 reads as

$$\int_M \big(\mathrm{S}_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3} - \sum\nolimits_i \|H_i\|^2 - \sum\nolimits_{i<j}\langle H_i,\, H_j\rangle + \frac12\sum\nolimits_i(\|h_i\|^2 - \|T_i\|^2)$$
$$+\frac12\sum\nolimits_{i<j}(\|h_{ij}\|^2 - \|T_{ij}\|^2)\big)\,\mathrm{d}\,\mathrm{vol}_g = 0.$$

The above formula was obtained in [4] by long direct calculations of $\operatorname{div}(H_1+H_2+H_3)$.

2. Splitting and isometric immersions of manifolds

Here, we use results of Section 1 to prove some splitting and non-existence of immersions results for Riemannian almost multi-product manifolds. We say that $(M,g;\mathcal{D}_1,\ldots,\mathcal{D}_k)$ *splits* if all distributions $\mathcal{D}_i$ are integrable and M is locally the direct product $M_1\times\ldots\times M_k$ with foliations tangent to $\mathcal{D}_i$. Recall that if a simply connected manifold splits then it is the direct product. In the next definition, we apply the submanifolds theory to Riemannian almost multi-product manifolds.

Definition 2.1. A pair $(\mathcal{D}_i,\mathcal{D}_j)$ with $i\ne j$ of distributions on $(M,g;\mathcal{D}_1,\ldots,\mathcal{D}_k)$ is called

a) *mixed totally geodesic*, if $h_{ij}(X,Y)=0$ for all $X\in\mathcal{D}_i$ and $Y\in\mathcal{D}_j$.

b) *mixed integrable*, if $T_{ij}(X,Y)=0$ for all $X\in\mathcal{D}_i$ and $Y\in\mathcal{D}_j$.

Lemma 2.1. *If each pair $(\mathcal{D}_i,\mathcal{D}_j)$ on $(M,g;\mathcal{D}_1,\ldots,\mathcal{D}_k)$ is*

a) mixed totally geodesic, then $h_q(X,Y)=0$ b) mixed integrable, then $T_q(X,Y)=0$ for all $q\in S(r,k)$, $2<r<k$ and $X\in\mathcal{D}_{q_1}$, $Y\in\mathcal{D}_{q_2}$.

Proof. This follows by mathematical induction. □

The next our results generalize [14, Theorem 6] and [17, Theorem 2] on case $k = 2$.

Theorem 2.1. *Let a Riemannian almost multi-product manifold $(M, g; \mathcal{D}_1, \dots, \mathcal{D}_k)$ has integrable harmonic distributions $\mathcal{D}_1, \dots, \mathcal{D}_k$. If $\mathrm{S}_{\mathcal{D}_1,\dots,\mathcal{D}_k} \ge 0$, and each pair $(\mathcal{D}_i, \mathcal{D}_j)$ is mixed integrable, then M splits.*

Proof. From the equality $H_{1\dots r} = P_{r+1\dots k}(H_1 + \dots + H_r)$ it follows that $H_i = 0$ for all $i \in \{1, \dots, k\}$, then $H_q = 0$ for all $q \in S(r,k)$ and $2 \le r < k$. Similarly (by Lemma 2.1), if $T_{ij} = 0$ for all $i \in \{1, \dots, k\}$, then $T_q = 0$ for all $q \in S(r,k)$ and $2 \le r < k$. By conditions and (1.2),

$$2\,\mathrm{S}_{\mathcal{D}_1,\dots,\mathcal{D}_k} + \sum\nolimits_{r\in\{1,k-1\}} \sum\nolimits_{q\in S(r,k)} (\|h_q\|^2 = 0.$$

Thus, $h_q = 0$ for all $q \in S(r,k)$ with $r \in \{1, k-1\}$, in particular, $h_i = 0$ $(1 \le i \le k)$. By well-known de Rham decomposition theorem, (M, g) splits. □

Modifying Stokes' theorem on a complete open manifold (M, g) yields the following.

Lemma 2.2 (see Proposition 1 in [5]). *Let (M^n, g) be a complete open Riemannian manifold endowed with a vector field ξ such that* $\operatorname{div}\xi \ge 0$. *If the norm $\|\xi\|_g \in \mathrm{L}^1(M, g)$ then* $\operatorname{div}\xi \equiv 0$.

Theorem 2.2. *Let a complete open Riemannian almost multi-product manifold $(M, g; \mathcal{D}_1, \dots, \mathcal{D}_k)$ has totally umbilical distributions such that each pair $(\mathcal{D}_i, \mathcal{D}_j)$ is mixed totally geodesic and $\langle H_i, H_j\rangle = 0$ for $i \ne j$. If $\mathrm{S}_{\mathcal{D}_1,\dots,\mathcal{D}_k} \le 0$ and $\|H_i\| \in \mathrm{L}^1(M, g)$ for $1 \le i \le k$, then M splits.*

Proof. By assumptions, from (1.2) we get

$$\operatorname{div}\xi = 2\,\mathrm{S}_{\mathcal{D}_1,\dots,\mathcal{D}_k} + \sum\nolimits_{r\in\{1,k-1\}} \sum\nolimits_{q\in S(r,k)} \big(\|h_q\|^2 - \|H_q\|^2 - \|T_q\|^2\big), \tag{2.1}$$

where $\xi = \sum_{r\in\{1,k-1\}} \sum_{q\in S(r,k)} H_q$. By conditions, and since $\|H_q\| \le \sum_{i=1}^k \|H_{q_i}\|$, we get $\|H_q\| \in \mathrm{L}^1(M, g)$. Since $\|\xi\| \le \sum_{r\in\{1,k-1\}} \sum_{q\in S(r,k)} \|H_q\|$, then also $\|\xi\| \in \mathrm{L}^1(M, g)$. Next, by conditions, for any $q = (q_1, \dots, q_r) \in S(r,k)$ with $r \ge 1$ we have

$$\|h_q\|^2 - \|H_q\|^2 = -\sum\nolimits_{i=1}^r \frac{n_{q_i} - 1}{n_{q_i}} \|\widehat{P}_q H_{q_i}\|^2 \le 0,$$

where $\widehat{P}_q$ is the orthoprojector on the distribution $\bigoplus_{j\notin q} \mathcal{D}_j$. Hence, from $\mathrm{S}_{\mathcal{D}_1,\dots,\mathcal{D}_k} \le 0$ and (2.1) we get $\operatorname{div}\xi \le 0$. By conditions and Lemma 2.2, $\operatorname{div}\xi = 0$. Thus, see (2.1), $\mathrm{S}_{\mathcal{D}_1,\dots,\mathcal{D}_k} = 0$ and T_i and h_i vanish. By de Rham decomposition theorem, (M, g) splits. □

Example 2.1. Totally umbilical integrable distributions appear on multiply warped products. Let $F_1, \dots, F_k$ be k Riemannian manifolds and let $M = F_1 \times \dots \times F_k$ be their direct product. A *multiply twisted product* $F_1 \times_{u_2} F_2 \times \dots \times_{u_k} F_k$ is M with the metric $g = g_{F_1} \oplus u_2^2 g_{F_2} \oplus \dots \oplus u_k^2 g_{F_k}$, where $u_i : F_1 \times F_i \to (0, \infty)$ for $i = 2, \dots, k$ are smooth functions. Twisted products ($k = 1$) and *multiply warped product*, i.e., $u_i : F_1 \to (0, \infty)$, are special cases of multiply twisted products. Let $\mathcal{D}_i$ be the distribution on M obtained from the vectors tangent to (the horizontal lifts of) F_i, e.g., [7]. The leaves (i.e., tangent to $\mathcal{D}_i$, $i \ge 2$) are totally umbilical submanifolds, with

$$H_i = -n_i P_1 \nabla(\log u_i),$$

and the fibers (i.e., tangent to $\mathcal{D}_1$) are totally geodesic submanifolds. Since

$$\operatorname{div}\, H_i = -n_i\,(\Delta_1\, u_i)/u_i - (n_i^2 - n_i)\,\|P_1 \nabla u_i\|^2/u_i^2,$$

where Δ_1 is the Laplacian on $C^2(F_1)$, and we have

$$\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \sum\nolimits_{i\ge 2} n_i\,(\Delta_1\, u_i)/u_i\,. \tag{2.2}$$

Corollary 2.1. *Let a multiply twisted product $(M,g) = F_1\times_{u_2} F_2\times\ldots\times_{u_k} F_k$ be complete open and $\langle H_i, H_j\rangle = 0$ for $i\ne j$. If $\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k}\le 0$ and $\|H_i\|\in \mathrm{L}^1(M,g)$ for $1\le i\le k$, then (M,g) is the direct product.*

The following theorem on isometric immersions of multiply warped products, see [7, Theorem 10.2], is related to the question by B.-Y. Chen: *"What can we conclude from an arbitrary isometric immersion of a warped product into a Riemannian manifold with arbitrary codimension?"*

Theorem A *Let $f : F_1\times_{u_2} F_2\times\ldots\times_{u_k} F_k \to \tilde M$ be an isometric immersion of a multiply warped product $(M,g) := F_1\times_{u_2} F_2\times\ldots\times_{u_k} F_k$ into an arbitrary Riemannian manifold. Then*

$$\sum\nolimits_{i=2}^k n_i\,(\Delta u_i)/u_i \le \frac{n^2(k-1)}{2\,k}\,H^2 + n_1(n-n_1)\max\tilde K,\quad n=\sum\nolimits_{j=1}^k n_j, \tag{2.3}$$

where $H^2 = \langle H, H\rangle$ is the squared mean curvature of f, $\max\tilde K(x)$ is the maximum of the sectional curvature of $\tilde M$ restricted to 2-plane sections of the tangent space T_xM of M at $x = (x_1,\ldots,x_k)$. The equality sign of (2.3) *holds identically if and only if the following statements hold*:

1) *f is a mixed totally geodesic immersion satisfying equalities $\mathrm{Tr}_g\, h_1 = \ldots = \mathrm{Tr}_g\, h_k$;*

2) *at each point $x\in M$, the sectional curvature function $\tilde K$ of $\tilde M$ satisfies $\tilde K(X,Y) = \max\tilde K(x)$ for each unit vector $X\in T_{x_1}(F_1)$ and each unit vector Y in $T_{x_2,\ldots,x_k}(F_2\times\ldots\times F_k)$.*

On a multiply warped product with $k>2$ each pair of distributions is mixed totally geodesic. Indeed, such manifold is diffeomorphic to the direct product and the Lie bracket does not depend on metric.

Using Theorem A and our results above, we obtain the following.

Theorem 2.3 (Non-existence of immersions)**.** *Let $f : F_1\times_{u_2} F_2\times\ldots\times_{u_k} F_k \to \tilde M$ be an isometric immersion of a closed multiply warped product $(M,g) := F_1\times_{u_2} F_2\times\ldots\times_{u_k} F_k$ into an arbitrary Riemannian manifold. Let, in addition, $\langle H_i, H_j\rangle = 0$ for $i\ne j$. Then there are no isometric immersions $f : F_1\times_{u_2} F_2\times\ldots\times_{u_k} F_k\to\tilde M$ satisfying the inequality*

$$\frac{n^2(k-1)}{2\,k}H^2 < -n_1(n-n_1)\max\tilde K; \tag{2.4}$$

in particular, there are no minimal isometric immersions f when $\tilde K<0$.

Proof. From (2.2), (2.1) and (2.3), applying the Stokes' Theorem, we get

$$\begin{aligned} 0 &\le (1/2)\int_M \sum\nolimits_{r\in\{1,k-1\}}\sum\nolimits_{q\in S(r,k)}\sum\nolimits_{i=1}^r \frac{n_{q_i}-1}{n_{q_i}}\,\|\widehat{P}_q H_{q_i}\|^2\,\mathrm{d}\,\mathrm{vol}_g\\ &\le \frac{n^2(k-1)}{2\,k}\int_M H^2\,\mathrm{d}\,\mathrm{vol}_g + n_1(n-n_1)\max\tilde K\cdot \mathrm{Vol}(M,g). \end{aligned} \tag{2.5}$$

The equality in (2.5) holds if and only if conclusions 1)–2) of Theorem A are satisfied. If such isometric immersion exists, then the inequality (2.4) yields a contradiction $\square$

Remark 2.4. Theorem 2.3 can be improved replacing $\max \tilde K$ by $\bar S(n_1,\ldots,n_k)$ (with certain factor) defined below. Let L_i $(i=1,2)$ be two subspaces of $T_x\bar M$ at some point $x\in\bar M$, and $\{E'_i\}$ and $\{\mathcal{E}'_j\}$ some orthonormal frames of these subspaces. The following quantity:

$$\bar K(L_1,L_2)=\sum\nolimits_{i,j}\langle\bar R(E'_i,\mathcal{E}'_j)\,E'_i,\,\mathcal{E}'_j\rangle$$

does not depend on the choice of frames. For any $n_1,\ldots,n_k\in\mathbb{N}$ with $\sum_i n_i\le\dim\bar M$, let $\Gamma_x(n_1,\ldots,n_k)$ be the set of k-tuples of pairwise orthogonal subspaces $(L_1,\ldots,L_k)$ with $\dim L_i=n_i$ $(1\le i\le k)$ of $T_x\bar M$ at some point $x\in\bar M$. Set $\bar S(n_1,\ldots,n_k)=\sup_{x\in\bar M}\bar S_x(n_1,\ldots,n_k)$, where

$$\bar S_x(n_1,\ldots,n_k)=\max\{\sum\nolimits_{i<j}\bar K(L_i,L_j):\ (L_1,\ldots,L_k)\in\Gamma_x(n_1,\ldots,n_k)\}.$$

3. Hypersurfaces with k distinct principal curvatures

Let M^n be a transversely orientable hypersurface in a Riemannian manifold $\bar M$. Denote by A the shape operator of M with respect to a unit normal vector field N, and let $\lambda_1\le\ldots\le\lambda_n$ be the principal curvatures (eigenvalues of A) – continuous functions on M. If at any point of M all principal curvatures are equal, then M is totally umbilical.
It is known that *a compact Dupin hypersurface M in a space form $\bar M(c)$ $(c\ge0)$ has 1, 2, 3, 4 or 6 distinct principal curvatures*, e.g., [6], that is we have $(M,g,\mathcal{D}_1,\ldots,\mathcal{D}_k)$ with $k=2,3,4,6$.
Suppose that there exists $k\ge2$ distinct principal curvatures (μ_i) on M of multiplicities n_i, and let $\mathcal{D}_i$ $(1\le i\le k)$ be corresponding eigen-distributions – subbundles of TM.

$$\mu_1:=\lambda_1=\ldots=\lambda_{n_1}<\ \ldots\ <\mu_k:=\lambda_{n-n_k+1}=\ldots=\lambda_n.$$

Such hypersurface is a Riemannian almost multi-product manifold, see Section 1.
For a hypersurface M in a Riemannian manifold $\bar M(c)$ of constant curvature c, it is known the following, see [6]: if $n_i>1$ for some i, then the function $\mu_i:M\to\mathbb{R}$ is differentiable and $\mathcal{D}_i$ is integrable and its leaves are totally umbilical in $\bar M(c)$.
The curvature tensor $\bar R$ of $\bar M(c)$ has the well-known view

$$\bar R(X,Y)Z=c\,(\langle X,Z\rangle\,Y-\langle Y,Z\rangle\,X).$$

Let $X_i\in\mathfrak{X}_i$ be local unit vector fields on $\mathcal{D}_i$ $(i\le k)$. For the sectional curvature of M we get

$$K(X_i,X_j)=c+\mu_i\,\mu_j,\quad i\ne j. \tag{3.1}$$

Example 3.1. For a hypersurface M in $\bar M(c)$ with $k=2$, using equalities

$$\|h_i\|^2-\|H_i\|^2=n_i(1-n_i)\,\frac{\|\nabla\mu_i\|^2}{(\mu_i-\mu_j)^2},\quad \mathrm{S}_{\mathcal{D}_1,\mathcal{D}_2}=n_1n_2(c+\mu_1\mu_2),$$

formula (0.3) can be rewritten in terms of A and μ_i $(i=1,2)$ as follows, see [17]:

$$\operatorname{div}(H_1+H_2)=n_1n_2(c+\mu_1\mu_2)+\frac{n_1(1-n_1)\,\|\nabla\mu_1\|^2+n_2(1-n_2)\,\|\nabla\mu_2\|^2}{(\mu_2-\mu_1)^2}, \tag{3.2}$$

The next **problem** was posed by P. Walczak [17]: “To search for formulae analogous to (3.2) in the case of a hypersurface in $\bar M(c)$ of $k>2$ distinct principal curvatures of constant multiplicities”.
By symmetry of A and the Codazzi equation for a hypersurface in $\bar M(c)$,

$$(\nabla_X A)(Y)=(\nabla_Y A)(X), \tag{3.3}$$

the following tensor $\mathcal{A}$ of rank 3 is totally symmetric:

$$\mathcal{A}(X,Y,Z) = \langle (\nabla_X A)(Y),\, Z\rangle, \quad X,Y,Z \in \mathfrak{X}_M.$$

Remark 3.1. A submanifold M in a Riemannian manifold $\bar M$ is called *curvature-invariant* if $R(X,Y)Z \in T_xM$ for any $X,Y,Z \in T_xM$ and any $x \in M$, e.g., [12]. Examples are arbitrary submanifolds in $\bar M(c)$. For a curvature-invariant hypersurface M in $\bar M$, the Codazzi equation (3.3) is satisfied, thus, results of this section on a hypersurface in $\bar M(c)$ can be extended for curvature-invariant hypersurfaces.

Lemma 3.1. *For a hypersurface M in $\bar M(c)$ with $k > 2$ distinct principal curvatures, we have*

$$\begin{aligned} \mathcal{A}(X_i,X_j,X_l) &= (\mu_j-\mu_l)\langle \nabla_{X_i}X_j,\, X_l\rangle, \quad j\ne l,\\ \mathcal{A}(X_i,X_j,X_j) &= X_i(\mu_j), \quad \forall\, i,j. \end{aligned} \tag{3.4}$$

Proof. After differentiating the equality $AX_j = \mu_j X_j$ and using the symmetry of A, we obtain

$$\begin{aligned} \langle \nabla_{X_i}(AX_j),\, X_l\rangle &= \langle (\nabla_{X_i}A)(X_j),\, X_l\rangle + \langle A\nabla_{X_i}X_j,\, X_l\rangle\\ &= \mathcal{A}(X_i,X_j,X_l) + \mu_l\langle \nabla_{X_i}X_j,\, X_l\rangle,\\ \langle \nabla_{X_i}(\mu_j X_j),\, X_l\rangle &= \mu_j\,\langle \nabla_{X_i}X_j,\, X_l\rangle + X_i(\mu_j)\,\delta_{jl}. \end{aligned}$$

This implies the equalities in (3.4). □

Corollary 3.1. *For a hypersurface M in $\bar M(c)$ with $k > 2$ distinct principal curvatures, we have*

$$(\mu_j-\mu_l)\,\langle \nabla_{X_i}X_j,\, X_l\rangle = (\mu_i-\mu_l)\,\langle \nabla_{X_j}X_i,\, X_l\rangle, \quad i\ne j\ne l. \tag{3.5}$$

Proof. This follows from (3.4) and symmetry of $\mathcal{A}$. Alternatively, one can use direct derivation,

$$\begin{aligned} 0 = \langle \bar R(X_i,X_j)N,\, X_l\rangle &= \langle \mu_{X_i}(\mu_j X_j) - \nabla_{X_j}(\mu_i X_i) - \langle [X_i,X_j],\, X_l\rangle\langle \mu_l X_l,\, X_l\rangle\\ &= (\mu_j-\mu_l)\,\langle \nabla_{X_i}X_j,\, X_l\rangle - (\mu_i-\mu_l)\,\langle \nabla_{X_j}X_i,\, X_l\rangle, \end{aligned}$$

which implies (3.5). □

Theorem 5.13 in [6] states that "*each point of a hypersurface M^n in $\bar M(c)$ with 2, 3, 4 or 6 distinct principal curvatures has a principal coordinate neighborhood if and only if each $\mathcal{D}_i^\perp$ is integrable*". We complete this result by the following.

Proposition 3.1. *Let M^n be a hypersurface in $\bar M(c)$ with $k > 2$ distinct principal curvatures at each point. Then each $\mathcal{D}_i^\perp$ is integrable if and only if*

$$\mathcal{A}(X_i,X_j,X_l) = 0$$

for $X_i \in \mathcal{D}_i$, $X_j \in \mathcal{D}_j$ and $X_l \in \mathcal{D}_l$ with $i<j<l$ on M^n.

Next, we study the problem (posed in [17]) for $k=3$ (the case of $k>3$ is similar).

Theorem 3.2. *For a hypersurface M in $\bar M(c)$ with $k=3$ distinct principal curvatures, we have*

$$\begin{aligned} &\operatorname{div}\sum\nolimits_i n_i\Big(\frac{P_j\nabla\mu_i}{\mu_i-\mu_j} + \frac{P_l\nabla\mu_i}{\mu_i-\mu_l}\Big)\\ &= \frac12\sum\nolimits_{j<l} n_j\, n_l(c+\mu_j\mu_l) + \sum\nolimits_i n_i(1-n_i)\Big(\frac{\|P_j\nabla\mu_i\|^2}{(\mu_i-\mu_j)^2} + \frac{\|P_l\nabla\mu_i\|^2}{(\mu_i-\mu_l)^2}\Big), \end{aligned} \tag{3.6}$$

where $(i,j,l) \in \{(1,2,3),(2,1,3),(3,1,2)\}$.

Proof. By (1.5) and $T_i = 0$, we have

$$\begin{aligned} \operatorname{div}\big(\sum_i H_i + \sum_{i<j} H_{ij}\big) \ &= \ 2\,\mathrm{S}_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3} + \sum_i(\|h_i\|^2 - \|H_i\|^2) \\ &+ \ \sum_{i<j}(\|h_{ij}\|^2 - \|H_{ij}\|^2 - \|T_{ij}\|^2). \end{aligned} \tag{3.7}$$

By (3.1), the mixed scalar curvature of M with $k = 3$ is

$$\mathrm{S}_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3} = \sum_{j<l} n_j\, n_l(c + \mu_j\mu_l).$$

Note that $\|h_i\|^2 = \|H_i\|^2/n_i$, where

$$h_i(X,Y) = \langle X, Y\rangle\Big(\frac{P_j\nabla\mu_i}{\mu_i - \mu_j} + \frac{P_l\nabla\mu_i}{\mu_i - \mu_l}\Big), \quad H_i = n_i\Big(\frac{P_j\nabla\mu_i}{\mu_i - \mu_j} + \frac{P_l\nabla\mu_i}{\mu_i - \mu_l}\Big).$$

Thus (similarly to Example 3.1), we obtain

$$\|h_i\|^2 - \|H_i\|^2 = n_i(1-n_i)\Big(\frac{\|P_j\nabla\mu_i\|^2}{(\mu_i - \mu_j)^2} + \frac{\|P_l\nabla\mu_i\|^2}{(\mu_i - \mu_l)^2}\Big).$$

Next, we consider $\|h_{ij}\|^2$ and $\|T_{ij}\|^2$ for $i \ne j$ along three subsets of $TM \oplus TM$: $\mathcal{D}_i \oplus \mathcal{D}_i$, $\mathcal{D}_j \oplus \mathcal{D}_j$ and $V_{ij} = (\mathcal{D}_i \oplus \mathcal{D}_j) \cup (\mathcal{D}_j \oplus \mathcal{D}_i)$. Using equalities $h_{ij} = \frac{n_i\, P_l\nabla\mu_i}{\mu_i - \mu_l} + \frac{n_j\, P_l\nabla\mu_j}{\mu_j - \mu_l}$ for $i \ne j \ne l$, we obtain

$$\|h_{ij}\|^2 - \|H_{ij}\|^2 = n_i(1-n_i)\frac{\|P_l\nabla\mu_i\|^2}{(\mu_i - \mu_l)^2} + n_j(1-n_j)\frac{\|P_l\nabla\mu_j\|^2}{(\mu_j - \mu_l)^2} + \|h_{ij}|_{V_{ij}}\|^2.$$

Since $\mathcal{D}_i$ are integrable, we have $\|T_{ij}\|^2 = \|T_{ij}|_{V_{ij}}\|^2$. Given $i \ne j \ne l$, let (e_α), (ε_β), (e_γ) be local orthonormal frames of $\mathcal{D}_i, \mathcal{D}_j$ and $\mathcal{D}_l$, respectively. Using (3.4), and

$$\|h_{ij}|_{V_{ij}}\|^2 = \frac{1}{4}\sum_{\alpha,\beta,\gamma}\langle\nabla_{e_\alpha}\varepsilon_\beta + \nabla_{\varepsilon_\beta} e_\alpha,\, e_\gamma\rangle^2, \quad \|T_{ij}|_{V_{ij}}\|^2 = \frac{1}{4}\sum_{\alpha,\beta,\gamma}\langle\nabla_{e_\alpha}\varepsilon_\beta - \nabla_{\varepsilon_\beta} e_\alpha,\, e_\gamma\rangle^2,$$

we find for $(i,j) \in \{(1,2),\,(1,3),\,(2,3)\}$ the equality

$$\|h_{ij}|_{V_{ij}}\|^2 - \|T_{ij}|_{V_{ij}}\|^2 = \sum_{\alpha,\beta,\gamma}\langle\nabla_{e_\alpha}\varepsilon_\beta,\, e_\gamma\rangle\,\langle\nabla_{\varepsilon_\beta} e_\alpha,\, e_\gamma\rangle = \frac{\sum_{\alpha,\beta,\gamma}\|\mathcal{A}(e_\alpha,\,\varepsilon_\beta,\,e_\gamma)\|^2}{(\mu_i - \mu_l)(\mu_j - \mu_l)}\,.$$

Since

$$\frac{1}{(\mu_2 - \mu_3)(\mu_1 - \mu_3)} + \frac{1}{(\mu_1 - \mu_2)(\mu_3 - \mu_2)} + \frac{1}{(\mu_2 - \mu_1)(\mu_3 - \mu_1)} = 0,$$

the sum $\sum_{i<j}\big(\|h_{ij}|_{V_{ij}}\|^2 - \|T_{ij}|_{V_{ij}}\|^2\big)$ vanishes. From the above, we have

$$\begin{aligned} &\sum_i(\|h_i\|^2 - \|H_i\|^2) + \sum_{i<j}(\|h_{ij}\|^2 - \|H_{ij}\|^2 - \|T_{ij}\|^2) \\ &= 2n_1(1-n_1)\Big(\frac{\|P_2\nabla\mu_1\|^2}{(\mu_1 - \mu_2)^2} + \frac{\|P_3\nabla\mu_1\|^2}{(\mu_1 - \mu_3)^2}\Big) \\ &+ 2\,n_2(1-n_2)\Big(\frac{\|P_1\nabla\mu_2\|^2}{(\mu_2 - \mu_1)^2} + \frac{\|P_3\nabla\mu_2\|^2}{(\mu_2 - \mu_3)^2}\Big) \\ &+ 2\,n_3(1-n_3)\Big(\frac{\|P_1\nabla\mu_3\|^2}{(\mu_3 - \mu_1)^2} + \frac{\|P_2\nabla\mu_3\|^2}{(\mu_3 - \mu_2)^2}\Big). \end{aligned}$$

Note that the terms under the divergence in (3.7) are

$$\sum\nolimits_i H_i + \sum\nolimits_{i<j} H_{ij} = \sum\nolimits_i n_i\Big(\frac{P_j\nabla\mu_i}{\mu_i-\mu_j} + \frac{P_l\nabla\mu_i}{\mu_i-\mu_l}\Big) + \sum\nolimits_{i<j}\Big(n_i\,\frac{P_l\nabla\mu_i}{\mu_i-\mu_l} + n_j\,\frac{P_l\nabla\mu_j}{\mu_j-\mu_l}\Big)$$
$$= 2\,n_1\Big(\frac{P_2\nabla\mu_1}{\mu_1-\mu_2} + \frac{P_3\nabla\mu_1}{\mu_1-\mu_3}\Big) + 2\,n_2\Big(\frac{P_1\nabla\mu_2}{\mu_2-\mu_1} + \frac{P_3\nabla\mu_2}{\mu_2-\mu_3}\Big) + 2\,n_3\Big(\frac{P_1\nabla\mu_3}{\mu_3-\mu_1} + \frac{P_2\nabla\mu_3}{\mu_3-\mu_2}\Big).$$

Thus, the above and (3.7) imply (3.6). □

Using Stokes' theorem to (3.6) on a closed M, we obtain the following integral formula.

Corollary 3.2. *In conditions of Theorem 3.2, for a closed hypersurface $M\subset\bar M(c)$, we have*

$$\int_M \Big[\sum\nolimits_{j<l} n_j\,n_l(c+\mu_j\mu_l) + 2\sum\nolimits_i n_i(1-n_i)\Big(\frac{\|P_j\nabla\mu_i\|^2}{(\mu_i-\mu_j)^2} + \frac{\|P_l\nabla\mu_i\|^2}{(\mu_i-\mu_l)^2}\Big)\Big]\,\mathrm{d\,vol}_g = 0,$$

where $(i,j,l)\in\{(1,2,3),(2,1,3),(3,1,2)\}$.

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