



A NEW FORMULATION OF THE PARALLELISM IN THE EUCLIDEAN GEOMETRY

NICOLA D'ALFONSO

ABSTRACT. In this paper, I introduce the primary term of "direction" that replaces the primary term of "line". Reformulating the parallel postulate with respect to this notion, I develop a more intuitive treatment of parallel lines.

1. INTRODUCTION

The easiest way to proceed is to refer to the Hilbert's axiomatization and replaces the following axioms:

- (1) *Given two points there is one and only one straight line (joining them)*
- (2) *Given a straight line and a point not on that line, there is one and only one straight line passing through that point, and having no point in common with it*

with these others:

- (1) *Given two points there is one and only one direction (capable to spatially connect them)*
- (2) *Given a point and a direction there is one and only one straight line (passing through that point and having that direction)*

The new axiomatization can be considered equivalent to the previous one, in the sense that from the axioms introduced follow those replaced.

Postulate 1.1. *Given two points there is one and only one direction (capable to spatially connect them). We can observe, with regard to this, the following figure 1:*

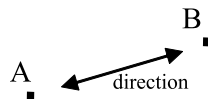


FIGURE 1. Direction determined by two points

Postulate 1.2. *Given a point and a direction there is one and only one straight line (passing through that point and having that direction). We can observe, with regard to this, the following figure 2:*

Date: 1 February 2012 .

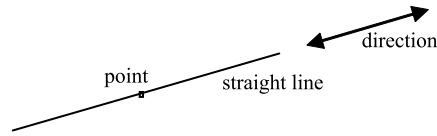


FIGURE 2. Straight line associated with one point and one direction

Theorem 1. *Given two points there is one and only one straight line (joining them). We can observe, with regard to this, the following figure 3:*

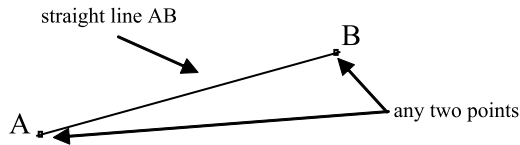


FIGURE 3. Straight line passing through two distinct points

Proof. The proof of this property derives directly from the following considerations. The postulate 1.1 on the preceding page ensure us that the two given points A and B are enough to identify any direction. Moreover, the postulate of 1.2 on the previous page ensure us that this direction will exactly be that one of any straight lines passing through the above-mentioned points. This means that there will be one and only one straight line able to passing through A and B and to have the direction identified. In fact, if there were more straight lines able to do that, it would be no more true that for a given point (A or B) and the same direction we can define one and only one straight line, contradicting the postulate 1.2 on the preceding page.

■

Theorem 2. *Given a straight line and a point not on that line, there is one and only one straight line passing through that point and having its direction. We can observe, with regard to this, the following figure 4:*

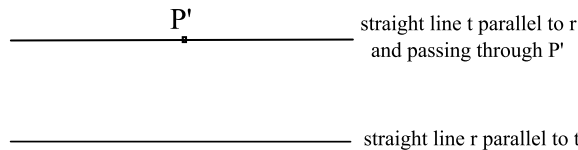


FIGURE 4. Straight line parallel to another given line and passing through a point not on it

Proof. The proof of this property derives directly from the postulate 1.2 on the previous page. In fact, given the point P' and the direction of the straight line r , we will identify one and only one straight line.

This property can be considered equivalent to the replaced axiom because as we will see later (theorem 12) two distinct straight lines having the same direction can have no point in common.

■

2. NEW FORMULATION OF THE PARALLELISM

Definition 3. *Straight lines that have the same direction are called parallel lines. We can observe, with regard to this, the following figure 5:*

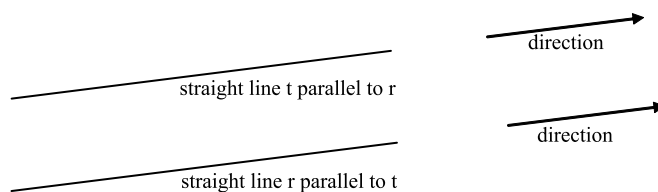


FIGURE 5. Direzione di due rette parallele

We note that the definition introduced here does not specify that the parallel lines should be distinct. This means that we must consider as parallel lines all the coincident straight lines, since they have the same direction.

Theorem 4. *Necessary and sufficient condition for the straight lines passing through the same point to be distinct, is that they have different directions. We can observe, with regard to this, the following figure 6:*

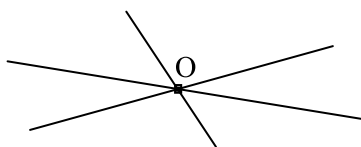


FIGURE 6. Straight lines passing through the same point

Proof. The proof of this property derives directly from the following considerations.

If the straight lines passing through the same point are distinct, they will necessarily have different directions. In fact, if they could have the same directions, they would not be distinct by the postulate 1.2 on page 1.

On the other hand, starting from the hypothesis that the straight lines passing through the same point have different directions, we can immediately conclude that they are distinct. In fact, if they were coincident, and therefore had all points in common, they would also have the same direction by the postulate 1.1 on page 1.

■

Theorem 5. *Given the point of intersection between two straight lines, the angles formed are univocally determined by the directions of the above-mentioned lines. We can observe, with regard to this, the following figure 7:*

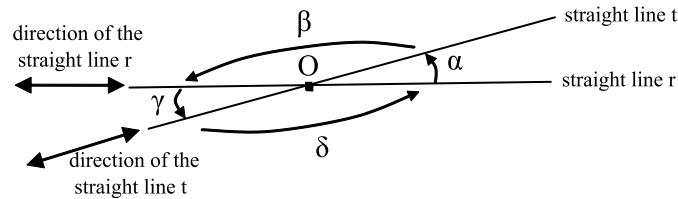


FIGURE 7. Angles formed by two intersecting straight lines

Proof. The proof of this property derives directly from the theorem 4 on the preceding page. In fact we can build an intersection equal to that of lines r and t , only drawing from the point O two straight lines having their directions.

■

Theorem 6. *Necessary and sufficient condition for two straight lines to be parallel is that they form with a transversal equal corresponding angles. We can observe, with regard to this, the following figure 8:*

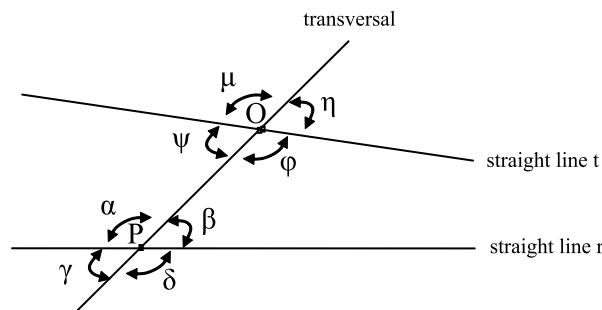


FIGURE 8. Angles formed by a transversal and two parallel straight lines

Proof. The proof of this property derives directly from the following considerations.

If the straight lines r and t are parallel, the points O and P represent the intersection of straight lines having the same directions, and therefore by the theorem 5 on the previous page they will identify the same angles. This means that the corresponding angles will be equal, that is $\alpha = \mu$, $\beta = \eta$, $\gamma = \psi$, $\delta = \varphi$.

On the other hand, starting from the hypothesis that the corresponding angles are equal, it is easy to verify that the straight lines r and t are parallel. In fact, by the theorem 5 on the preceding page, the points O and P can form equal angles starting from a straight line with the same direction (the transversal), only if even the other straight line whose are the intersection will have the same direction.

The theorem just proved allows us to consider the notion of parallelism between two straight lines as the presence of transversal lines that cut them at each of their points, forming equal corresponding angles.

■

Theorem 7. *Necessary and sufficient condition for two straight lines to be parallel is that a line perpendicular to one of them is also perpendicular to the other. We can observe, with regard to this, the following figure 9:*

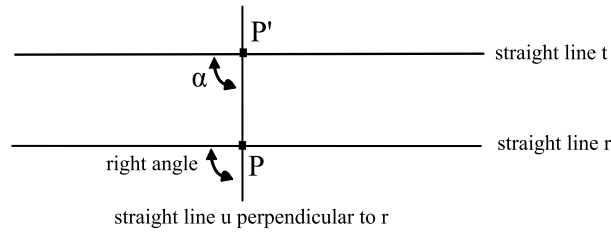


FIGURE 9. Straight line perpendicular to another given straight line

Proof. The proof of this property derives directly from the following considerations. If the straight lines r and t are parallel, by the theorem 6 on the preceding page, the angle α of the line t must be right, and therefore the line u will also be perpendicular to the line t .

On the other hand, starting from the hypothesis that the straight line u , perpendicular to the line r , is also perpendicular to the straight line t , we identify two equal corresponding angles, and therefore by the theorem 6 on the facing page we could consider the lines r and t parallel.

The theorem just proved allows us to consider the notion of parallelism between two straight lines as the presence of transversal lines that cut them at each of their points, forming right angles.

■

Theorem 8. *Necessary and sufficient condition for two straight lines to be parallel is that they are perpendicular to a third line. We can observe, with regard to this, the following figure 10:*

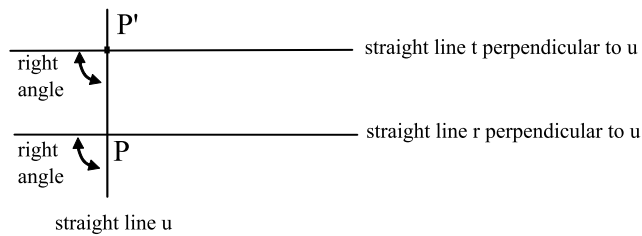


FIGURE 10. Two straight lines perpendicular to another given straight line

Proof. The proof of this property derives directly from the following considerations. If the straight lines r and t are perpendicular to the same straight line u , then they form equal corresponding angles, and therefore by the theorem 6 on the preceding page we could consider them parallel.

On the other hand, starting from the hypothesis that the lines r and t are parallel, it is easy to verify that if one of them is perpendicular to a third line, this line must also be perpendicular to the other, having to satisfy the equality among corresponding angles established by the theorem 6 on the facing page.

The theorem just proved allows us to consider the notion of parallelism between two straight lines as the ability of these lines to have in common the same perpendicular lines.

■

Theorem 9. *Necessary and sufficient condition for two straight lines to be parallel is that one of them has each point distant from the other of a same value, provided that the distances do not belong to both half-planes defined by that parallel. We can observe, with regard to this, the following figure 11:*

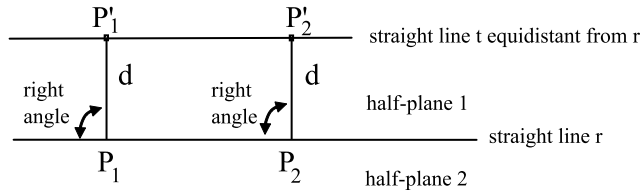


FIGURE 11. A straight line equidistant from another given straight line

Proof. The proof of this property derives directly from the following considerations. If the straight lines r and t are parallel, we must make reference to any point P'_1 lying on one of them. We know that there is only one perpendicular to the other line passing through that point, as we can see in [1, p.53,n.155]. Let P_1 be the foot of this perpendicular and d_1 be the distance between the two points so determined. Let us take another point P'_2 lying on the same line of P'_1 , but at an arbitrary distance from it, and let P_2 be the foot of the perpendicular to the other line passing through it, and d_2 be the distance between these two points. We can observe, with regard to this, the following figure 12:

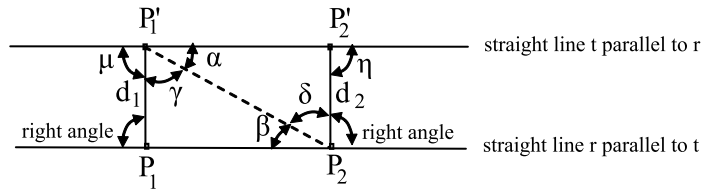


FIGURE 12. Graphical representation of the steps listed above

By the theorem 7 on page 4 we know that the lines perpendicular to the line r will also be perpendicular to the line t parallel to it. As a consequence we could consider the angles μ and η right. Moreover, by the theorem 6 on page 4 we know that the angles α and β formed by the transversal P'_1P_2 will be equal as well. This means that we could also consider the angles γ and δ equal because they are obtained by subtracting the same value from the straight angle formed by the lines r and t in the points P'_1 and P_2 . It follows that the triangles $P'_1P_1P_2$ and $P'_1P'_2P_2$ have the common side and the two adjacent angles equal, and therefore for the criteria of congruence of triangles that can be consulted in [1, p.26, n.105] we could consider them equal. But if the triangles $P'_1P_1P_2$ and $P'_1P'_2P_2$ are equal, so are the distances d_1 and d_2 .

By repeating this procedure, fixing the point P'_1 and changing the point P'_2 we prove that the line t is equidistant from r .

We note that not only the parallel lines are equidistant from each other, but they are equidistant of the same value because the perpendicular lines on which we determine the distances are perpendicular to both the lines, as we can deduce by the theorem 8 on the previous page. We can immediately reach this conclusion by observing that the

properties we use to calculate the distance between two parallel lines can be applied in the same way for all their points. Therefore, whatever be the point on which we determine the distance of a straight line from its parallel line, we will always end up to get the same value, because if we act in the same way on points having the same properties, we cannot obtain different results.

We note that this theorem is also valid in the specific case of two parallel lines coinciding with each other. In this case, the above-mentioned distance should always be considered equal to zero.

On the other hand, starting from the hypothesis according to which the straight lines r and t are equidistant from each other, we must make reference to the figure 11. The theorem 6 on page 4 ensures us that there is a straight line parallel to r passing through P'_1 . In fact, to identify this parallel line is enough to draw a straight line u passing through P'_1 which forms a right angle with the perpendicular line used to determine the distance of P'_1 from the line r . Let Q'' be the point of u whose distance from the line r is taken on the perpendicular to r passing through P_2 . We can observe, with regard to this, the following figure 13:

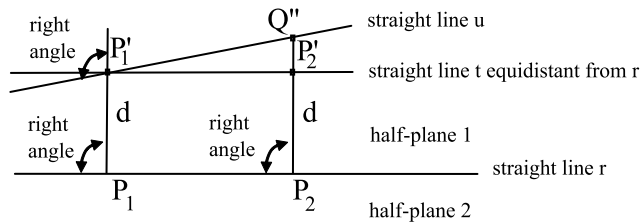


FIGURE 13. Graphical representation of the steps listed above

As we have seen in the first part of this proof, since the straight line u , parallel to r , passes through the point P'_1 distant d from r , it will have all other points to that distance from r , including Q'' . This means that the points P'_2 and Q'' , both located in the same side of the plane with respect to r and both distant d from it, will have to coincide. It follows that the line u and the line t will have two points in common, and by the theorem 1 on page 2 we could consider them coincident, and therefore conclude that the line t is parallel to r . The theorem just proved allows us to consider the notion of parallelism between two straight lines as the ability of these lines to be equidistant from each other.

We should note that since for two points there is one and only one straight line (theorem 1 on page 2), the above-mentioned condition of equivalence is satisfied by taking two lines of which one has two points equidistant from the other.

■

Theorem 10. *Given a straight line and a point on that line, there is one and only one parallel to this line passing through that point, and it will coincide with the given line. We can observe, with regard to this, the following figure 14:*

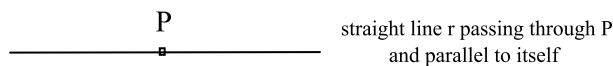


FIGURE 14. Straight line parallel to itself

Proof. The proof of this property derives directly from the following considerations. Given that any straight line passing through P would have in that point a distance equal to zero from the line r , by the theorem 9 on page 6 we could consider it a parallel of r , provided that all its points have a distance equal to zero from it. Therefore, any straight line parallel to r and passing through P will have to coincide with r . Another way of expressing this theorem is to say that given a straight line and a point on that line, there is no other parallel line passing through that point and at the same time being distinct from it.

■

Theorem 11. *Each segment of straight line is gradually moving farther and farther or closer and closer to any straight line to which is not parallel, provided that it does not extend to both half-planes defined by that line. We can observe, with regard to this, the following figure 15:*

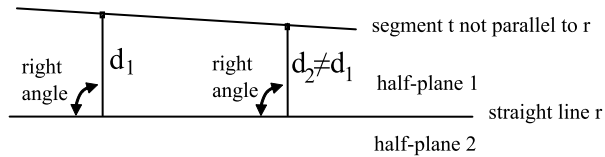


FIGURE 15. Segment of straight line not parallel to another given straight line

Proof. The proof of this property derives directly from the following considerations. Let us take into consideration for the line r the direction toward a right side, and assume that it is possible for the segment t moving both farther and closer to the line r . If this happens, there must be at least a point P' of t whose distance from r is equal to d , a point P'_1 preceding it and a point P'_2 succeeding it, whose corresponding distances d_1 and d_2 are both bigger than d (or both smaller, in which case the procedure is similar to the one presented here). We can observe, with regard to this, the following figure 16:

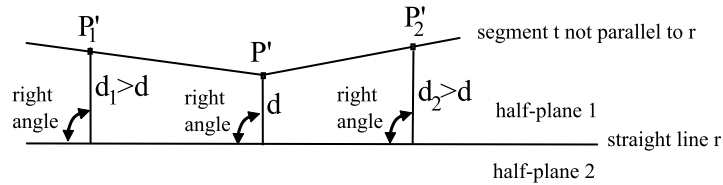


FIGURE 16. Graphical representation of the steps listed above

Let us draw the straight line parallel to r passing through P' (that we know to be existent by the theorem 2 on page 2 and distinct from the segment t), and call Q' and R' the points in which this parallel meets the lines d_1 and d_2 , located at a distance d from the line r by the theorem 9 on page 6.

We note that by construction the points Q' and R' will be different from those P'_1 and P'_2 located at a superior height. We can observe, with regard to this, the following figure 17:

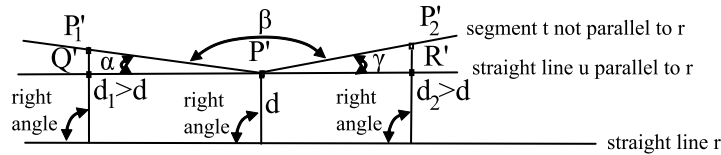


FIGURE 17. Graphical representation of the steps listed above

Given that the line u is distinct from the segment t , they may have only the point P' in common and therefore the figures $P'_1Q'P'$ and $P'_2R'P'$ will represent a triangle and the angles α and γ will be different from zero.

Since both points P'_1 and P'_2 have a distance from the line r that is bigger than the distance d shown by the points Q' and R' , the angles α and γ will occupy the same part of the plan with regard to the line $Q'R'$, making the angle β smaller than the straight angle that the line $Q'R'$ forms in the point P' , as shown in [1, p.15, n.66]. It follows that the segment t is not really a straight segment, and therefore will not exist any segment able to moving both farther and closer to a straight line of which is not parallel, provided that we avoid the segments that extend to both half-planes defined by that line.

■

Theorem 12. *Necessary and sufficient condition for two straight lines to be parallel is that although they may be extended indefinitely will not have any point in common. We can observe, with regard to this, the following figure 18:*

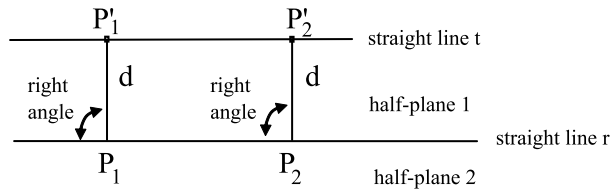


FIGURE 18. Two points of a straight line equidistant from another given straight line

Proof. The proof of this property derives directly from the following considerations. If the straight line t and r are parallel and distinct from each other, they can not have any point in common as far as we can extend them. In fact, if they had a point in common, they could not be distinct and having the same direction by the theorem 4 on page 3.

On the other hand, starting from the hypothesis that it is impossible to give a point in common to the straight lines t and r , even extending them indefinitely, it is easy to verify that they are parallel. Let us assume, in this regard, that they are not parallel, and therefore that the line t is moving farther and farther or closer and closer to the line r (if not, it would be possible to identify a segment inside t moving both farther and closer to the line r , which is impossible by the theorem 11 on the preceding page). In this case, if we could extend these lines toward the side on which they are moving closer and closer, their distance will inevitably end up to be zero, and they will meet

The theorem just proved allows us to consider the notion of parallelism between two straight lines as the impossibility for it to have points in common.

■

Theorem 13 (Fifth postulate of Euclid). *If a transversal to two straight lines makes the conjugate interior angles less than one straight angles, the two straight lines, if extended indefinitely, meet on that side on which are the conjugate interior angles less than the straight angles. We can observe, with regard to this, the following figure 19:*

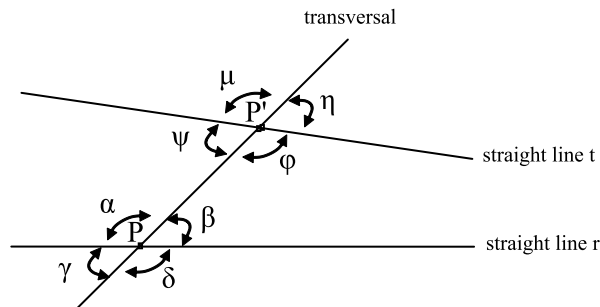


FIGURE 19. Two given straight lines cut by a transversal

Proof. The proof of this property derives directly from the following considerations. If the conjugate angles α and ψ are not supplementary (a similar argument it is valid for the conjugate angles β and φ), the angles γ and μ to which they are instead supplementary result distinct from the corresponding ψ and α , and therefore by the theorem 6 on page 4 we could conclude that the line t and r are not parallel. It follows that by the theorem 12 on the previous page the line t and r will have a point O in common. Since this point forms a triangle with the points P and P' , it will necessarily have to be on the side of the conjugate angles whose sum is smaller than one straight angle, as we can deduce by reading [1, p.52, n.153].

■

REFERENCES

- [1] Hart, C.A. and Feldman, D.D. and Tanner, J.H. and Snyder, V. , Plane geometry, American book company, London, 1911.

INDEPENDENT SCHOLAR, MILAN-ITALY
E-mail address: nicola.dalfonso@hotmail.com