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GEOMETRIC INVARIANTS OF THE TIME-SLICE OF THE MEAN CURVATURE FLOW

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Abstract. Let Σ be a submanifold of a Riemannian manifold M. The evolution equations of some extrinsic geometric quantities such as the norm of the second fundamental form, the norm of the mean curvature, etc., have been computed along the mean curvature flow and then used to investigate either the long-time existence of the flow or some extrinsic geometrical properties (minimality, totally geodesic, umbilicity, etc.) of the limiting submanifold (see e.g. [1, 2, 4, 5]). But few results are known about the intrinsic geometry of the evolving submanifold along the flow [3]. In the present work, assuming that the mean curvature flow exists, we compute for a submanifold of a Riemannian manifold, the evolution equations of the Riemannian curvature, the Ricci curvature and the scalar curvature which are intrinsic geometrical invariants of the submanifold. From the evolution equation of the scalar curvature, we derived that the graphical mean curvature flow favours positive scalar curvature ; a well known result in this area.

1. Introduction and motivations

The celebrated theory of J. F. Nash on isometric immersion of a Riemannian manifold into a suitable Euclidean space gives a very important and effective motivation to view each Riemannian manifold as a submanifold in an Euclidean space. Since ever, the problem of discovering simple and sharp relationships between intrinsic and extrinsic invariants of a Riemannian submanifold becomes one of the most fundamental problems in submanifolds theory. Extrinsic and intrinsic geometries are the often used two approaches to study the Riemannian geometry of a submanifold. In terms of geometrics quantities, these two approaches are equivalent since Riemannian geometry deals with the geometrical elements such as metric, curvatures, connections, etc. For the extrinsic approach, the study is carried out in reference to the geometry of the ambient manifold but any reference to another manifold is needed for the intrinsic approach [7]. The main intrinsic invariants are the scalar curvature and the Ricci curvature while the main extrinsic one are the second fundamental form and its relatives such as the mean curvature, the shape operator etc. Talking about geometric flow, where the purpose is to describe the evolution of the geometry of manifold under some flow, the widely known flows are the mean curvature flow and the Ricci flow for their interest and applications in mathematic and physic. Roughly

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speaking, the motion by the mean curvature is a process under which a submanifold deforms in the direction of its mean curvature vector. Essentially, the purpose in mean curvature flow is to study the behavior of a “moving” submanifold inside some ambient manifold ; thus the flow mainly describes its extrinsic geometry whereas the Ricci flow is based on the intrinsic geometry. Therefore, these properties turn out to be important to study when one submits a submanifold to the mean curvature flow in a Ricci flow background. In [3], the author computed the evolution equations of the intrinsic curvatures along the mean curvature flow of an embedded submanifold in a higher-dimensional flat space ; he also proved that the evolution equation of the scalar curvature of a Riemannian submanifold is similar to the Ricci flow. Later on he deduced that negativity rather than positivity of the scalar curvature is preserved, what does no longer hold according to our understanding. So, the present work compute the evolution equations of the curvatures for a submanifold in a (non necessarily) flat Riemannian manifold. These computations generalize those in [3]. Contrarily to the result in [3], this work proved that the positivity of the scalar curvature is preserved along the flow : a well known result in the area.
The mean curvature flow equation is a quasi-parabolic equation and it is well known that there is no theory which trivially guarantees the long time existence of the solution. However, from Huisken seminal work [2], one can prove using the monotonicity formula and the blow-up analysis, the long time existence for suitable initial data. The argument is based on the geometric structure of the ambient manifold ; thus, proving the long time existence heavily relies on the curvature. Therefore, computing such evolution equations is very illuminating.
In the sequel, we assemble in Section 2 some useful materials for the study of the geometry of Riemannian submanifold along the mean curvature flow by recalling the evolution equation of the induced first fundamental form, of a frame on the normal bundle and the time derivative of the mean curvature vector. Thereafter, we compute in Section 3 the evolution equation of the $(0,4)$-tensor Riemann curvature and derive those of the Ricci and the scalar curvatures. In Section, 4 we deal with the special case of constant curvature of the ambient manifold where we rewrite the evolution equation of the scalar curvature of the evolving submanifold.

2. Preliminaries

The motion by the mean curvature of a submanifold Σ in a Riemannian manifold M is a one parameter family of submanifolds $\left(\Sigma_t\right)_{t\in I}$, for some time interval I containing 0, which admits a parametrization $F(t) \equiv F_t : \Sigma \longrightarrow \Sigma_t \subset M$ over a fixed (the initial) submanifold $\Sigma_0 = \Sigma$ with velocity equals to the mean curvature vector; namely

$$\begin{cases} \frac{\partial}{\partial t}F_t(P) & = H_{F_t(P)} \\ F_0(P) & = P, \qquad \forall P \in \Sigma \end{cases} \tag{2.1}$$

where $H_{F_t(P)}$ stands for the mean curvature vector at $F_t(P)$.
Let $\Sigma^n \subset M^m$ be a n-dimensional submanifold of a Riemannian manifold M of dimension m and $F : \Sigma \times [0,T) \to M$, a mean curvature flow of Σ in M. Choose a local orthonormal frame $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_m\}$ of M along Σ_t such that $\{e_1, \cdots, e_n\}$ consists of tangent vectors to Σ_t whereas $e_{n+1}, \cdots, e_m$ are in the normal bundle over Σ_t. In the sequel we will use the following indexes convention :

$$1 \leqslant i, j, k, l, \cdots \leqslant n, \quad \text{and} \quad n+1 \leqslant \alpha, \beta, \gamma \leqslant m,$$

i.e lowercase of Latin (resp. Greek) letters from 1 to n (resp. from $n+1$ to m). We also use Einstein summation convention for repeated indexes.
If we denote by $\{\partial_i\}_{1\leqslant i\leqslant n}$ the partial derivatives with respect to the local coordinates in Σ and by $\langle\cdot,\cdot\rangle \equiv \overline{g}$ the metric on M, the induced metric on Σ is given by

$$g_{ij} = \langle \partial_i F, \partial_j F\rangle \tag{2.2}$$

and we write the second fundamental form and the mean curvature vector as follows :

$$\begin{aligned} A &= A^\alpha e_\alpha = h^\alpha_{ij} e_\alpha, \qquad (2.3)\\ H &= H^\alpha e_\alpha = trace_g A = g^{ij} h^\alpha_{ij} e_\alpha = h^\alpha_{ii} e_\alpha \qquad (2.4)\end{aligned}$$

where $h^\alpha_{ij} = \langle \overline{\nabla}_{e_i} e_j, e_\alpha\rangle = -\langle \overline{\nabla}_{e_i} e_\alpha, e_j\rangle = -\langle e_i, \overline{\nabla}_{e_j} e_\alpha\rangle = \langle \overline{\nabla}_{e_j} e_i, e_\alpha\rangle = h^\alpha_{ji}$ are the components of the symmetric matrix A^α.
Since Gauss, Ricci and Codazzi-Mainardi equations play a key role in the study of the geometry of submanifold, let us recall them in local coordinates for their use in the sequel. We have :

$$\begin{aligned} R_{ijkl} &= \overline{R}_{ijkl} + h^\alpha_{il} h^\alpha_{jk} - h^\alpha_{ik} h^\alpha_{jl}, \qquad (2.5)\\ h^\alpha_{ij,k} - h^\alpha_{kj,i} &= -\overline{R}_{\alpha jki} \qquad (2.6)\end{aligned}$$

where $\overline{R}(U,V,X,Y) = \overline{g}(\overline{\nabla}_U\overline{\nabla}_V X + \overline{\nabla}_V\overline{\nabla}_U X - \overline{\nabla}_{[U,V]}X, Y)$,
$R(X,Y,Z,T) = g(\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, T)$ and $h^\alpha_{rp,q} = \overline{g}\Big((\widetilde{\nabla}_{e_q} A)(e_r, e_p), e_\alpha\Big)$.

Lemma 2.1. [5]
Along the mean curvature flow, the induced metric on Σ_t satisfies :

$$\frac{\partial}{\partial t}\Big|_{t=0} g_{ij} = -2H^\alpha h^\alpha_{ij}, \qquad \text{and} \qquad \frac{\partial}{\partial t}\Big|_{t=0} g^{ij} = 2H^\alpha h^\alpha_{ij}. \tag{2.7}$$

In the following lemma, we derive the evolution equation of the frame on the normal bundle $N\Sigma$. Let us denote by $b^\beta_\alpha = \langle \frac{\partial}{\partial t} e_\alpha, e_\beta\rangle = \langle \overline{\nabla}_H e_\alpha, e_\beta\rangle$ and observe that $b^\alpha_\beta = -b^\beta_\alpha$.

Lemma 2.2.
The evolution of the normal frame can be written as :

$$\frac{\partial}{\partial t} e_\alpha = -\nabla H^\alpha - H^\beta \langle \overline{\nabla}_{e_i} e_\beta, e_\alpha\rangle e_i + b^\beta_\alpha e_\beta \tag{2.8}$$

where ∇H^α is the covariant differentiation of H^α with respect to the induced connection on Σ_t.

Proof. A straightforward computation yields

$$\begin{aligned}\frac{\partial}{\partial t}e_\alpha &= \Big\langle \frac{\partial}{\partial t}e_\alpha, e_i\Big\rangle e_i + \Big\langle \frac{\partial}{\partial t}e_\alpha, e_\beta\Big\rangle e_\beta = \langle \overline{\nabla}_H e_\alpha, e_i\rangle e_i + b_\alpha^\beta e_\beta \\ &= -\langle e_\alpha, \overline{\nabla}_H \partial_i F\rangle e_i + b_\alpha^\beta e_\beta \\ &= -\langle e_\alpha, \overline{\nabla}_{e_i} H\rangle e_i + b_\alpha^\beta e_\beta \\ &= -\langle e_\alpha, \overline{\nabla}_{e_i}(H^\beta e_\beta)\rangle e_i + b_\alpha^\beta e_\beta \\ &= -\langle e_\alpha, (\overline{\nabla}_{e_i} H^\beta) e_\beta + H^\beta \overline{\nabla}_{e_i} e_\beta\rangle e_i + b_\alpha^\beta e_\beta \\ &= -(\overline{\nabla}_{e_i} H^\alpha) e_i - H^\beta \langle e_\alpha, \overline{\nabla}_{e_i} e_\beta\rangle e_i + b_\alpha^\beta e_\beta \\ &= -\nabla H^\alpha - H^\beta \langle e_\alpha, \overline{\nabla}_{e_i} e_\beta\rangle e_i + b_\alpha^\beta e_\beta.\end{aligned}$$

The third equality use the fact that $\frac{\partial}{\partial t}\frac{\partial}{\partial x_i}F = \overline{\nabla}_H\left(\frac{\partial}{\partial x_i}F\right) = \frac{\partial}{\partial x_i}\frac{\partial}{\partial t}F = \overline{\nabla}_{e_i}H.$ □

Lemma 2.3.
The time derivative of the mean curvature vector is given by :

$$\frac{\partial}{\partial t}H = (\triangle H^\alpha + H^\beta h_{ik}^\beta h_{jk}^\alpha) e_\alpha - H^\alpha \nabla H^\alpha - H^\alpha H^\beta \langle \overline{\nabla}_{e_i} e_\beta, e_\alpha\rangle e_i. \tag{2.9}$$

Proof. From $H = H^\alpha e_\alpha$ we get :

$$\begin{aligned}\frac{\partial}{\partial t}H &= \frac{\partial}{\partial t}(H^\alpha e_\alpha) = \left(\frac{\partial}{\partial t}H^\alpha\right) e_\alpha + H^\alpha \left(\frac{\partial}{\partial t}e_\alpha\right) \\ &= \frac{\partial}{\partial t}\left(g^{ij}h_{ij}^\alpha e_\alpha\right) = \left(\frac{\partial}{\partial t}g^{ij}h_{ij}^\alpha\right) e_\alpha + g^{ij}h_{ij}^\alpha \left(\frac{\partial}{\partial t}e_\alpha\right).\end{aligned}$$

Using Lemma 2.1 we obtain :

$$\begin{aligned}\frac{\partial}{\partial t}H^\alpha &= \frac{\partial}{\partial t}\left(g^{ij}h_{ij}^\alpha\right) = \left(\frac{\partial}{\partial t}g^{ij}\right) h_{ij}^\alpha + g^{ij}\left(\frac{\partial}{\partial t}h_{ij}^\alpha\right) \\ &= 2H^\beta h_{ij}^\beta h_{ij}^\alpha + g^{ij}\left(H_{,ji}^\alpha - H^\beta h_{ik}^\beta h_{jk}^\alpha + H^\beta \overline{R}_{\beta ij\alpha} + h_{ij}^\beta \langle e_\beta, \overline{\nabla}_H e_\alpha\rangle\right) \\ &= 2H^\beta h_{ij}^\beta h_{ij}^\alpha + \triangle H^\alpha - H^\beta h_{ij}^\beta h_{ij}^\alpha + H^\beta g^{ij}\overline{R}_{\beta ij\alpha} + H^\beta \langle e_\beta, \overline{\nabla}_H e_\alpha\rangle \\ &= \triangle H^\alpha + H^\beta h_{ij}^\beta h_{ij}^\alpha + H^\beta g^{ij}\overline{R}_{\beta ij\alpha} + H^\beta \langle e_\beta, \overline{\nabla}_H e_\alpha\rangle \\ &= \triangle H^\alpha + H^\beta h_{ij}^\beta h_{ij}^\alpha + H^\beta g^{ij}\overline{R}_{\beta ij\alpha} + H^\beta b_\alpha^\beta.\end{aligned}$$

Combining this with Lemma 2.2 gives :

$$\begin{aligned}\frac{\partial}{\partial t}H &= \left(\triangle H^\alpha + H^\beta h_{ij}^\beta h_{ij}^\alpha + H^\beta g^{ij}\overline{R}_{\beta ij\alpha} + H^\beta b_\alpha^\beta\right) e_\alpha - H^\alpha \nabla H^\alpha - H^\alpha H^\beta \langle \overline{\nabla}_{e_i} e_\beta, e_\alpha\rangle e_i + H^\alpha b_\alpha^\beta e_\beta \\ &= \left(\triangle H^\alpha + H^\beta h_{ij}^\beta h_{ij}^\alpha + H^\beta g^{ij}\overline{R}_{\beta ij\alpha} + H^\beta b_\alpha^\beta\right) e_\alpha - H^\alpha \nabla H^\alpha - H^\alpha H^\beta \langle \overline{\nabla}_{e_i} e_\beta, e_\alpha\rangle e_i + H^\beta b_\beta^\alpha e_\alpha \\ &= \left(\triangle H^\alpha + H^\beta h_{ij}^\beta h_{ij}^\alpha + H^\beta g^{ij}\overline{R}_{\beta ij\alpha}\right) e_\alpha - H^\alpha \nabla H^\alpha - H^\alpha H^\beta \langle \overline{\nabla}_{e_i} e_\beta, e_\alpha\rangle e_i.\end{aligned}$$

In the second and the third equality, used have been made of : $H^\alpha b_\alpha^\beta e_\beta = H^\beta b_\beta^\alpha e_\alpha = -H^\beta b_\alpha^\beta e_\alpha$. □

3. Intrinsic geometry along the mean curvature flow

In this section we study the proper geometric objects of the evolving submanifold assuming that the flow exists. First of all, we compute the evolution equation of the Riemannian curvature from which we derive those of the Ricci curvature and the scalar curvature of the moving submanifold along the flow.
Let us recall the Riemannian, the Ricci and the scalar curvature in local coordinates

$$\begin{aligned} R_{ijkl} &= R(e_i,e_j,e_k,e_l) = \overline{R}(e_i,e_j,e_k,e_l) + \langle A(e_i,e_l), A(e_j,e_k)\rangle - \langle A(e_i,e_k), A(e_j,e_l)\rangle, \\ R_{ij} &= Ric(e_i,e_j) = g^{kl}R_{kijl}, \\ S &= g^{ij}R_{ij}. \end{aligned}$$

3.1. The Riemannian curvature equation.

Theorem 3.1. Along the mean curvature flow, the curvature tensor of Σ_t satisfies the following equation :

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \triangle_{g_t}\right) R_{ijkl} = {} & (\overline{\nabla}_H \overline{R})\left(e_i,e_j,e_k,e_l\right) - \overline{R}_{ijkl,pp} + h^\alpha_{ik}\left(\overline{R}_{\alpha plp,j} + \overline{R}_{\alpha ljp,p}\right) \\ & -h^\alpha_{il}\left(\overline{R}_{\alpha pkp,j}+\overline{R}_{\alpha kjp,p}\right) - h^\alpha_{jk}\left(\overline{R}_{\alpha plp,i}+\overline{R}_{\alpha lip,p}\right) + h^\alpha_{jl}\left(\overline{R}_{\alpha pkp,i}+\overline{R}_{\alpha kip,p}\right) \\ & -2\left(h^\alpha_{pi}\overline{R}_{\alpha jkl,p} + h^\alpha_{pj}\overline{R}_{i\alpha kl,p} + h^\alpha_{pk}\overline{R}_{ij\alpha l,p} + h^\alpha_{pl}\overline{R}_{ijk\alpha,p}\right) \\ & +h^\alpha_{pi}\left(h^\alpha_{pr}\overline{R}_{rjkl} - 2h^\beta_{pj}\overline{R}_{\alpha\beta kl} - 2h^\beta_{pk}\overline{R}_{\alpha j\beta l} - 2h^\beta_{pl}\overline{R}_{\alpha jk\beta}\right) \\ & +h^\alpha_{pj}\left(h^\alpha_{pr}\overline{R}_{irkl} - 2h^\beta_{pk}\overline{R}_{i\alpha\beta l} - 2h^\beta_{pl}\overline{R}_{i\alpha k\beta}\right) + h^\alpha_{pk}\left(h^\alpha_{pr}\overline{R}_{ijrl} - 2h^\beta_{pl}\overline{R}_{ij\alpha\beta}\right) \\ & +h^\alpha_{pl}h^\alpha_{pr}\overline{R}_{ijkr} + h^\alpha_{il}h^\alpha_{mp}\overline{R}_{mkjp} - h^\alpha_{ik}h^\alpha_{mp}\overline{R}_{mljp} + h^\alpha_{mp}h^\alpha_{jk}\overline{R}_{mlip} \\ & -h^\alpha_{mp}h^\alpha_{jl}\overline{R}_{mkip} - H^\alpha h^\alpha_{pi}\overline{R}_{pjkl} - H^\alpha h^\alpha_{pj}\overline{R}_{ipkl} - H^\alpha h^\alpha_{pk}\overline{R}_{ijpl} \\ & -H^\alpha h^\alpha_{pl}\overline{R}_{ijkp} + H^\beta h^\alpha_{il}\overline{R}_{\beta jk\alpha} - H^\beta h^\alpha_{ik}\overline{R}_{\beta jl\alpha} + H^\beta h^\alpha_{jk}\overline{R}_{\beta il\alpha} \\ & -H^\beta h^\alpha_{jl}\overline{R}_{\beta ik\alpha} + \overline{R}_{\alpha jkl}\overline{R}_{\alpha ppi} + \overline{R}_{i\alpha kl}\overline{R}_{\alpha ppj} + \overline{R}_{ij\alpha l}\overline{R}_{\alpha ppk} + \overline{R}_{ijk\alpha}\overline{R}_{\alpha ppl} \\ & +\left(h^\beta_{lp}h^\alpha_{jk} - h^\beta_{kp}h^\alpha_{jl}\right)\overline{R}_{\beta\alpha ip} + \left(h^\alpha_{il}h^\beta_{kp} - h^\alpha_{ik}h^\beta_{lp}\right)\overline{R}_{\beta\alpha jp} \\ & +\left(h^\alpha_{lm}h^\alpha_{jk} - h^\alpha_{km}h^\alpha_{jl}\right)\overline{R}_{mpip} + \left(h^\alpha_{il}h^\alpha_{km} - h^\alpha_{ik}h^\alpha_{lm}\right)\overline{R}_{mpjp} \\ & -H^\beta h^\alpha_{jk}\left(h^\beta_{lm}h^\alpha_{im} + h^\beta_{in}h^\alpha_{ln}\right) - H^\beta h^\alpha_{il}\left(h^\beta_{km}h^\alpha_{jm} + h^\beta_{jn}h^\alpha_{kn}\right) \\ & +H^\beta h^\alpha_{jl}\left(h^\beta_{km}h^\alpha_{im} + h^\beta_{in}h^\alpha_{kn}\right) + H^\beta h^\alpha_{ik}\left(h^\beta_{lm}h^\alpha_{jm} + h^\beta_{jn}h^\alpha_{ln}\right) \\ & +h^\beta_{mp}h^\alpha_{im}\left(h^\beta_{lp}h^\alpha_{jk} - h^\beta_{kp}h^\alpha_{jl}\right) + h^\beta_{mp}h^\beta_{pi}\left(h^\alpha_{lm}h^\alpha_{jk} - h^\alpha_{km}h^\alpha_{jl}\right) \\ & +h^\beta_{mp}h^\alpha_{jm}\left(h^\alpha_{il}h^\beta_{kp} - h^\alpha_{ik}h^\beta_{lp}\right) + h^\beta_{mp}h^\beta_{pj}\left(h^\alpha_{il}h^\alpha_{km} - h^\alpha_{ik}h^\alpha_{lm}\right) \\ & +h^\beta_{mp}h^\alpha_{mp}\left(h^\beta_{li}h^\alpha_{jk} + h^\alpha_{il}h^\beta_{kj} - h^\beta_{ki}h^\alpha_{jl} - h^\alpha_{ik}h^\beta_{lj}\right) + 2h^\beta_{im}h^\alpha_{mp}\left(h^\beta_{kp}h^\alpha_{jl} - h^\beta_{lp}h^\alpha_{jk}\right) \\ & +2h^\beta_{jm}h^\alpha_{mp}\left(h^\alpha_{ik}h^\beta_{lp} - h^\alpha_{il}h^\beta_{kp}\right) - 2h^\alpha_{il,p}h^\alpha_{jk,p} + 2h^\alpha_{ik,p}h^\alpha_{jl,p}. \end{aligned} \tag{3.1}$$

Proof. We start by computing the (rough) Laplacian of the curvature.

$$\begin{aligned}
(\nabla_{e_p}R)\Big(e_i,e_j,e_k,e_l\Big) &= e_p\cdot R(e_i,e_j,e_k,e_l)-R(\nabla_{e_p}e_i,e_j,e_k,e_l)-R(e_i,\nabla_{e_p}e_j,e_k,e_l)\\
&\quad -R(e_i,e_j,\nabla_{e_p}e_k,e_l)-R(e_i,e_j,e_k,\nabla_{e_p}e_l)\\
&= (\overline{\nabla}_{e_p}\overline{R})\Big(e_i,e_j,e_k,e_l\Big)+h^\alpha_{pi}\overline{R}\Big(e_\alpha,e_j,e_k,e_l\Big)+h^\alpha_{pj}\overline{R}\Big(e_i,e_\alpha,e_k,e_l\Big)\\
&\quad +h^\alpha_{pk}\overline{R}\Big(e_i,e_j,e_\alpha,e_l\Big)+h^\alpha_{pl}\overline{R}\Big(e_i,e_j,e_k,e_\alpha\Big)-\langle A(\nabla_{e_p}e_i,e_l),A(e_j,e_k)\rangle\\
&\quad +\langle A(\nabla_{e_p}e_i,e_k),A(e_j,e_l)\rangle-\langle A(e_i,e_l),A(\nabla_{e_p}e_j,e_k)\rangle\\
&\quad +\langle A(e_i,e_k),A(\nabla_{e_p}e_j,e_l)\rangle-\langle A(e_i,\nabla_{e_p}e_l),A(e_j,e_k)\rangle\\
&\quad +\langle A(e_i,e_k),A(e_j,\nabla_{e_p}e_l)\rangle-\langle A(e_i,e_l),A(e_j,\nabla_{e_p}e_k)\rangle\\
&\quad +\langle A(e_i,\nabla_{e_p}e_k),A(e_j,e_l)\rangle+\overline{\nabla}_{e_p}\Big(h^\alpha_{il}h^\alpha_{jk}\Big)-\overline{\nabla}_{e_p}\Big(h^\alpha_{ik}h^\alpha_{jl}\Big). \qquad (3.2)
\end{aligned}$$

The last two terms can be computed as follows :

$$\begin{aligned}
\overline{\nabla}_{e_p}\Big(h^\alpha_{il}h^\alpha_{jk}\Big) &= \overline{\nabla}_{e_p}\Big(\langle A(e_i,e_l),A(e_j,e_k)\rangle\Big)=\langle\overline{\nabla}_{e_p}A(e_i,e_l),A(e_j,e_k)\rangle+\langle A(e_i,e_l),\overline{\nabla}_{e_p}A(e_j,e_k)\rangle\\
&= \langle(\widetilde{\nabla}_{e_p}A)(e_i,e_l)+A(\nabla_{e_p}e_i,e_l)+A(e_i,\nabla_{e_p}e_l),A(e_j,e_k)\rangle\\
&\quad +\langle A(e_i,e_l),(\widetilde{\nabla}_{e_p}A)(e_j,e_k)+A(\nabla_{e_p}e_j,e_k)+A(e_j,\nabla_{e_p}e_k)\rangle
\end{aligned}$$

and

$$\begin{aligned}
\overline{\nabla}_{e_p}\Big(h^\alpha_{ik}h^\alpha_{jl}\Big) &= \overline{\nabla}_{e_p}\Big(\langle A(e_i,e_k),A(e_j,e_l)\rangle\Big)=\langle\overline{\nabla}_{e_p}A(e_i,e_k),A(e_j,e_l)\rangle+\langle A(e_i,e_k),\overline{\nabla}_{e_p}A(e_j,e_l)\rangle\\
&= \langle(\widetilde{\nabla}_{e_p}A)(e_i,e_k)+A(\nabla_{e_p}e_i,e_k)+A(e_i,\nabla_{e_p}e_k),A(e_j,e_l)\rangle\\
&\quad +\langle A(e_i,e_k),(\widetilde{\nabla}_{e_p}A)(e_j,e_l)+A(\nabla_{e_p}e_j,e_l)+A(e_j,\nabla_{e_p}e_l)\rangle.
\end{aligned}$$

Plugging them back into equation (3.2) gives :

$$\begin{aligned}
(\nabla_{e_p}R)\Big(e_i,e_j,e_k,e_l\Big) &= (\overline{\nabla}_{e_p}\overline{R})\Big(e_i,e_j,e_k,e_l\Big)+h^\alpha_{pi}\overline{R}\Big(e_\alpha,e_j,e_k,e_l\Big)+h^\alpha_{pj}\overline{R}\Big(e_i,e_\alpha,e_k,e_l\Big)\\
&\quad +h^\alpha_{pk}\overline{R}\Big(e_i,e_j,e_\alpha,e_l\Big)+h^\alpha_{pl}\overline{R}\Big(e_i,e_j,e_k,e_\alpha\Big)+\langle(\widetilde{\nabla}_{e_p}A)(e_i,e_l),A(e_j,e_k)\rangle\\
&\quad +\langle A(e_i,e_l),(\widetilde{\nabla}_{e_p}A)(e_j,e_k)\rangle-\langle(\widetilde{\nabla}_{e_p}A)(e_i,e_k),A(e_j,e_l)\rangle\\
&\quad -\langle A(e_i,e_k),(\widetilde{\nabla}_{e_p}A)(e_j,e_l)\rangle. \qquad (3.3)
\end{aligned}$$

Let us abbreviate using the following notations :

$$\overline{R}_{ijkl,p}=(\overline{\nabla}_{e_p}\overline{R})\Big(e_i,e_j,e_k,e_l\Big) \qquad \text{and} \qquad R_{ijkl,p}=(\nabla_{e_p}R)\Big(e_i,e_j,e_k,e_l\Big).$$

Then the equation (3.3) can be rewritten as :

$$\begin{aligned}
R_{ijkl,p}\equiv(\nabla_{e_p}R)\Big(e_i,e_j,e_k,e_l\Big) &= \overline{R}_{ijkl,p}+h^\alpha_{pi}\overline{R}_{\alpha jkl}+h^\alpha_{pj}\overline{R}_{i\alpha kl}+h^\alpha_{pk}\overline{R}_{ij\alpha l}+h^\alpha_{pl}\overline{R}_{ijk\alpha}\\
&\quad +h^\alpha_{il,p}h^\alpha_{jk}+h^\alpha_{il}h^\alpha_{jk,p}-h^\alpha_{ik,p}h^\alpha_{jl}-h^\alpha_{ik}h^\alpha_{jl,p}. \qquad (3.4)
\end{aligned}$$

Put $Q = \nabla_{e_p} R$. A straightforward computation gives :

$$\begin{aligned}
\Big(\nabla_{e_q} Q\Big)(e_i, e_j, e_k, e_l) &= Q_{ijkl,q} = R_{ijkl,pq} \\
&= e_q \cdot Q(e_i, e_j, e_k, e_l) - Q\Big(\nabla_{e_q} e_i, e_j, e_k, e_l\Big) - Q\Big(e_i, \nabla_{e_q} e_j, e_k, e_l\Big) \\
&\quad - Q\Big(e_i, e_j, \nabla_{e_q} e_k, e_l\Big) - Q\Big(e_i, e_j, e_k, \nabla_{e_q} e_l\Big). \qquad (3.5)
\end{aligned}$$

The first term in the right hand side of (3.5) can be expressed as :

$$\begin{aligned}
e_q \cdot Q(e_i, e_j, e_k, e_l) &= e_q \cdot \overline{R}_{ijkl,p} + e_q \cdot \Big(h^\alpha_{pi} \overline{R}_{\alpha jkl}\Big) + e_q \cdot \Big(h^\alpha_{pj} \overline{R}_{i\alpha kl}\Big) \\
&\quad + e_q \cdot \Big(h^\alpha_{pk} \overline{R}_{ij\alpha l}\Big) + e_q \cdot \Big(h^\alpha_{pl} \overline{R}_{ijk\alpha}\Big) + e_q \cdot \Big(h^\alpha_{il,p} h^\alpha_{jk}\Big) \\
&\quad + e_q \cdot \Big(h^\alpha_{il} h^\alpha_{jk,p}\Big) - e_q \cdot \Big(h^\alpha_{ik,p} h^\alpha_{jl}\Big) - e_q \cdot \Big(h^\alpha_{ik} h^\alpha_{jl,p}\Big).
\end{aligned}$$

Thus

$$\begin{aligned}
R_{ijkl,pq} &= (\nabla_{e_q} \nabla_{e_p} R)\Big(e_i, e_j, e_k, e_l\Big) \\
&= e_q \cdot \overline{R}_{ijkl,p} + e_q \cdot \Big(h^\alpha_{pi} \overline{R}_{\alpha jkl}\Big) + e_q \cdot \Big(h^\alpha_{pj} \overline{R}_{i\alpha kl}\Big) + e_q \cdot \Big(h^\alpha_{pk} \overline{R}_{ij\alpha l}\Big) \\
&\quad + e_q \cdot \Big(h^\alpha_{pl} \overline{R}_{ijk\alpha}\Big) + e_q \cdot \Big(h^\alpha_{il,p} h^\alpha_{jk}\Big) + e_q \cdot \Big(h^\alpha_{il} h^\alpha_{jk,p}\Big) - e_q \cdot \Big(h^\alpha_{ik,p} h^\alpha_{jl}\Big) \\
&\quad - e_q \cdot \Big(h^\alpha_{ik} h^\alpha_{jl,p}\Big) - Q\Big(\nabla_{e_q} e_i, e_j, e_k, e_l\Big) - Q\Big(e_i, \nabla_{e_q} e_j, e_k, e_l\Big) \\
&\quad - Q\Big(e_i, e_j, \nabla_{e_q} e_k, e_l\Big) - Q\Big(e_i, e_j, e_k, \nabla_{e_q} e_l\Big). \qquad (3.6)
\end{aligned}$$

We have :

$$\begin{aligned}
e_q \cdot \Big(\overline{R}_{ijkl,p}\Big) &= e_q \cdot \Bigg((\overline{\nabla}_{e_p} \overline{R})\Big(e_i, e_j, e_k, e_l\Big)\Bigg) \\
&= (\overline{\nabla}_{e_q} \overline{\nabla}_{e_p} \overline{R})\Big(e_i, e_j, e_k, e_l\Big) + (\overline{\nabla}_{e_p} \overline{R})\Big(\overline{\nabla}_{e_q} e_i, e_j, e_k, e_l\Big) + (\overline{\nabla}_{e_p} \overline{R})\Big(e_i, \overline{\nabla}_{e_q} e_j, e_k, e_l\Big) \\
&\quad + (\overline{\nabla}_{e_p} \overline{R})\Big(e_i, e_j, \overline{\nabla}_{e_q} e_k, e_l\Big) + (\overline{\nabla}_{e_p} \overline{R})\Big(e_i, e_j, e_k, \overline{\nabla}_{e_q} e_l\Big)
\end{aligned}$$

and

$$\begin{aligned}
e_q \cdot \Big(h^\alpha_{pi} \overline{R}_{\alpha jkl}\Big) &= e_q \cdot \Bigg(\langle A(e_p, e_i), e_\alpha \rangle \overline{R}\Big(e_\alpha, e_j, e_k, e_l\Big)\Bigg) \\
&= \overline{R}_{\alpha jkl}\Big(h^\alpha_{pq,i} - \overline{R}_{\alpha pqi}\Big) + h^\alpha_{pi}\Big(\overline{R}_{\alpha jkl,q} - h^\alpha_{qr} \overline{R}_{rjkl}\Big) \\
&\quad + \overline{R}_{\alpha jkl}\Big(\langle A(\nabla_{e_q} e_p, e_i), e_\alpha \rangle + \langle A(e_p, \nabla_{e_q} e_i), e_\alpha \rangle\Big) \\
&\quad + h^\alpha_{pi}\Bigg(\overline{R}\Big(e_\alpha, \overline{\nabla}_{e_q} e_j, e_k, e_l\Big) + \overline{R}\Big(e_\alpha, e_j, \overline{\nabla}_{e_q} e_k, e_l\Big) + \overline{R}\Big(e_\alpha, e_j, e_k, \overline{\nabla}_{e_q} e_l\Big)\Bigg)
\end{aligned}$$

where used have been made of Codazzi - Mainardi equation, precisely $h^\alpha_{ip,q} - h^\alpha_{qp,i} = -\overline{R}_{\alpha pqi}$.

We also have :

$$\begin{aligned} e_q\cdot\left(h^{\alpha}_{il,p}h^{\alpha}_{jk}\right) &= \overline{\nabla}_{e_q}\left(h^{\alpha}_{il,p}h^{\alpha}_{jk}\right)=\overline{\nabla}_{e_q}\left(\langle(\widetilde{\nabla}_{e_p}A)(e_i,e_l),A(e_j,e_k)\rangle\right)\\ &= \left\langle(\widetilde{\nabla}_{e_q}\widetilde{\nabla}_{e_p}A)(e_i,e_l)+(\widetilde{\nabla}_{e_p}A)(\nabla_{e_q}e_i,e_l)+(\widetilde{\nabla}_{e_p}A)(e_i,\nabla_{e_q}e_l),A(e_j,e_k)\right\rangle\\ &\quad+\left\langle(\widetilde{\nabla}_{e_p})A(e_i,e_l),(\widetilde{\nabla}_{e_q}A)(e_j,e_k)+A(\nabla_{e_q}e_j,e_k)+A(e_j,\nabla_{e_q}e_k)\right\rangle\\ &= h^{\alpha}_{il,pq}h^{\alpha}_{jk}+h^{\alpha}_{il,p}h^{\alpha}_{jk,q}+\left\langle(\widetilde{\nabla}_{e_p}A)(\nabla_{e_q}e_i,e_l)+(\widetilde{\nabla}_{e_p}A)(e_i,\nabla_{e_q}e_l),A(e_j,e_k)\right\rangle\\ &\quad+\left\langle(\widetilde{\nabla}_{e_p})A(e_i,e_l),A(\nabla_{e_q}e_j,e_k)+A(e_j,\nabla_{e_q}e_k)\right\rangle, \end{aligned} \tag{3.7}$$

and similarly interchanging k and l ($k\leftrightarrow l$), we obtain :

$$\begin{aligned} e_q\cdot\left(h^{\alpha}_{ik,p}h^{\alpha}_{jl}\right) &= \overline{\nabla}_{e_q}\left(h^{\alpha}_{ik,p}h^{\alpha}_{jl}\right)=\overline{\nabla}_{e_q}\left(\langle(\widetilde{\nabla}_{e_p}A)(e_i,e_k),A(e_j,e_l)\rangle\right)\\ &= h^{\alpha}_{ik,pq}h^{\alpha}_{jl}+h^{\alpha}_{ik,p}h^{\alpha}_{jl,q}+\left\langle(\widetilde{\nabla}_{e_p}A)(\nabla_{e_q}e_i,e_k)+(\widetilde{\nabla}_{e_p}A)(e_i,\nabla_{e_q}e_k),A(e_j,e_l)\right\rangle\\ &\quad+\left\langle(\widetilde{\nabla}_{e_p})A(e_i,e_k),A(\nabla_{e_q}e_j,e_l)+A(e_j,\nabla_{e_q}e_l)\right\rangle. \end{aligned} \tag{3.8}$$

Besides

$$\begin{aligned} Q(\nabla_{e_q}e_i,e_j,e_k,e_l) &= (\overline{\nabla}_{e_p}\overline{R})\left(\nabla_{e_q}e_i,e_j,e_k,e_l\right)+\langle A(\nabla_{e_q}e_i,e_p),e_\alpha\rangle\overline{R}\left(e_\alpha,e_j,e_k,e_l\right)\\ &\quad+h^{\alpha}_{pj}\overline{R}\left(\nabla_{e_q}e_i,e_\alpha,e_k,e_l\right)+h^{\alpha}_{pk}\overline{R}\left(\nabla_{e_q}e_i,e_j,e_\alpha,e_l\right)+h^{\alpha}_{pl}\overline{R}\left(\nabla_{e_q}e_i,e_j,e_k,e_\alpha\right)\\ &\quad+\left\langle(\widetilde{\nabla}_{e_p}A)(\nabla_{e_q}e_i,e_l),A(e_j,e_k)\right\rangle+\left\langle A(\nabla_{e_q}e_i,e_l),(\widetilde{\nabla}_{e_p}A)(e_j,e_k)\right\rangle\\ &\quad-\left\langle(\widetilde{\nabla}_{e_p}A)(\nabla_{e_q}e_i,e_k),A(e_j,e_l)\right\rangle-\left\langle A(\nabla_{e_q}e_i,e_k),(\widetilde{\nabla}_{e_p}A)(e_j,e_l)\right\rangle. \end{aligned} \tag{3.9}$$

Analogously, we compute the remaining terms in the right hand side of equation (3.6). One may choose arbitrary coordinates around a point $P\in\Sigma$ as we did so far, but the equations are quite simpler in normal coordinates. Let us now choose normal coordinates system with center P. Therefore $g_{ij}(P)=\delta_{ij}$, $g_{ij,k}(P)=0=\Gamma^k_{ij}(P)$ for all i,j,k and we obtain :

$$\begin{aligned} R_{ijkl,pq} &= \overline{R}_{ijkl,pq}+h^{\alpha}_{qi}\overline{R}_{\alpha jkl,p}+h^{\alpha}_{qj}\overline{R}_{i\alpha kl,p}+h^{\alpha}_{qk}\overline{R}_{ij\alpha l,p}+h^{\alpha}_{ql}\overline{R}_{ijk\alpha,p}+\overline{R}_{\alpha jkl}\left(h^{\alpha}_{pq,i}-\overline{R}_{\alpha pqi}\right)\\ &\quad+h^{\alpha}_{pi}\left(\overline{R}_{\alpha jkl,q}-h^{\alpha}_{qr}\overline{R}_{rjkl}+h^{\beta}_{qj}\overline{R}_{\alpha\beta kl}+h^{\beta}_{qk}\overline{R}_{\alpha j\beta l}+h^{\beta}_{ql}\overline{R}_{\alpha jk\beta}\right)+\overline{R}_{i\alpha kl}\left(h^{\alpha}_{pq,j}-\overline{R}_{\alpha pqj}\right)\\ &\quad+h^{\alpha}_{pj}\left(\overline{R}_{i\alpha kl,q}-h^{\alpha}_{qr}\overline{R}_{irkl}+h^{\beta}_{qi}\overline{R}_{\beta\alpha kl}+h^{\beta}_{qk}\overline{R}_{i\alpha\beta l}+h^{\beta}_{ql}\overline{R}_{i\alpha k\beta}\right)+\overline{R}_{ij\alpha l}\left(h^{\alpha}_{pq,k}-\overline{R}_{\alpha pqk}\right)\\ &\quad+h^{\alpha}_{pk}\left(\overline{R}_{ij\alpha l,q}-h^{\alpha}_{qr}\overline{R}_{ijrl}+h^{\beta}_{qi}\overline{R}_{\beta j\alpha l}+h^{\beta}_{qj}\overline{R}_{i\beta\alpha l}+h^{\beta}_{ql}\overline{R}_{ij\alpha\beta}\right)+\overline{R}_{ijk\alpha}\left(h^{\alpha}_{pq,l}-\overline{R}_{\alpha pql}\right)\\ &\quad+h^{\alpha}_{pl}\left(\overline{R}_{ijk\alpha,q}-h^{\alpha}_{qr}\overline{R}_{ijkr}+h^{\beta}_{qi}\overline{R}_{\beta jk\alpha}+h^{\beta}_{qj}\overline{R}_{i\beta k\alpha}+h^{\beta}_{qk}\overline{R}_{ij\beta\alpha}\right)+h^{\alpha}_{il,pq}h^{\alpha}_{jk}+h^{\alpha}_{il,p}h^{\alpha}_{jk,q}\\ &\quad+h^{\alpha}_{il}h^{\alpha}_{jk,pq}+h^{\alpha}_{il,q}h^{\alpha}_{jk,p}-h^{\alpha}_{ik,pq}h^{\alpha}_{jl}-h^{\alpha}_{ik,p}h^{\alpha}_{jl,q}-h^{\alpha}_{ik}h^{\alpha}_{jl,pq}-h^{\alpha}_{ik,q}h^{\alpha}_{jl,p}. \end{aligned} \tag{3.10}$$

Taking its trace with respect to the induced metric gives the (rough) Laplacian of the curvature

$$\begin{aligned}
R_{ijkl,pp} &= \overline{R}_{ijkl,pp} + \overline{R}_{\alpha jkl}\left(h^{\alpha}_{pp,i} - \overline{R}_{\alpha ppi}\right) + h^{\alpha}_{pi}\left(2\overline{R}_{\alpha jkl,p} - h^{\alpha}_{pr}\overline{R}_{rjkl} + h^{\beta}_{pj}\overline{R}_{\alpha\beta kl} + h^{\beta}_{pk}\overline{R}_{\alpha j\beta l} + h^{\beta}_{pl}\overline{R}_{\alpha jk\beta}\right) \\
&\quad + \overline{R}_{i\alpha kl}\left(h^{\alpha}_{pp,j} - \overline{R}_{\alpha ppj}\right) + h^{\alpha}_{pj}\left(2\overline{R}_{i\alpha kl,p} - h^{\alpha}_{pr}\overline{R}_{irkl} + h^{\beta}_{pi}\overline{R}_{\beta\alpha kl} + h^{\beta}_{pk}\overline{R}_{i\alpha\beta l} + h^{\beta}_{pl}\overline{R}_{i\alpha k\beta}\right) \\
&\quad + \overline{R}_{ij\alpha l}\left(h^{\alpha}_{pp,k} - \overline{R}_{\alpha ppk}\right) + h^{\alpha}_{pk}\left(2\overline{R}_{ij\alpha l,p} - h^{\alpha}_{pr}\overline{R}_{ijrl} + h^{\beta}_{pi}\overline{R}_{\beta j\alpha l} + h^{\beta}_{pj}\overline{R}_{i\beta\alpha l} + h^{\beta}_{pl}\overline{R}_{ij\alpha\beta}\right) \\
&\quad + \overline{R}_{ijk\alpha}\left(h^{\alpha}_{pp,l} - \overline{R}_{\alpha ppl}\right) + h^{\alpha}_{pl}\left(2\overline{R}_{ijk\alpha,p} - h^{\alpha}_{pr}\overline{R}_{ijkr} + h^{\beta}_{pi}\overline{R}_{\beta jk\alpha} + h^{\beta}_{pj}\overline{R}_{i\beta k\alpha} + h^{\beta}_{pk}\overline{R}_{ij\beta\alpha}\right) \\
&\quad + h^{\alpha}_{il,pp}h^{\alpha}_{jk} + 2h^{\alpha}_{il,p}h^{\alpha}_{jk,p} + h^{\alpha}_{il}h^{\alpha}_{jk,pp} - h^{\alpha}_{ik,pp}h^{\alpha}_{jl} - 2h^{\alpha}_{ik,p}h^{\alpha}_{jl,p} - h^{\alpha}_{ik}h^{\alpha}_{jl,pp}. \qquad (3.11)
\end{aligned}$$

The time derivative of the curvature is given by :

$$\begin{aligned}
\frac{\partial}{\partial t}R_{ijkl} &\equiv \frac{\partial}{\partial t}\left(R\left(e_i, e_j, e_k, e_l\right)\right) \\
&= \frac{\partial}{\partial t}\left(\overline{R}(e_i, e_j, e_k, e_l) + \langle A(e_i, e_l), A(e_j, e_k)\rangle - \langle A(e_i, e_k), A(e_j, e_l)\rangle\right) \\
&= \left(\overline{\nabla}_H\overline{R}\right)\left(e_i, e_j, e_k, e_l\right) + \overline{R}\left(\overline{\nabla}_{e_i}H, e_j, e_k, e_l\right) + \overline{R}\left(e_i, \overline{\nabla}_{e_j}H, e_k, e_l\right) \\
&\quad + \overline{R}\left(e_i, e_j, \overline{\nabla}_{e_k}H, e_l\right) + \overline{R}\left(e_i, e_j, e_k, \overline{\nabla}_{e_l}H\right) + \frac{\partial}{\partial t}\left(h^{\alpha}_{il}h^{\alpha}_{jk}\right) - \frac{\partial}{\partial t}\left(h^{\alpha}_{ik}h^{\alpha}_{jl}\right) \\
&= \left(\overline{\nabla}_H\overline{R}\right)\left(e_i, e_j, e_k, e_l\right) + \overline{R}\left(\langle\overline{\nabla}_{e_i}H, e_\alpha\rangle e_\alpha + \langle\overline{\nabla}_{e_i}H, e_p\rangle e_p, e_j, e_k, e_l\right) \\
&\quad + \overline{R}\left(e_i, \langle\overline{\nabla}_{e_j}H, e_\alpha\rangle e_\alpha + \langle\overline{\nabla}_{e_j}H, e_p\rangle e_p, e_k, e_l\right) + \overline{R}\left(e_i, e_j, \langle\overline{\nabla}_{e_k}H, e_\alpha\rangle e_\alpha + \langle\overline{\nabla}_{e_k}H, e_p\rangle e_p, e_l\right) \\
&\quad + \overline{R}\left(e_i, e_j, e_k, \langle\overline{\nabla}_{e_l}H, e_\alpha\rangle e_\alpha + \langle\overline{\nabla}_{e_l}H, e_p\rangle e_p\right) + \frac{\partial}{\partial t}\left(h^{\alpha}_{il}h^{\alpha}_{jk}\right) - \frac{\partial}{\partial t}\left(h^{\alpha}_{ik}h^{\alpha}_{jl}\right) \\
&= \left(\overline{\nabla}_H\overline{R}\right)\left(e_i, e_j, e_k, e_l\right) + \overline{R}\left(H^{\alpha}_{,i}e_\alpha - H^{\alpha}h^{\alpha}_{pi}e_p, e_j, e_k, e_l\right) \\
&\quad + \overline{R}\left(e_i, H^{\alpha}_{,j}e_\alpha - H^{\alpha}h^{\alpha}_{pj}e_p, e_k, e_l\right) + \overline{R}\left(e_i, e_j, H^{\alpha}_{,k}e_\alpha - H^{\alpha}h^{\alpha}_{pk}e_p, e_l\right) \\
&\quad + \overline{R}\left(e_i, e_j, e_k, H^{\alpha}_{,l}e_\alpha - H^{\alpha}h^{\alpha}_{pl}e_p\right) + \frac{\partial}{\partial t}\left(h^{\alpha}_{il}h^{\alpha}_{jk}\right) - \frac{\partial}{\partial t}\left(h^{\alpha}_{ik}h^{\alpha}_{jl}\right) \\
&= \left(\overline{\nabla}_H\overline{R}\right)\left(e_i, e_j, e_k, e_l\right) + H^{\alpha}_{,i}\overline{R}_{\alpha jkl} - H^{\alpha}h^{\alpha}_{pi}\overline{R}_{pjkl} + H^{\alpha}_{,j}\overline{R}_{i\alpha kl} - H^{\alpha}h^{\alpha}_{pj}\overline{R}_{ipkl} \\
&\quad + H^{\alpha}_{,k}\overline{R}_{ij\alpha l} - H^{\alpha}h^{\alpha}_{pk}\overline{R}_{ijpl} + H^{\alpha}_{,l}\overline{R}_{ijk\alpha} - H^{\alpha}h^{\alpha}_{pl}\overline{R}_{ijkp} + \frac{\partial}{\partial t}\left(h^{\alpha}_{il}h^{\alpha}_{jk}\right) - \frac{\partial}{\partial t}\left(h^{\alpha}_{ik}h^{\alpha}_{jl}\right). \qquad (3.12)
\end{aligned}$$

From these, we can write the evolution equation of the curvature as follows :

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)R_{ijkl} &= \left(\overline{\nabla}_H\overline{R}\right)\left(e_i,e_j,e_k,e_l\right)+H^\alpha_{,i}\overline{R}_{\alpha jkl}-H^\alpha h^\alpha_{pi}\overline{R}_{pjkl}+H^\alpha_{,j}\overline{R}_{i\alpha kl}\\
&-H^\alpha h^\alpha_{pj}\overline{R}_{ipkl}+H^\alpha_{,k}\overline{R}_{ij\alpha l}-H^\alpha h^\alpha_{pk}\overline{R}_{ijpl}+H^\alpha_{,l}\overline{R}_{ijk\alpha}-H^\alpha h^\alpha_{pl}\overline{R}_{ijkp}\\
&-\overline{R}_{ijkl,pp}-\overline{R}_{\alpha jkl}\left(h^\alpha_{pp,i}-\overline{R}_{\alpha ppi}\right)-\overline{R}_{i\alpha kl}\left(h^\alpha_{pp,j}-\overline{R}_{\alpha ppj}\right)\\
&-\overline{R}_{ij\alpha l}\left(h^\alpha_{pp,k}-\overline{R}_{\alpha ppk}\right)-\overline{R}_{ijk\alpha}\left(h^\alpha_{pp,l}-\overline{R}_{\alpha ppl}\right)\\
&-h^\alpha_{pi}\left(2\overline{R}_{\alpha jkl,p}-h^\alpha_{pr}\overline{R}_{rjkl}+h^\beta_{pj}\overline{R}_{\alpha\beta kl}+h^\beta_{pk}\overline{R}_{\alpha j\beta l}+h^\beta_{pl}\overline{R}_{\alpha jk\beta}\right)\\
&-h^\alpha_{pj}\left(2\overline{R}_{i\alpha kl,p}-h^\alpha_{pr}\overline{R}_{irkl}+h^\beta_{pi}\overline{R}_{\beta\alpha kl}+h^\beta_{pk}\overline{R}_{i\alpha\beta l}+h^\beta_{pl}\overline{R}_{i\alpha k\beta}\right)\\
&-h^\alpha_{pk}\left(2\overline{R}_{ij\alpha l,p}-h^\alpha_{pr}\overline{R}_{ijrl}+h^\beta_{pi}\overline{R}_{\beta j\alpha l}+h^\beta_{pj}\overline{R}_{i\beta\alpha l}+h^\beta_{pl}\overline{R}_{ij\alpha\beta}\right)\\
&-h^\alpha_{pl}\left(2\overline{R}_{ijk\alpha,p}-h^\alpha_{pr}\overline{R}_{ijkr}+h^\beta_{pi}\overline{R}_{\beta jk\alpha}+h^\beta_{pj}\overline{R}_{i\beta k\alpha}+h^\beta_{pk}\overline{R}_{ij\beta\alpha}\right)\\
&-h^\alpha_{il,pp}h^\alpha_{jk}-2h^\alpha_{il,p}h^\alpha_{jk,p}-h^\alpha_{il}h^\alpha_{jk,pp}+h^\alpha_{ik,pp}h^\alpha_{jl}+2h^\alpha_{ik,p}h^\alpha_{jl,p}+h^\alpha_{ik}h^\alpha_{jl,pp}\\
&+\frac{\partial}{\partial t}\left(h^\alpha_{il}h^\alpha_{jk}\right)-\frac{\partial}{\partial t}\left(h^\alpha_{ik}h^\alpha_{jl}\right) \qquad (3.13)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)R_{ijkl} &= \left(\overline{\nabla}_H\overline{R}\right)\left(e_i,e_j,e_k,e_l\right)-H^\alpha h^\alpha_{pi}\overline{R}_{pjkl}-H^\alpha h^\alpha_{pj}\overline{R}_{ipkl}-H^\alpha h^\alpha_{pk}\overline{R}_{ijpl}\\
&-H^\alpha h^\alpha_{pl}\overline{R}_{ijkp}+\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)\left(h^\alpha_{il}h^\alpha_{jk}\right)-\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)\left(h^\alpha_{ik}h^\alpha_{jl}\right)\\
&-h^\alpha_{pi}\left(2\overline{R}_{\alpha jkl,p}-h^\alpha_{pr}\overline{R}_{rjkl}+2h^\beta_{pj}\overline{R}_{\alpha\beta kl}+2h^\beta_{pk}\overline{R}_{\alpha j\beta l}+2h^\beta_{pl}\overline{R}_{\alpha jk\beta}\right)\\
&-h^\alpha_{pj}\left(2\overline{R}_{i\alpha kl,p}-h^\alpha_{pr}\overline{R}_{irkl}+2h^\beta_{pk}\overline{R}_{i\alpha\beta l}+2h^\beta_{pl}\overline{R}_{i\alpha k\beta}\right)\\
&-h^\alpha_{pk}\left(2\overline{R}_{ij\alpha l,p}-h^\alpha_{pr}\overline{R}_{ijrl}+2h^\beta_{pl}\overline{R}_{ij\alpha\beta}\right)-h^\alpha_{pl}\left(2\overline{R}_{ijk\alpha,p}-h^\alpha_{pr}\overline{R}_{ijkr}\right)\\
&-\overline{R}_{ijkl,pp}+\overline{R}_{\alpha jkl}\overline{R}_{\alpha ppi}+\overline{R}_{i\alpha kl}\overline{R}_{\alpha ppj}+\overline{R}_{ij\alpha l}\overline{R}_{\alpha ppk}+\overline{R}_{ijk\alpha}\overline{R}_{\alpha ppl} \qquad (3.14)
\end{aligned}$$

Using the evolution equation of the second fundamental form, we get :

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)\left(h^\alpha_{il}h^\alpha_{jk}\right) &= -h^\alpha_{jk}\overline{R}_{\alpha lip,p}-h^\alpha_{jk}\overline{R}_{\alpha plp,i}+h^\beta_{lp}h^\alpha_{jk}\overline{R}_{\beta\alpha ip}+h^\alpha_{mp}h^\alpha_{jk}\overline{R}_{mlip}+h^\alpha_{lm}h^\alpha_{jk}\overline{R}_{mpip}\\
&-H^\beta h^\alpha_{jk}\left(h^\beta_{lm}h^\alpha_{im}+h^\beta_{in}h^\alpha_{ln}\right)+H^\beta h^\alpha_{jk}\overline{R}_{\beta il\alpha}-H^\beta h^\alpha_{il}\left(h^\beta_{km}h^\alpha_{jm}+h^\beta_{jn}h^\alpha_{kn}\right)\\
&+H^\beta h^\alpha_{il}\overline{R}_{\beta jk\alpha}+h^\beta_{mp}h^\alpha_{im}h^\beta_{lp}h^\alpha_{jk}-h^\beta_{im}h^\alpha_{mp}h^\beta_{lp}h^\alpha_{jk}+h^\beta_{mp}h^\beta_{li}h^\alpha_{mp}h^\alpha_{jk}\\
&-h^\beta_{im}h^\beta_{lp}h^\alpha_{mp}h^\alpha_{jk}+h^\beta_{mp}h^\beta_{pi}h^\alpha_{lm}h^\alpha_{jk}-h^\alpha_{il}\overline{R}_{\alpha kjp,p}-h^\alpha_{il}\overline{R}_{\alpha pkp,j}+h^\alpha_{il}h^\beta_{kp}\overline{R}_{\beta\alpha jp}\\
&+h^\alpha_{il}h^\alpha_{mp}\overline{R}_{mkjp}+h^\alpha_{il}h^\alpha_{km}\overline{R}_{mpjp}+h^\alpha_{il}h^\beta_{mp}h^\alpha_{jm}h^\beta_{kp}-h^\alpha_{il}h^\beta_{jm}h^\alpha_{mp}h^\beta_{kp}\\
&+h^\alpha_{il}h^\beta_{mp}h^\beta_{kj}h^\alpha_{mp}-h^\alpha_{il}h^\beta_{jm}h^\beta_{kp}h^\alpha_{mp}+h^\alpha_{il}h^\beta_{mp}h^\beta_{pj}h^\alpha_{km}-2h^\alpha_{il,p}h^\alpha_{jk,p} \qquad (3.15)
\end{aligned}$$

and similarly

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)\left(h^\alpha_{ik}h^\alpha_{jl}\right) &= -h^\alpha_{jl}\overline{R}_{\alpha kip,p}-h^\alpha_{jl}\overline{R}_{\alpha pkp,i}+h^\beta_{kp}h^\alpha_{jl}\overline{R}_{\beta\alpha ip}+h^\alpha_{mp}h^\alpha_{jl}\overline{R}_{mkip}+h^\alpha_{km}h^\alpha_{jl}\overline{R}_{mpip}\\ &\quad -H^\beta h^\alpha_{jl}\left(h^\beta_{km}h^\alpha_{im}+h^\beta_{in}h^\alpha_{kn}\right)+H^\beta h^\alpha_{jl}\overline{R}_{\beta ik\alpha}-H^\beta h^\alpha_{ik}\left(h^\beta_{lm}h^\alpha_{jm}+h^\beta_{jn}h^\alpha_{ln}\right)\\ &\quad +H^\beta h^\alpha_{ik}\overline{R}_{\beta jl\alpha}+h^\beta_{mp}h^\alpha_{im}h^\beta_{kp}h^\alpha_{jl}-h^\beta_{im}h^\alpha_{mp}h^\beta_{kp}h^\alpha_{jl}+h^\beta_{mp}h^\beta_{ki}h^\alpha_{mp}h^\alpha_{jl}\\ &\quad -h^\beta_{im}h^\beta_{kp}h^\alpha_{mp}h^\alpha_{jl}+h^\beta_{mp}h^\beta_{pi}h^\alpha_{km}h^\alpha_{jl}-h^\alpha_{ik}\overline{R}_{\alpha ljp,p}-h^\alpha_{ik}\overline{R}_{\alpha plp,j}+h^\alpha_{ik}h^\beta_{lp}\overline{R}_{\beta\alpha jp}\\ &\quad +h^\alpha_{ik}h^\alpha_{mp}\overline{R}_{mljp}+h^\alpha_{ik}h^\alpha_{lm}\overline{R}_{mpjp}+h^\alpha_{ik}h^\beta_{mp}h^\alpha_{jm}h^\beta_{lp}-h^\alpha_{ik}h^\beta_{jm}h^\alpha_{mp}h^\beta_{lp}\\ &\quad +h^\alpha_{ik}h^\beta_{mp}h^\beta_{lj}h^\alpha_{mp}-h^\alpha_{ik}h^\beta_{jm}h^\beta_{lp}h^\alpha_{mp}+h^\alpha_{ik}h^\beta_{mp}h^\beta_{pj}h^\alpha_{lm}-2h^\alpha_{ik,p}h^\alpha_{jl,p}. \qquad (3.16)\end{aligned}$$

Gathering everything together gives :

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)R_{ijkl} &= \left(\overline{\nabla}_H\overline{R}\right)\left(e_i,e_j,e_k,e_l\right)-H^\alpha h^\alpha_{pi}\overline{R}_{pjkl}-H^\alpha h^\alpha_{pj}\overline{R}_{ipkl}-H^\alpha h^\alpha_{pk}\overline{R}_{ijpl}-H^\alpha h^\alpha_{pl}\overline{R}_{ijkp}\\ &\quad -h^\alpha_{pi}\left(2\overline{R}_{\alpha jkl,p}-h^\alpha_{pr}\overline{R}_{rjkl}+2h^\beta_{pj}\overline{R}_{\alpha\beta kl}+2h^\beta_{pk}\overline{R}_{\alpha j\beta l}+2h^\beta_{pl}\overline{R}_{\alpha jk\beta}\right)\\ &\quad -h^\alpha_{pj}\left(2\overline{R}_{i\alpha kl,p}-h^\alpha_{pr}\overline{R}_{irkl}+2h^\beta_{pk}\overline{R}_{i\alpha\beta l}+2h^\beta_{pl}\overline{R}_{i\alpha k\beta}\right)-2h^\alpha_{il,p}h^\alpha_{jk,p}+2h^\alpha_{ik,p}h^\alpha_{jl,p}\\ &\quad -h^\alpha_{pk}\left(2\overline{R}_{ij\alpha l,p}-h^\alpha_{pr}\overline{R}_{ijrl}+2h^\beta_{pl}\overline{R}_{ij\alpha\beta}\right)-h^\alpha_{pl}\left(2\overline{R}_{ijk\alpha,p}-h^\alpha_{pr}\overline{R}_{ijkr}\right)\\ &\quad -\overline{R}_{ijkl,pp}+\overline{R}_{\alpha jkl}\overline{R}_{\alpha ppi}+\overline{R}_{i\alpha kl}\overline{R}_{\alpha ppj}+\overline{R}_{ij\alpha l}\overline{R}_{\alpha ppk}+\overline{R}_{ijk\alpha}\overline{R}_{\alpha ppl}.\\ &\quad -h^\alpha_{jk}\overline{R}_{\alpha lip,p}-h^\alpha_{jk}\overline{R}_{\alpha plp,i}+h^\beta_{lp}h^\alpha_{jk}\overline{R}_{\beta\alpha ip}+h^\alpha_{mp}h^\alpha_{jk}\overline{R}_{mlip}+h^\alpha_{lm}h^\alpha_{jk}\overline{R}_{mpip}\\ &\quad -H^\beta h^\alpha_{jk}\left(h^\beta_{lm}h^\alpha_{im}+h^\beta_{in}h^\alpha_{ln}\right)+H^\beta h^\alpha_{jk}\overline{R}_{\beta il\alpha}-H^\beta h^\alpha_{il}\left(h^\beta_{km}h^\alpha_{jm}+h^\beta_{jn}h^\alpha_{kn}\right)\\ &\quad +H^\beta h^\alpha_{il}\overline{R}_{\beta jk\alpha}+h^\beta_{mp}h^\alpha_{im}h^\beta_{lp}h^\alpha_{jk}-h^\beta_{im}h^\alpha_{mp}h^\beta_{lp}h^\alpha_{jk}+h^\beta_{mp}h^\beta_{li}h^\alpha_{mp}h^\alpha_{jk}-h^\beta_{im}h^\beta_{lp}h^\alpha_{mp}h^\alpha_{jk}\\ &\quad +h^\beta_{mp}h^\beta_{pi}h^\alpha_{lm}h^\alpha_{jk}-h^\alpha_{il}\overline{R}_{\alpha kjp,p}-h^\alpha_{il}\overline{R}_{\alpha pkp,j}+h^\alpha_{il}h^\beta_{kp}\overline{R}_{\beta\alpha jp}+h^\alpha_{il}h^\alpha_{mp}\overline{R}_{mkjp}\\ &\quad +h^\alpha_{il}h^\alpha_{km}\overline{R}_{mpjp}+h^\alpha_{il}h^\beta_{mp}h^\alpha_{jm}h^\beta_{kp}-h^\alpha_{il}h^\beta_{jm}h^\alpha_{mp}h^\beta_{kp}+h^\alpha_{il}h^\beta_{mp}h^\beta_{kj}h^\alpha_{mp}-h^\alpha_{il}h^\beta_{jm}h^\beta_{kp}h^\alpha_{mp}\\ &\quad +h^\alpha_{il}h^\beta_{mp}h^\beta_{pj}h^\alpha_{km}+h^\alpha_{jl}\overline{R}_{\alpha kip,p}+h^\alpha_{jl}\overline{R}_{\alpha pkp,i}-h^\beta_{kp}h^\alpha_{jl}\overline{R}_{\beta\alpha ip}-h^\alpha_{mp}h^\alpha_{jl}\overline{R}_{mkip}\\ &\quad -h^\alpha_{km}h^\alpha_{jl}\overline{R}_{mpip}+H^\beta h^\alpha_{jl}\left(h^\beta_{km}h^\alpha_{im}+h^\beta_{in}h^\alpha_{kn}\right)-H^\beta h^\alpha_{jl}\overline{R}_{\beta ik\alpha}\\ &\quad +H^\beta h^\alpha_{ik}\left(h^\beta_{lm}h^\alpha_{jm}+h^\beta_{jn}h^\alpha_{ln}\right)-H^\beta h^\alpha_{ik}\overline{R}_{\beta jl\alpha}-h^\beta_{mp}h^\alpha_{im}h^\beta_{kp}h^\alpha_{jl}+h^\beta_{im}h^\alpha_{mp}h^\beta_{kp}h^\alpha_{jl}\\ &\quad -h^\beta_{mp}h^\beta_{ki}h^\alpha_{mp}h^\alpha_{jl}+h^\beta_{im}h^\beta_{kp}h^\alpha_{mp}h^\alpha_{jl}-h^\beta_{mp}h^\beta_{pi}h^\alpha_{km}h^\alpha_{jl}+h^\alpha_{ik}\overline{R}_{\alpha ljp,p}+h^\alpha_{ik}\overline{R}_{\alpha plp,j}\\ &\quad -h^\alpha_{ik}h^\beta_{lp}\overline{R}_{\beta\alpha jp}-h^\alpha_{ik}h^\alpha_{mp}\overline{R}_{mljp}-h^\alpha_{ik}h^\alpha_{lm}\overline{R}_{mpjp}-h^\alpha_{ik}h^\beta_{mp}h^\alpha_{jm}h^\beta_{lp}+h^\alpha_{ik}h^\beta_{jm}h^\alpha_{mp}h^\beta_{lp}\\ &\quad -h^\alpha_{ik}h^\beta_{mp}h^\beta_{lj}h^\alpha_{mp}+h^\alpha_{ik}h^\beta_{jm}h^\beta_{lp}h^\alpha_{mp}-h^\alpha_{ik}h^\beta_{mp}h^\beta_{pj}h^\alpha_{lm} \qquad (3.17)\end{aligned}$$

this completes the proof of the theorem after simplification. □

3.2. Evolution equation of the Ricci and the scalar curvatures. In the following theorem, we derive the evolution equation of the Ricci curvature from the equation of the Riemannian curvature (3.1) by taking its trace with respect to the induced metric g. In the same way, from the evolution equation of the Ricci curvature, we derive the equation of the scalar curvature.

Theorem 3.2. Under the mean curvature flow, the Ricci curvature of the induced metric satisfies:

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \triangle_{g_t}\right) R_{ij} = {} & \left(\overline{\nabla}_H \overline{R}\right)\left(e_k, e_i, e_j, e_k\right) - \overline{R}_{kijk,pp} - h^\alpha_{ij}\overline{R}_{\alpha pkk,p} - H^\alpha \overline{R}_{\alpha jip,p} - H^\alpha \overline{R}_{\alpha pjp,i} \\
& + h^\alpha_{ik}\left(\overline{R}_{\alpha jkp,p} + \overline{R}_{\alpha pjp,k}\right) + h^\alpha_{kj}\left(\overline{R}_{\alpha kip,p} + \overline{R}_{\alpha pkp,i}\right) - 2h^\alpha_{pk}\left(\overline{R}_{\alpha ijk,p} + \overline{R}_{kij\alpha,p}\right) \\
& - 2h^\alpha_{pi}\overline{R}_{k\alpha jk,p} - 2h^\alpha_{pj}\overline{R}_{ki\alpha k,p} - h^\alpha_{pi}\left(2h^\beta_{pj}\overline{R}_{k\alpha\beta k} + 2h^\beta_{pk}(\overline{R}_{k\alpha j\beta} + \overline{R}_{\beta\alpha jk}) - h^\alpha_{pr}\overline{R}_{krjk}\right) \\
& - h^\alpha_{pk}\left(h^\beta_{pj}(\overline{R}_{\alpha i\beta k} + 3\overline{R}_{\beta ik\alpha}) + 2h^\beta_{pk}\overline{R}_{\alpha ij\beta} - 2h^\alpha_{pr}\overline{R}_{rijk}\right) + h^\alpha_{pj}h^\alpha_{pr}\overline{R}_{kirk} \\
& + \overline{R}_{\alpha ijk}\overline{R}_{\alpha ppk} + \overline{R}_{k\alpha jk}\overline{R}_{\alpha ppi} + \overline{R}_{ki\alpha k}\overline{R}_{\alpha ppj} + \overline{R}_{kij\alpha}\overline{R}_{\alpha ppk} - h^\alpha_{kj}h^\alpha_{mp}\overline{R}_{mkip} \\
& - h^\alpha_{mp}h^\alpha_{ik}\overline{R}_{mjkp} - h^\alpha_{jm}h^\alpha_{ik}\overline{R}_{mpkp} + \left(h^\beta_{kp}h^\alpha_{ij} - h^\beta_{jp}h^\alpha_{ik}\right)\overline{R}_{\beta\alpha kp} \\
& - H^\alpha h^\beta_{jk}\overline{R}_{\beta ik\alpha} + H^\beta h^\alpha_{ij}\overline{R}_{\beta kk\alpha} - H^\beta h^\alpha_{ik}\overline{R}_{\beta kj\alpha} + \left(H^\alpha h^\alpha_{jm} - h^\alpha_{kj}h^\alpha_{km}\right)\overline{R}_{mpip} \\
& - H^\alpha\left(h^\alpha_{pi}\overline{R}_{kpjk} + h^\alpha_{pj}\overline{R}_{kipk} - h^\alpha_{pk}\overline{R}_{pijk}\right) - H^\beta H^\alpha\left(2h^\beta_{jm}h^\alpha_{im} - \overline{R}_{\beta ij\alpha}\right) \\
& + 2H^\beta h^\alpha_{km}\left(h^\alpha_{ik}h^\beta_{jm} + h^\alpha_{kj}h^\beta_{im} + h^\beta_{km}h^\alpha_{ij}\right) + 2h^\beta_{im}h^\alpha_{mp}\left(h^\alpha_{kj}h^\beta_{kp} - H^\alpha h^\beta_{jp}\right) \\
& - h^\beta_{mp}h^\alpha_{km}(h^\beta_{pi}h^\alpha_{kj} + h^\beta_{jp}h^\alpha_{ik}) - h^\alpha_{ik}h^\alpha_{jm}(h^\beta_{mp}h^\beta_{pk} + h^\beta_{kp}h^\beta_{mp}) \\
& - h^\beta_{mp}h^\alpha_{mp}\left(h^\beta_{jk}h^\alpha_{ik} + h^\alpha_{kj}h^\beta_{ki}\right) + 2h^\beta_{km}h^\alpha_{mp}h^\beta_{jp}h^\alpha_{ik} - 2H^\alpha_{,p}h^\alpha_{ij,p} + 2h^\alpha_{ik,p}h^\alpha_{jk,p} \qquad (3.18)
\end{aligned}$$

while the scalar curvature evolves as :

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \triangle_{g_t}\right) S = {} & \left(\overline{\nabla}_H \overline{R}\right)\left(e_k, e_i, e_i, e_k\right) - \overline{R}_{kiik,pp} + 2h^\alpha_{ik}\left(\overline{R}_{\alpha ikp,p} + \overline{R}_{\alpha pip,k}\right) \\
& - h^\alpha_{pk}\left(8\overline{R}_{\alpha iik,p} + h^\beta_{pi}(6\overline{R}_{\alpha\beta ik} + 4\overline{R}_{\alpha i\beta k}) + 4h^\beta_{pk}\overline{R}_{\alpha ii\beta}\right) \\
& + 4\overline{R}_{\alpha ppk}\overline{R}_{\alpha iik} + 2H^\alpha h^\beta_{ip}\left(\overline{R}_{\beta\alpha ip} + \overline{R}_{\beta i\alpha p}\right) + 6h^\alpha_{ki}h^\alpha_{km}\overline{R}_{mppi} \\
& + 2H^\beta H^\alpha\left(\overline{R}_{\beta ii\alpha} + h^\beta_{ip}h^\alpha_{ip}\right) + 2\sum_{\alpha,\beta,i,p}\left(\sum_m h^\alpha_{im}h^\beta_{mp} - h^\beta_{im}h^\alpha_{mp}\right)^2 \\
& - 2h^\beta_{mp}h^\alpha_{mp}h^\beta_{ik}h^\alpha_{ik} - 2|\nabla H|^2 + 2|\widetilde{\nabla} A|^2. \qquad (3.19)
\end{aligned}$$

Proof. From the definition of the Ricci curvature we have :

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)R_{ij} &= \left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)\left(g^{kl}R_{kijl}\right)\\ &= 2H^\alpha h^\alpha_{kl}R_{kijl}+g^{kl}\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)R_{kijl}\\ &= 2H^\alpha h^\alpha_{kl}\left(\overline{R}_{kijl}+h^\beta_{kl}h^\beta_{ij}-h^\beta_{kj}h^\beta_{il}\right)+g^{kl}\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)R_{kijl} \quad (3.20)\end{aligned}$$

Using the evolution equation of the curvature we get :

$$\begin{aligned}g^{kl}\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)R_{kijl} &= \left(\overline{\nabla}_H\overline{R}\right)\left(e_k,e_i,e_j,e_k\right)-\overline{R}_{kijk,pp}-h^\alpha_{ij}\overline{R}_{\alpha kkp,p}-h^\alpha_{ij}\overline{R}_{\alpha pkp,k}-h^\alpha_{kk}\overline{R}_{\alpha jip,p}\\ &-h^\alpha_{kk}\overline{R}_{\alpha pjp,i}+h^\alpha_{ik}\overline{R}_{\alpha jkp,p}+h^\alpha_{ik}\overline{R}_{\alpha pjp,k}+h^\alpha_{kj}\overline{R}_{\alpha kip,p}+h^\alpha_{kj}\overline{R}_{\alpha pkp,i}\\ &-h^\alpha_{pk}\left(2\overline{R}_{\alpha ijk,p}-h^\alpha_{pr}\overline{R}_{rijk}+2h^\beta_{pi}\overline{R}_{\alpha\beta jk}+2h^\beta_{pj}\overline{R}_{\alpha i\beta k}+2h^\beta_{pk}\overline{R}_{\alpha ij\beta}\right)\\ &-h^\alpha_{pi}\left(2\overline{R}_{k\alpha jk,p}-h^\alpha_{pr}\overline{R}_{krjk}+2h^\beta_{pj}\overline{R}_{k\alpha\beta k}+2h^\beta_{pk}\overline{R}_{k\alpha j\beta}\right)\\ &-h^\alpha_{pj}\left(2\overline{R}_{ki\alpha k,p}-h^\alpha_{pr}\overline{R}_{kirk}+2h^\beta_{pk}\overline{R}_{ki\alpha\beta}\right)-h^\alpha_{pk}\left(2\overline{R}_{kij\alpha,p}-h^\alpha_{pr}\overline{R}_{kijr}\right)\\ &+\overline{R}_{\alpha ijk}\overline{R}_{\alpha ppk}+\overline{R}_{k\alpha jk}\overline{R}_{\alpha ppi}+\overline{R}_{ki\alpha k}\overline{R}_{\alpha ppj}+\overline{R}_{kij\alpha}\overline{R}_{\alpha ppk}\\ &+h^\alpha_{kk}h^\alpha_{mp}\overline{R}_{mjip}+h^\alpha_{mp}h^\alpha_{ij}\overline{R}_{mkkp}-h^\alpha_{kj}h^\alpha_{mp}\overline{R}_{mkip}-h^\alpha_{mp}h^\alpha_{ik}\overline{R}_{mjkp}\\ &-H^\alpha h^\alpha_{pk}\overline{R}_{pijk}-H^\alpha h^\alpha_{pi}\overline{R}_{kpjk}-H^\alpha h^\alpha_{pj}\overline{R}_{kipk}-H^\alpha h^\alpha_{pk}\overline{R}_{kijp}\\ &+\left(h^\beta_{kp}h^\alpha_{ij}-h^\beta_{jp}h^\alpha_{ik}\right)\overline{R}_{\beta\alpha kp}+\left(h^\alpha_{km}h^\alpha_{ij}-h^\alpha_{jm}h^\alpha_{ik}\right)\overline{R}_{mpkp}\\ &+\left(h^\alpha_{kk}h^\beta_{jp}-h^\alpha_{kj}h^\beta_{kp}\right)\overline{R}_{\beta\alpha ip}+\left(h^\alpha_{kk}h^\alpha_{jm}-h^\alpha_{kj}h^\alpha_{km}\right)\overline{R}_{mpip}\\ &-H^\beta h^\alpha_{ij}\left(h^\beta_{km}h^\alpha_{km}+h^\beta_{kn}h^\alpha_{kn}-\overline{R}_{\beta kk\alpha}\right)-H^\beta h^\alpha_{kk}\left(h^\beta_{jm}h^\alpha_{im}+h^\beta_{in}h^\alpha_{jn}-\overline{R}_{\beta ij\alpha}\right)\\ &+H^\beta h^\alpha_{ik}\left(h^\beta_{jm}h^\alpha_{km}+h^\beta_{kn}h^\alpha_{jn}-\overline{R}_{\beta kj\alpha}\right)+H^\beta h^\alpha_{kj}\left(h^\beta_{km}h^\alpha_{im}+h^\beta_{in}h^\alpha_{kn}-\overline{R}_{\beta ik\alpha}\right)\\ &+h^\beta_{mp}h^\alpha_{km}\left(h^\beta_{kp}h^\alpha_{ij}-h^\beta_{jp}h^\alpha_{ik}\right)+h^\beta_{mp}h^\beta_{pk}\left(h^\alpha_{km}h^\alpha_{ij}-h^\alpha_{jm}h^\alpha_{ik}\right)\\ &+h^\beta_{mp}h^\alpha_{im}\left(h^\alpha_{kk}h^\beta_{jp}-h^\alpha_{kj}h^\beta_{kp}\right)+h^\beta_{mp}h^\beta_{pi}\left(h^\alpha_{kk}h^\alpha_{jm}-h^\alpha_{kj}h^\alpha_{km}\right)\\ &+h^\beta_{mp}h^\alpha_{mp}\left(h^\beta_{kk}h^\alpha_{ij}+h^\alpha_{kk}h^\beta_{ji}-h^\beta_{jk}h^\alpha_{ik}-h^\alpha_{kj}h^\beta_{ki}\right)-2h^\beta_{km}h^\alpha_{mp}h^\beta_{kp}h^\alpha_{ij}\\ &-2h^\alpha_{kk}h^\beta_{im}h^\alpha_{mp}h^\beta_{jp}+2h^\beta_{km}h^\alpha_{mp}h^\beta_{jp}h^\alpha_{ik}+2h^\alpha_{kj}h^\beta_{im}h^\alpha_{mp}h^\beta_{kp}-2H^\alpha_{,p}h^\alpha_{ij,p}+2h^\alpha_{ik,p}h^\alpha_{jk,p}.\end{aligned}$$

Therefore

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)R_{ij} &= 2H^{\alpha}h^{\alpha}_{kl}\left(\overline{R}_{kijl}+h^{\beta}_{kl}h^{\beta}_{ij}-h^{\beta}_{kj}h^{\beta}_{il}\right)+\left(\overline{\nabla}_{H}\overline{R}\right)\left(e_k,e_i,e_j,e_k\right)-\overline{R}_{kijk,pp}\\
&+h^{\alpha}_{ik}\left(\overline{R}_{\alpha jkp,p}+\overline{R}_{\alpha pjp,k}\right)+h^{\alpha}_{kj}\left(\overline{R}_{\alpha kip,p}+\overline{R}_{\alpha pkp,i}\right)-h^{\alpha}_{ij}\overline{R}_{\alpha kkp,p}-h^{\alpha}_{ij}\overline{R}_{\alpha pkp,k}\\
&-H^{\alpha}\overline{R}_{\alpha jip,p}-H^{\alpha}\overline{R}_{\alpha pjp,i}-h^{\alpha}_{pi}\left(2\overline{R}_{k\alpha jk,p}-h^{\alpha}_{pr}\overline{R}_{krjk}+2h^{\beta}_{pj}\overline{R}_{k\alpha\beta k}+2h^{\beta}_{pk}\overline{R}_{k\alpha j\beta}\right)\\
&-h^{\alpha}_{pj}\left(2\overline{R}_{ki\alpha k,p}-h^{\alpha}_{pr}\overline{R}_{kirk}+2h^{\beta}_{pk}\overline{R}_{ki\alpha\beta}\right)-h^{\alpha}_{pk}\left(2\overline{R}_{kij\alpha,p}-h^{\alpha}_{pr}\overline{R}_{kijr}\right)\\
&-h^{\alpha}_{pk}\left(2\overline{R}_{\alpha ijk,p}-h^{\alpha}_{pr}\overline{R}_{rijk}+2h^{\beta}_{pi}\overline{R}_{\alpha\beta jk}+2h^{\beta}_{pj}\overline{R}_{\alpha i\beta k}+2h^{\beta}_{pk}\overline{R}_{\alpha ij\beta}\right)\\
&+\overline{R}_{\alpha ijk}\overline{R}_{\alpha ppk}+\overline{R}_{k\alpha jk}\overline{R}_{\alpha ppi}+\overline{R}_{ki\alpha k}\overline{R}_{\alpha ppj}+\overline{R}_{kij\alpha}\overline{R}_{\alpha ppk}-2H^{\alpha}_{,p}h^{\alpha}_{ij,p}+2h^{\alpha}_{ik,p}h^{\alpha}_{jk,p}\\
&+H^{\alpha}\left(h^{\alpha}_{mp}\overline{R}_{mjip}-h^{\alpha}_{pk}\overline{R}_{pijk}-h^{\alpha}_{pi}\overline{R}_{kpjk}-h^{\alpha}_{pj}\overline{R}_{kipk}-h^{\alpha}_{pk}\overline{R}_{kijp}\right)\\
&+h^{\alpha}_{mp}h^{\alpha}_{ij}\overline{R}_{mkkp}-h^{\alpha}_{kj}h^{\alpha}_{mp}\overline{R}_{mkip}-h^{\alpha}_{mp}h^{\alpha}_{ik}\overline{R}_{mjkp}+\left(h^{\beta}_{kp}h^{\alpha}_{ij}-h^{\beta}_{jp}h^{\alpha}_{ik}\right)\overline{R}_{\beta\alpha kp}\\
&+\left(h^{\alpha}_{km}h^{\alpha}_{ij}-h^{\alpha}_{jm}h^{\alpha}_{ik}\right)\overline{R}_{mpkp}+\left(H^{\alpha}h^{\beta}_{jp}-h^{\alpha}_{kj}h^{\beta}_{kp}\right)\overline{R}_{\beta\alpha ip}+\left(H^{\alpha}h^{\alpha}_{jm}-h^{\alpha}_{kj}h^{\alpha}_{km}\right)\overline{R}_{mpip}\\
&-H^{\beta}h^{\alpha}_{ij}\left(2h^{\beta}_{km}h^{\alpha}_{km}-\overline{R}_{\beta kk\alpha}\right)-H^{\beta}H^{\alpha}\left(2h^{\beta}_{jm}h^{\alpha}_{im}-\overline{R}_{\beta ij\alpha}\right)\\
&+H^{\beta}h^{\alpha}_{ik}\left(h^{\beta}_{jm}h^{\alpha}_{km}+h^{\beta}_{kn}h^{\alpha}_{jn}-\overline{R}_{\beta kj\alpha}\right)+H^{\beta}h^{\alpha}_{kj}\left(h^{\beta}_{km}h^{\alpha}_{im}+h^{\beta}_{in}h^{\alpha}_{kn}-\overline{R}_{\beta ik\alpha}\right)\\
&-h^{\beta}_{mp}h^{\alpha}_{km}h^{\beta}_{jp}h^{\alpha}_{ik}+h^{\beta}_{mp}h^{\beta}_{pk}\left(2h^{\alpha}_{km}h^{\alpha}_{ij}-h^{\alpha}_{jm}h^{\alpha}_{ik}\right)+h^{\beta}_{mp}h^{\alpha}_{im}\left(H^{\alpha}h^{\beta}_{jp}-h^{\alpha}_{kj}h^{\beta}_{kp}\right)\\
&+h^{\beta}_{mp}h^{\beta}_{pi}\left(H^{\alpha}h^{\alpha}_{jm}-h^{\alpha}_{kj}h^{\alpha}_{km}\right)+h^{\beta}_{mp}h^{\alpha}_{mp}\left(2H^{\beta}h^{\alpha}_{ij}-h^{\beta}_{jk}h^{\alpha}_{ik}-h^{\alpha}_{kj}h^{\beta}_{ki}\right)\\
&+2h^{\beta}_{km}h^{\alpha}_{mp}\left(h^{\beta}_{jp}h^{\alpha}_{ik}-h^{\beta}_{kp}h^{\alpha}_{ij}\right)+2h^{\beta}_{im}h^{\alpha}_{mp}\left(h^{\alpha}_{kj}h^{\beta}_{kp}-H^{\alpha}h^{\beta}_{jp}\right). \qquad (3.21)
\end{aligned}$$

In the same way, we derive the evolution equation of the scalar curvature :

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)S &= \left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)\left(g^{ij}R_{ij}\right)\\
&= 2H^{\alpha}h^{\alpha}_{ij}R_{ij}+g^{ij}\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)R_{ij}\\
&= 2H^{\alpha}h^{\alpha}_{ij}\left(\overline{R}_{kijk}+H^{\beta}h^{\beta}_{ij}+h^{\beta}_{kj}h^{\beta}_{ik}\right)+g^{ij}\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)R_{ij}. \qquad (3.22)
\end{aligned}$$

Taking the trace with respect to the induced metric of the evolution equation of the Ricci curvature yields the following :

$$
\begin{aligned}
g^{ij}\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)R_{ij} &= 2H^{\alpha}h^{\alpha}_{kl}\left(\overline{R}_{kiil}+h^{\beta}_{kl}h^{\beta}_{ii}-h^{\beta}_{ki}h^{\beta}_{il}\right)+\left(\overline{\nabla}_H\overline{R}\right)\left(e_k,e_i,e_i,e_k\right)-\overline{R}_{kiik,pp}-2H^{\alpha}\overline{R}_{\alpha iip,p}\\
&\quad -2H^{\alpha}\overline{R}_{\alpha pip,i}+2h^{\alpha}_{ik}\left(\overline{R}_{\alpha ikp,p}+\overline{R}_{\alpha pip,k}\right)+H^{\alpha}\left(2h^{\alpha}_{mp}\overline{R}_{miip}-4h^{\alpha}_{pk}\overline{R}_{piik}\right)\\
&\quad -h^{\alpha}_{pk}\left(8\overline{R}_{\alpha iik,p}-4h^{\alpha}_{pr}\overline{R}_{riik}+4h^{\beta}_{pi}(\overline{R}_{\alpha\beta ik}+\overline{R}_{\alpha i\beta k})+4h^{\beta}_{pk}\overline{R}_{\alpha ii\beta}\right)\\
&\quad +4\overline{R}_{\alpha ppk}\overline{R}_{\alpha iik}-h^{\alpha}_{ki}h^{\alpha}_{mp}\left(\overline{R}_{mkip}-\overline{R}_{mikp}\right)+2\left(H^{\alpha}h^{\beta}_{ip}-h^{\alpha}_{ki}h^{\beta}_{kp}\right)\overline{R}_{\beta\alpha ip}\\
&\quad +2\left(H^{\alpha}h^{\alpha}_{im}-h^{\alpha}_{ki}h^{\alpha}_{km}\right)\overline{R}_{mpip}-2H^{\beta}H^{\alpha}\left(2h^{\beta}_{im}h^{\alpha}_{im}-\overline{R}_{\beta ii\alpha}\right)\\
&\quad +2H^{\beta}h^{\alpha}_{ik}\left(2h^{\beta}_{im}h^{\alpha}_{km}-\overline{R}_{\beta ki\alpha}\right)+2h^{\beta}_{mp}h^{\beta}_{pk}\left(H^{\alpha}h^{\alpha}_{km}-h^{\alpha}_{im}h^{\alpha}_{ik}\right)\\
&\quad +2h^{\beta}_{mp}h^{\alpha}_{im}\left(H^{\alpha}h^{\beta}_{ip}-h^{\alpha}_{ki}h^{\beta}_{kp}\right)+2h^{\beta}_{mp}h^{\alpha}_{mp}\left(H^{\beta}H^{\alpha}-h^{\beta}_{ik}h^{\alpha}_{ik}\right)\\
&\quad +4h^{\beta}_{km}h^{\alpha}_{mp}\left(h^{\beta}_{ip}h^{\alpha}_{ik}-H^{\alpha}h^{\beta}_{kp}\right)-2H^{\alpha}_{,p}H^{\alpha}_{,p}+2h^{\alpha}_{ik,p}h^{\alpha}_{ik,p}.
\end{aligned}
\tag{3.23}
$$

Finally we obtain :

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\triangle_{g_t}\right)S &= \left(\overline{\nabla}_H\overline{R}\right)\left(e_k,e_i,e_i,e_k\right)-\overline{R}_{kiik,pp}+2h^{\alpha}_{ik}\left(\overline{R}_{\alpha ikp,p}+\overline{R}_{\alpha pip,k}\right)\\
&\quad -h^{\alpha}_{pk}\left(8\overline{R}_{\alpha iik,p}+h^{\beta}_{pi}(6\overline{R}_{\alpha\beta ik}+4\overline{R}_{\alpha i\beta k})+4h^{\beta}_{pk}\overline{R}_{\alpha ii\beta}\right)\\
&\quad +4\overline{R}_{\alpha ppk}\overline{R}_{\alpha iik}+2H^{\alpha}h^{\beta}_{ip}\left(\overline{R}_{\beta\alpha ip}+\overline{R}_{\beta i\alpha p}\right)+6h^{\alpha}_{ki}h^{\alpha}_{km}\overline{R}_{mppi}\\
&\quad +2H^{\beta}H^{\alpha}\left(\overline{R}_{\beta ii\alpha}+h^{\beta}_{ip}h^{\alpha}_{ip}\right)-4h^{\beta}_{mp}h^{\beta}_{pk}h^{\alpha}_{im}h^{\alpha}_{ik}\\
&\quad +4h^{\beta}_{km}h^{\alpha}_{mp}h^{\beta}_{ip}h^{\alpha}_{ik}-2h^{\beta}_{mp}h^{\alpha}_{mp}h^{\beta}_{ik}h^{\alpha}_{ik}-2H^{\alpha}_{,p}H^{\alpha}_{,p}+2h^{\alpha}_{ik,p}h^{\alpha}_{ik,p}.
\end{aligned}
$$

□

To the author knowledge the long time existence of the mean curvature flow heavily relies on the curvature of the manifold and is proven only when the ambient manifold is (almost) Kaehler-Einstein manifold. Therefore we study the case when the ambient manifold has constant curvature and rewrite the evolution equation of the scalar curvature.

4. Constant curvature case

We assume now that M is of constant curvature, say c. The Riemannian curvature of M can be expressed as :

$$\overline{R}_{ijkl}=\overline{g}\left(\overline{R}(e_i,e_j)e_k,e_l\right)=c\left(\overline{g}_{il}\overline{g}_{jk}-\overline{g}_{ik}\overline{g}_{jl}\right). \tag{4.1}$$

Therefore

$$\begin{aligned}
\overline{R}_{\alpha\beta ik} &= c\Big(\overline{g}_{\alpha k}\overline{g}_{\beta i} - \overline{g}_{\alpha i}\overline{g}_{\beta k}\Big) = 0,\\
\overline{R}_{\alpha i\beta k} &= c\Big(\overline{g}_{\alpha k}\overline{g}_{i\beta} - \overline{g}_{\alpha\beta}\overline{g}_{ik}\Big) = -c\overline{g}_{\alpha\beta}\overline{g}_{ik}\\
\overline{R}_{\alpha ii\beta} &= c\Big(\overline{g}_{\alpha\beta}\overline{g}_{ii} - \overline{g}_{\alpha i}\overline{g}_{i\beta}\Big) = c\overline{g}_{\alpha\beta}\overline{g}_{ii} = cn\overline{g}_{\alpha\beta}\\
\overline{R}_{\alpha iik} &= c\Big(\overline{g}_{\alpha k}\overline{g}_{ii} - \overline{g}_{\alpha i}\overline{g}_{ik}\Big) = 0\\
\overline{R}_{ijjk} &= c\Big(\overline{g}_{ik}\overline{g}_{jj} - \overline{g}_{ij}\overline{g}_{jk}\Big) = c\Big(n\overline{g}_{ik} - \overline{g}_{ij}\overline{g}_{jk}\Big)
\end{aligned}$$

Let us remark that under this hypothesis Σ is curvature-invariant i.e. $\overline{R}_{\alpha ijk} = 0$, for any $i, j, k \in \{1, \cdots, n\}$ and any $\alpha \in \{n+1, \cdots, m\}$

Proposition 4.1. As long as the mean curvature flow $(\Sigma_t)_t$ of $\Sigma \hookrightarrow M$ exists, the scalar curvature of Σ_t satisfies:

$$\left(\frac{\partial}{\partial t} - \triangle_{g_t}\right)S \;\geqslant\; 2c(n-1)\Big(|A|^2 + |H|^2\Big) \tag{4.2}$$

Proof. Under the assumption of constant curvature, the equation (3.19) reduces to :

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \triangle_{g_t}\right)S &= 4\overline{R}_{\alpha ppk}\overline{R}_{\alpha iik} - h^{\alpha}_{pk}\Big(h^{\beta}_{pi}(6\overline{R}_{\alpha\beta ik} + 4\overline{R}_{\alpha i\beta k}) + 4h^{\beta}_{pk}\overline{R}_{\alpha ii\beta}\Big)\\
&\quad +2H^{\alpha}h^{\beta}_{ip}\Big(\overline{R}_{\beta\alpha ip} + \overline{R}_{\beta i\alpha p}\Big) + 6h^{\alpha}_{ki}h^{\alpha}_{km}\overline{R}_{mppi} + 2H^{\beta}H^{\alpha}\Big(\overline{R}_{\beta ii\alpha} + h^{\beta}_{ip}h^{\alpha}_{ip}\Big)\\
&\quad -4h^{\beta}_{mp}h^{\beta}_{pk}h^{\alpha}_{im}h^{\alpha}_{ik} + 4h^{\beta}_{km}h^{\alpha}_{mp}h^{\beta}_{ip}h^{\alpha}_{ik} - 2h^{\beta}_{mp}h^{\alpha}_{mp}h^{\beta}_{ik}h^{\alpha}_{ik} - 2H^{\alpha}_{,p}H^{\alpha}_{,p} + 2h^{\alpha}_{ik,p}h^{\alpha}_{ik,p}\\
&= 4c\; h^{\alpha}_{pk}h^{\beta}_{pi}\overline{g}_{\alpha\beta}\overline{g}_{ik} - 4nc\; h^{\alpha}_{pk}h^{\beta}_{pk}\overline{g}_{\alpha\beta} + 6c\; h^{\alpha}_{ki}h^{\alpha}_{km}\Big(n\overline{g}_{mi} - \overline{g}_{mp}\overline{g}_{pi}\Big)\\
&\quad -2c\; H^{\alpha}h^{\beta}_{ip}\overline{g}_{\alpha\beta}\overline{g}_{ip} + 2nc\; H^{\beta}H^{\alpha}\overline{g}_{\alpha\beta} + 2H^{\beta}H^{\alpha}h^{\beta}_{ip}h^{\alpha}_{ip} - 4h^{\beta}_{mp}h^{\beta}_{pk}h^{\alpha}_{im}h^{\alpha}_{ik}\\
&\quad +4h^{\beta}_{km}h^{\alpha}_{mp}h^{\beta}_{ip}h^{\alpha}_{ik} - 2h^{\beta}_{mp}h^{\alpha}_{mp}h^{\beta}_{ik}h^{\alpha}_{ik} - 2H^{\alpha}_{,p}H^{\alpha}_{,p} + 2h^{\alpha}_{ik,p}h^{\alpha}_{ik,p}\\
&= 2c(n-1)\Big(|A|^2 + |H|^2\Big) + 2H^{\beta}H^{\alpha}h^{\beta}_{ip}h^{\alpha}_{ip} + 2\sum_{\alpha,\beta,i,p}\Big(\sum_m h^{\alpha}_{im}h^{\beta}_{mp} - h^{\beta}_{im}h^{\alpha}_{mp}\Big)^2\\
&\quad -2\sum_{i,k,m,p}\Big(\sum_{\alpha} h^{\alpha}_{ik}h^{\alpha}_{mp}\Big)^2 - 2|\nabla H|^2 + 2|\widetilde{\nabla}A|^2\\
&\geqslant 2c(n-1)\Big(|A|^2 + |H|^2\Big)
\end{aligned} \tag{4.3}$$

since we have

$$2\sum_{\alpha,\beta,i,p}\Big(\sum_m h^{\alpha}_{im}h^{\beta}_{mp} - h^{\beta}_{im}h^{\alpha}_{mp}\Big)^2 - 2\sum_{i,k,m,p}\Big(\sum_{\alpha} h^{\alpha}_{ik}h^{\alpha}_{mp}\Big)^2 \geqslant 0,$$

$|\widetilde{\nabla}A|^2 - |\nabla H|^2 \geqslant 0$ and we can write $H^{\beta}H^{\alpha}h^{\beta}_{ip}h^{\alpha}_{ip} = \langle H, A\rangle^2$. ☐

Consequently we have the following corollary

Corollary 4.1.
If the ambient manifold has non negative constant curvature then the flow preserves the positivity of the scalar curvature of the evolving submanifold.

The consequence follows from the maximum principle. This result is in contrast with the one proved in [3] since the concluding remark does not agree with the commonly known comment asserting that the mean curvature flow favors positive scalar curvature.

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References

[1] X. Han and J. Li, On symplectic mean curvature flows, Lecture notes.
[2] G. Huisken, Asymptotique behavior for singularity of the mean curvature flow, J. Diff. Geom., N. 31(1990) : 285-299.
[3] V. Tapia, Evolution of the curvature tensors under mean curvature flow, Revista Colombiana de Matemáticas, N. 43, no. 2 (2009) : 175-185.
[4] M.-T. Wang, Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension, Invent. math., N.148, no. 3 (2002) : 525-543.
[5] M.-T. Wang, Mean curvature flow of surfaces in four-dimension manifolds, J. Diff. Geom., N. 57 (2001) : 301-338
[6] M.-T. Wang, Lectures on mean curvature flows in higher codimensions, Handbook of geometric analysis. N.1 (2008), 525-543, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA.
[7] F. Warner, Foundations of Differentiable Manifolds and Lie Groups, Berlin, New York : Springer-Verlag, ISBN 978-0-387-90894-6 (1983).

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