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Elementary Geometric Determination of all Symmetry Types of Regular Skew Hexagons

Rolfdieter Frank & Heinz Schumann

Abstract. *By definition, a regular skew hexagon is equilateral, equiangular and not contained in a plane. We determine six different symmetry types of regular skew hexagons and show that there are no other types. Our proof only makes use of the classification of the isometries of Euclidean 3-space and of properties of the dihedral group D_6. For each of the six symmetry types, we describe a corresponding hexagon with its symmetries and vertex coordinates. – Such hexagons occure as kernel structures of cyclohexane molecules. Furthermore hexagons of four of the six types are flexible with invariant angles and sides, because they are associated with flexible Bricard octahedra.*

In our paper "Ueber 'regelmaeßige' raeumliche Polygone" (On 'regular' spatial polygons), Frank & Schumann 2019 ([1]), we raised the problem to determine all the types of different symmetry groups of such spatial polygons with a given number of vertices. For vertex number 6, we constructed examples of six different symmetry types and conjectured that there are no other types. The aim of the present paper is an elementary geometric proof of our conjecture. It only makes use of the classification of the isometries of Euclidean 3-space and of properties of the dihedral group $\mathbf{D_6}$. For each of the six possible symmetry types determined in the proof, we then describe a corresponding skew hexagon with its symmetries and vertex coordinates.

The determination

Definition A hexagon $A_1A_2A_3A_4A_5A_6$ is regular, if the distances A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_1 are equal and also the distances A_1A_3, A_2A_4, A_3A_5, A_4A_6, A_5A_1, A_6A_2, i. e. it is regular, if it is equilateral and equiangular. – It is skew if it is not contained in plane.

Theorem Only the following six groups can be the symmetry group of a regular skew hexagon: The group of the 12 symmetries of a regular 3-antiprism (type 1); the group consisting of the identity, the reflections in two orthogonal planes and the reflection in their line of intersection (type 2); the group of the 12 symmetries of a regular 3-prism (type 3); the group consisting of the identity and the reflections in a plane, in a line orthogonal to this plane and in their point of intersection (type 4); the group consisting of the identity and the reflections in three mutually orthogonal lines (type 5); the group consisting of the identity and the reflection in a line (type 6).
Each of the six groups contains a symmetry, which maps every vertex of the hexagon to its opposite vertex; this symmetry is a point reflection in type 1, a plane reflection in type 3 and a line reflection in all the other types.

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Proof of the theorem

Preparation: Let $A_1A_2A_3A_4A_5A_6$ be regular skew hexagon. Every symmetry of this hexagon induces a permutation of its vertex set $\{A_1, A_2, A_3, A_4, A_5, A_6\}$, which preserves edges and all distances. Conversely, every such permutation **p** can be extended in a unique way to an isometry $\bar{\mathbf{p}}$ of Euclidean 3-space. Then $\bar{\mathbf{p}}$ is a symmetry of the hexagon, and $\bar{\mathbf{p}}$ induces **p**.

Structure: In a first step, we will consider not the symmetries, but only the induced vertex permutations. We will find only three possible permutation groups, depending whether the three main diagonals A_1A_4, A_2A_5, A_3A_6 are pairwise of different length (case 1), or all of equal length (case 2) or whether only two of the main diagonals are of equal length (case 3). Afterwards we determine the symmetry groups, which may induce these permutation groups. It will turn out that in case 1 only type 6 is possible, in case 2 only types 1 and 3 and in case 3 only types 2, 4 and 5.
Now we denote by $\mathbf{D_6}$ the group of permutations of the vertex set of a regular plane hexagon, which is induced by its 12 symmetries. It consists precisely of those permutations, which preserve edges. Thus the set of permutations induced by the symmetries of a skew hexagon is a subgroup **U** of $\mathbf{D_6}$. The elements of **U** are precisely those permutations of $\{A_1, A_2, A_3, A_4, A_5, A_6\}$, which preserve edges and all distances.
Case 1: The elements of $\mathbf{U_1}$ are induced by those symmetries of the regular plane hexagon, which map every main diagonal to itself, hence by the identity and by the halfturn. It follows $\mathbf{U_1} = \{\text{id}, (A_1A_4)(A_2A_5)(A_3A_6)\}$.
Case 2: The elements of $\mathbf{U_2}$ are induced by all symmetries of the regular plane hexagon; it follows $\mathbf{U_2} = \mathbf{D_6}$.
Case 3: The elements $\mathbf{U_3}$ are induced by those symmetries of the regular plane hexagon, which map one fixed main diagonal to itself. Without loss of generality we may assume this main diagonal to be A_1A_4. Then we get $\mathbf{U_3} = \{\text{id}, (A_1A_4)(A_2A_5)(A_3A_6), (A_1A_4)(A_2A_3)(A_5A_6), (A_1)(A_4)(A_2A_6)(A_3A_5)\}$.

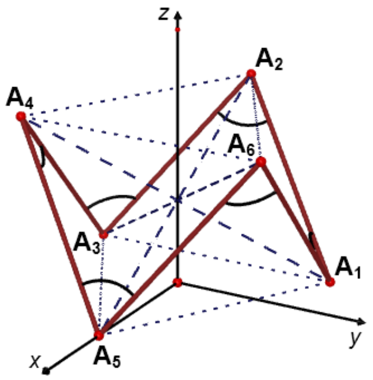
Execution: Now we are ready to determine the symmetry groups.
The order of the symmetry $\mathbf{s_1} := \overline{(A_1A_4)(A_2A_5)(A_3A_6)}$ is 2, since this is the order of the permutation $(A_1A_4)(A_2A_5)(A_3A_6)$. Hence $\mathbf{s_1}$ must be a reflection in a line, in a plane, or in a point. If $\mathbf{s_1}$ is a reflection in a plane, then $A_1A_4A_5A_2$, $A_5A_2A_3A_6$, $A_3A_6A_1A_4$ are congruent isosceles trapezia fitting cyclically together, hence congruent rectangles, and we have case 2.
If $\mathbf{s_1}$ is a reflection in a point, then we choose the origin to be the center of reflection and get, with italic writing of corresponding vectors, $A_1 = -A_4$, $A_2 = -A_5$, $A_3 = -A_6$. We conclude $(A_1 + A_3)^2 = (A_1 - A_6)^2 = (A_5 - A_6)^2 = (A_3 + A_5)^2 = (A_3 - A_2)^2 = (A_1 - A_2)^2 = (A_5 + A_1)^2$. We substract $(A_1 - A_3)^2 = (A_3 - A_5)^2 = (A_5 - A_1)^2$ from $(A_1 + A_3)^2 = (A_3 + A_5)^2 = (A_5 + A_1)^2$ and obtain $4A_1 \cdot A_3 = 4A_3 \cdot A_5 = 4A_5 \cdot A_1$. It follows $A_1^2 = A_3^2 = A_5^2$, and so we have again case 2.
Therefore $\mathbf{s_1}$ must be a line reflection in case 1, and so $\mathbf{U_1}$ is induced by a symmetry group of type 6.
In case 2, $(A_1A_2A_3A_4A_5A_6)$ is in $\mathbf{U_2}$, and $(A_1A_2A_3A_4A_5A_6)^3 = (A_1A_4)(A_2A_5)(A_3A_6)$ implies $\mathbf{s_2}^3 = \mathbf{s_1}$ for $\mathbf{s_2} := \overline{(A_1A_2A_3A_4A_5A_6)}$. If $\mathbf{s_1}$ were a line reflection, then $\mathbf{s_2}$ would be orientation preserving and hence a rotation and consequently $A_1, A_2, A_3, A_4, A_5, A_6$ were contained in a plane. If $\mathbf{s_1}$ is a plane reflection, then $\mathbf{s_2}$ must be a ±120°-rotary reflection, and consequently $A_1, A_2, A_3, A_4, A_5, A_6$ are the vertices of a regular 3-prism (type 3).

If $\mathbf{s_1}$ is a point reflection, then $\mathbf{s_2}$ must be a ±60°-rotary reflection, and consequently A_1, A_2, A_3, A_4, A_5, A_6 are the vertices of a regular 3-antiprism (type 1).
In case 3, $\mathbf{U_3}$ is isomorphic to the Klein 4-group, and as in case 1, $\mathbf{s_1}$ must be a line reflection. The symmetries $\mathbf{s_3}:= \overline{(A_1A_4)(A_2A_3)(A_5A_6)}$ and $\mathbf{s_4}:= \overline{(A_1)(A_4)(A_2A_6)(A_3A_5)}$ can only be reflections in a line, a plane, or a point, and $\mathbf{s_4}$ cannot be a point reflection because of its two fixed points A_1 and A_4. If $\mathbf{s_3}$ is a line reflection, then $\mathbf{s_4} = \mathbf{s_1} \circ \mathbf{s_3}$ orientation preserving, and hence a line reflection (type 5). If $\mathbf{s_3}$ is a reflection in a plane or in a point, then $\mathbf{s_4}$ is orientation reversing, and hence a plane reflection (type 2 or 4) **q. e. d.**

Existence of the six types of regular skew hexagons

By a survey we show selected representatives for the six types of regular skew hexagons with their vertex-coordinates und symmetries.

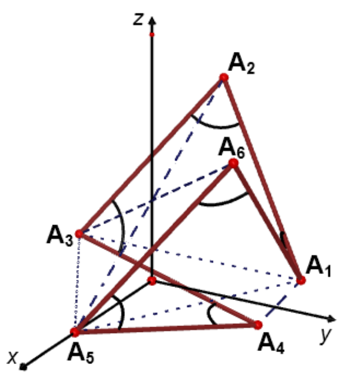
The hexagon-types with their vertex-coordinates and main diagonals
A hexagon of type 1 consists of the Petrie-polygon of a straight 3-antiprism with equilateral base.



$A_1(-\sqrt{3}/4, 3/4, 0)$
$A_2(-\sqrt{3}/2, 0, 1)$
$A_3(-\sqrt{3}/4, -3/4, 0)$
$A_4(\sqrt{3}/4, -3/4, 1)$
$A_5(\sqrt{3}/2, 0, 0)$
$A_6(\sqrt{3}/4, 3/4, 1)$
$|A_1A_4| = |A_2A_5| = |A_3A_6|$

Fig. 1.1 Hexagon of type 1 with its coordinates and main diagonals.

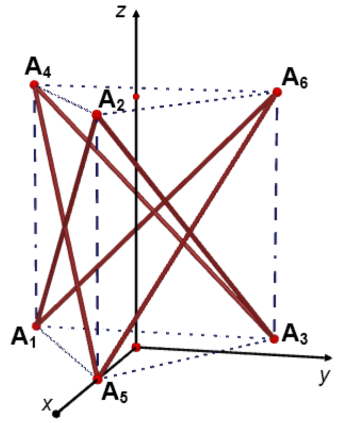
A hexagon of type 2 is to be constructed by reflection of A_4 in plane $A_2A_3A_5A_6$ in figure 1.1.



$A_1(-\sqrt{3}/4, 3/4, 0)$
$A_2(-\sqrt{3}/2, 0, 1)$
$A_3(-\sqrt{3}/4, -3/4, 0)$
$A_4(-5\sqrt{3}/28, 15/28, -2/7)$
$A_5(\sqrt{3}/2, 0, 0)$
$A_6(\sqrt{3}/4, 3/4, 1)$
$|A_1A_4| \neq |A_2A_5| = |A_3A_6|$

Fig. 2.1 Hexagon of type 2 with its coordinates and main diagonals.

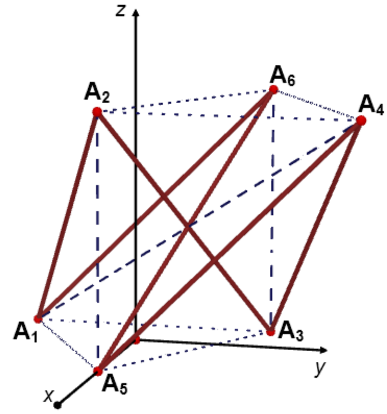
A hexagon of type 3 consists of the diagonals of a straight 3-prism with equilateral base.



$A_1(-1/2, -\sqrt{3}/2, 0)$
$A_2(1, 0, 1)$
$A_3(-1/2, \sqrt{3}/2, 0)$
$A_4(-1/2, -\sqrt{3}/2, 1)$
$A_5(1, 0, 0)$
$A_6(-1/2, \sqrt{3}/2, 1)$
$|A_1A_4| = |A_2A_5| = |A_3A_6|$

Fig. 3.1 Hexagon of type 3 with its coordinates and main diagonals.

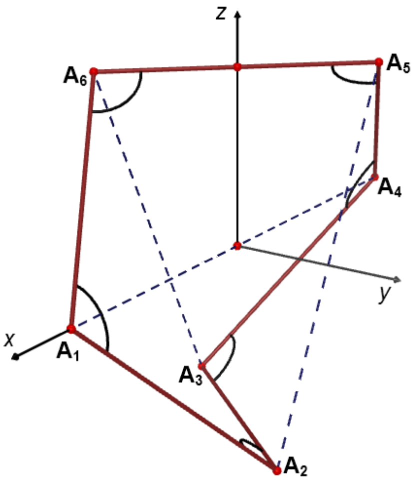
A hexagon of type 4 is to be constructed by reflection of A_4 in plane $A_2A_3A_5A_6$ in figure 3.1.



$A_1(-1/2, -\sqrt{3}/2, 0)$
$A_2(1, 0, 1)$
$A_3(1/2, \sqrt{3}/2, 0)$
$A_4(1, \sqrt{3}, \sqrt{3})$
$A_5(1, 0, 0)$
$A_6(-1/2, \sqrt{3}/2, 1)$
$|A_1A_4| \neq |A_2A_5| = |A_3A_6|$

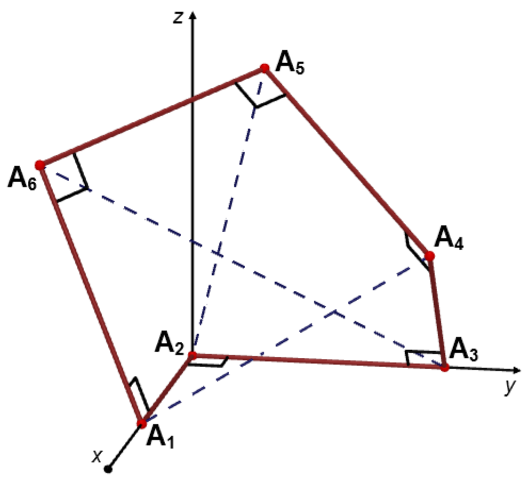
Fig. 4.1 Hexagon of type 4 with its coordinates and main diagonals.

Hexagons of types 5 and 6 are to be constructed by calculation with regard to their axial symmetry. – There exist non rectangular hexagons of type 6 too.



$A_1(1,0,0)$
$A_2(1/3, 2\sqrt{3}/9, -2/3)$
$A_3(-1/3, -2\sqrt{3}/9, -2/3)$
$A_4(-1,0,0)$
$A_5(-1/3, 2\sqrt{3}/9, 2/3)$
$A_6(1/3, -2\sqrt{3}/9, 2/3)$
$|A_2A_5| = |A_3A_6| = 2\sqrt{5}/3 \neq |A_1A_4|$

Fig. 5.1 Hexagon of type 5 with its coordinates and main diagonals.



$A_1(1,0,0)$
$A_2(0,0,0)$
$A_3(0,1,0)$
$A_4((3+\sqrt{65})/14,1,(3\sqrt{13}-\sqrt{5})/14)$
$A_5(1/3,1/3,\sqrt{13}/3)$
$A_6(1,(3-\sqrt{65})/14,(3\sqrt{13}+\sqrt{5})/14)$

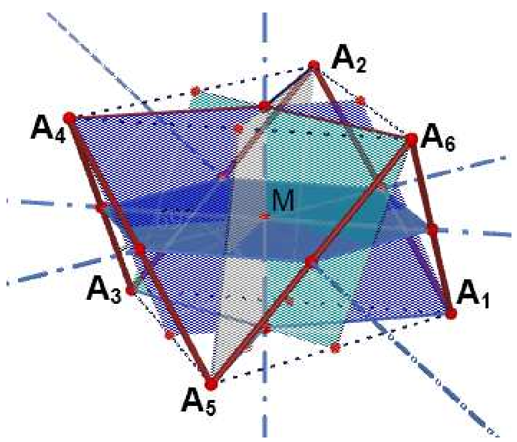
$|A_1A_4| = \sqrt{126-\sqrt{65}}/7 \approx 1{,}192$
$|A_2A_5| = \sqrt{15}/3 \approx 1{,}291$
$|A_3A_6| = \sqrt{126+\sqrt{65}}/7 \approx 1{,}930$

Fig. 6.1 Hexagon of type 6 with its coordinates and main diagonals.

Remark about the parameterization of the symmetry types
It is a difficult task to parameterize the hexagons of types 5 and 6.

The hexagon-types with their elements of symmetry and symmetries
In the following, the elements of symmetry resp. the symmetries of the six types of regular skew hexagons are described by means of the permutations of their vertices in cyclic notation. There we use the concept of the abstract dihedral group, which is isomorphic to the symmetry group of the regular plane hexagon as well as the abstract KLEIN 4-group, which consists of three elements of order 2 and the neutral element.

Symmetry-type 1
The 12 symmetries of a straight 3-antiprism with equilateral base are transferred to its Petrie-polygon.



2 Rotations:
$(A_1A_3A_5)(A_2A_4A_6)$,
$(A_1A_5A_3)(A_2A_6A_4)$

3 Reflections in axes (Halfturns):
$(A_1A_6)(A_2A_5)(A_3A_4)$,
$(A_1A_4)(A_2A_3)(A_5A_6)$,
$(A_1A_2)(A_3A_6)(A_4A_5)$

Reflection in a point:
$(A_1A_4)(A_2A_5)(A_3A_6)$

3 Reflections in planes:
$(A_1)(A_4)(A_2A_6)(A_3A_5)$,
$(A_2)(A_5)(A_1A_3)(A_4A_6)$,
$(A_3)(A_6)\ (A_1A_5)(A_2A_4)$

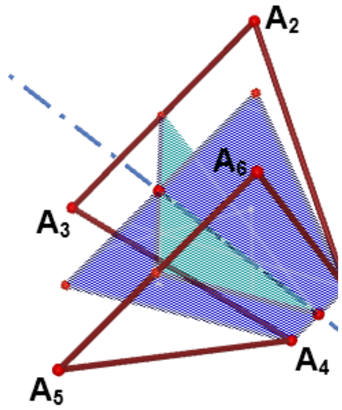
2 Rotary-Reflections:
$(A_1A_2A_3A_4A_5A_6)$,
$(A_6A_5A_4A_3A_2A_1)$

Fig. 1.2 Type 1 with its elements of symmetry and symmetries.

±120°-rotation around a threefold axis containing the centers of the base triangles; three line reflections (halfturns), in each case around an axis connecting the midpoints of opposite edges; reflection in the center of the antiprism; three reflections in planes, in each case determined by two opposite vertices and the threefold axis; the rotary-reflection consisting of ±60°-rotation around the threefold axis and the reflection in the plane through the midpoints of the edges. Together with the identity the eleven symmetries form a concrete group $\mathbf{D_6}$. – This hexagon is completely symmetric.

Symmetry-type 2

This hexagon is symmetric to the perpendicular bisector planes of the sides A_2A_3 resp. A_5A_6 and the segments A_3A_5 resp. A_2A_6, mutually perpendicular; therefore the hexagon is axial symmetric to the intersection of these planes. Together with the identity these three symmetries form a concrete KLEIN 4-group.



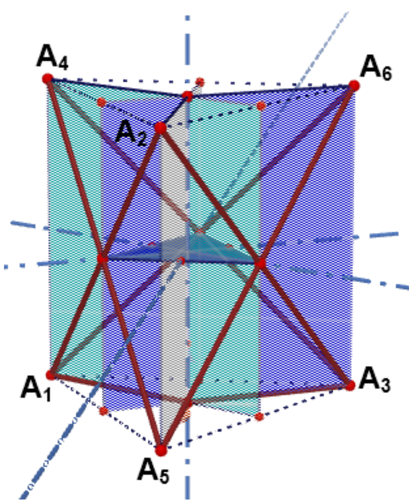
Reflection in an axis (Halfturn):
$(A_1A_4)(A_2A_5)(A_3A_6)$

2 Reflections in planes:
$(A_1A_4)(A_2A_3)(A_5A_5)$,
$(A_1)(A_4)(A_2A_6)(A_3A_5)$

Fig. 2.2 Type 2 with its elements of symmetry and symmetries.

Symmetrie-type 3

The symmetries of the type 3 are those of a straight 3-prism with regular base: ±120°-rotation around a threefold axis, 3 reflections in axes (halfturns) through midpoints of edges and centers of opposite prism sides, 3 reflections in planes through these rotational axes, 2 rotary-reflections consisting of ±60°-rotation around the threefold axis and a reflection in a plane orthogonal to these axis. Together with the identity the eleven symmetries form a concrete group $\mathbf{D_6}$. – This hexagon is completely symmetric.



2 Rotations:
$(A_1A_3A_5)(A_2A_4A_6)$,
$(A_1A_5A_3)(A_2A_6A_4)$

3 Reflections in axes (Halfturns):
$(A_1A_2)(A_3A_6)(A_4A_5)$,
$(A_2A_3)(A_4A_1)(A_5A_6)$,
$(A_3A_4)(A_5A_2)(A_6A_1)$

4 Reflections in planes:
$(A_1)(A_4)(A_2A_6)(A_3A_5)$,
$(A_2)(A_5)(A_1A_3)$ (A_4A_6),
$(A_3)(A_6)$ $(A_1A_5)(A_2A_4)$,
$(A_1A_4)(A_2A_5)(A_3A_6)$

2 Rotary-Reflections:
$(A_1A_2A_3A_4A_5A_6)$, $(A_6A_5A_4A_3A_2A_1)$

Fig. 3.2 Type 3 with its elements of symmetry and symmetries.

Symmetrie-type 4

The type 4 is symmetric to the intersection of the mutually intersecting sides A_2A_3, A_5A_6; it is symmetric to the perpendicular bisector plane of the sides A_2A_3 resp. A_5A_6 and to the axis through the midpoints of A_2A_5 and A_3A_6. Together with the identity these three symmetries form a concrete KLEIN 4-group.

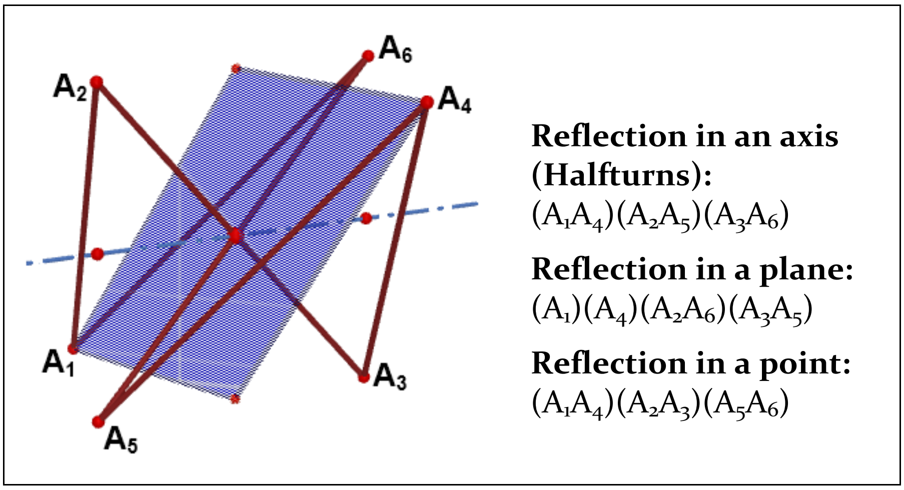


Fig. 4.2 Type 4 with its elements of symmetry and symmetries.

Symmetrie-type 5

The type 5 owns three reflections in axes (halfturns), one axis is a main diagonal, the second one connects the midpoints of two opposite sides, which are opposite to the above main diagonal, and the third goes through the intersection of the above two axes and is perpendicular to the plane spanned by these axes. Together with the identity these three symmetries form a concrete KLEIN 4-group.

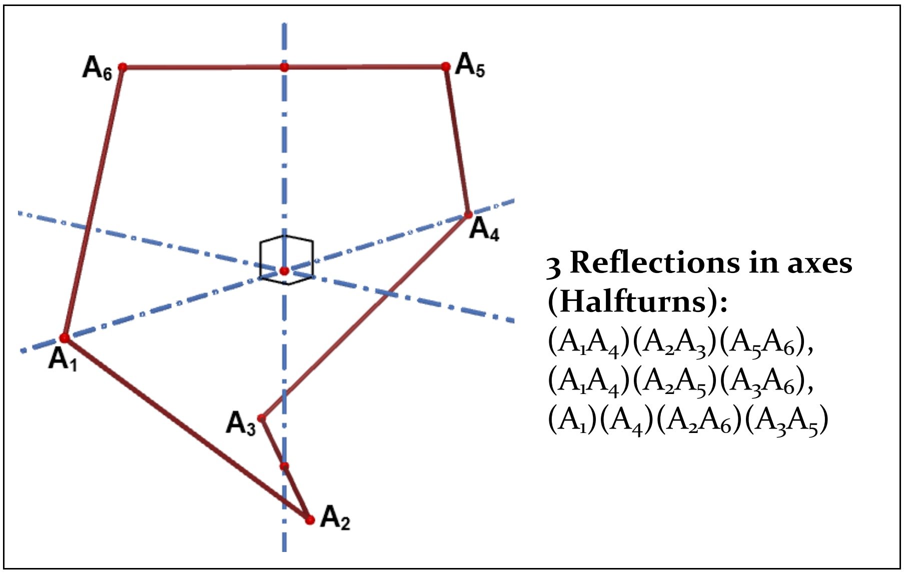


Fig. 5.2 Type 5 with its elements of symmetry and symmetries.

Symmetrie-type 6

The type 6 owns only one reflection in an axis (halfturn), which goes through the midpoints of its main diagonals. Together with the identity this symmetry form a group of order 2.

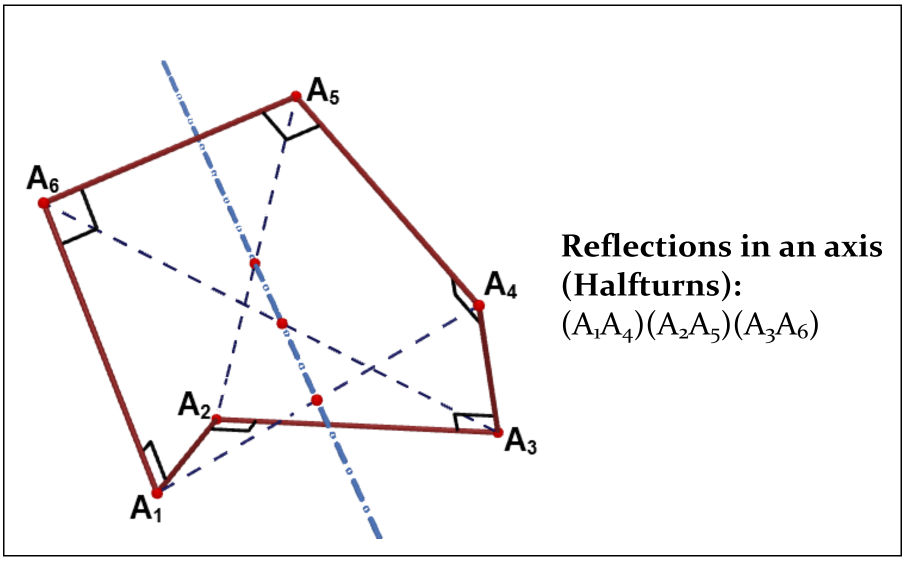


Fig. 6.2 Type 6 with its elements of symmetry and symmetries.

Final remarks

<u>Remark about an application of the symmetry types</u>

The symmetry types are important for the analysis of molecule structures of cyclohexane, a component of petroleum (Wirth 2018).

<u>Remark about the flexibiliy of the symmetry types</u>

Associated with each skew hexagon is an octahedron with triangular faces $A_1A_2A_3$, $A_2A_3A_4$, $A_3A_4A_5$, $A_4A_5A_6$, $A_5A_6A_1$, $A_6A_1A_2$, $A_1A_3A_5$, $A_2A_4A_6$. According to Wunderlich (1965), this octahedron is a flexible Bricard octahedron, if the permutation $(A_1A_4)(A_2A_5)(A_3A_6)$ is induced by a line reflection. This is the case with symmetry types 2, 4, 5 and 6. Hence regular skew hexagons of these types are flexible with invariant angles and sides. – According to Wirth (2018), flexible hexagons of types 2, 5 and 6 occure as transistory shapes of the cyclohexane molecule, which normally is in the stable shape of type 1.

<u>Remark about future tasks</u>

It remains a challenging task to find symmetry classifications for equilateral and equiangular skew polygons with more vertices than six.

Literature

[1] Frank, R.D. & Schumann, H. (2019): Über „regelmäßige" räumliche Polygone. In: Informationsblätter der Geometrie (IBDG), Jg. 38, 2, 23 – 30.

[2] Wirth, K. (2018): Aufgabe 1372. „Elemente der Mathematik", Vol. 73, 37 – 43.

Auswertung der eingesandten Lösungen von Aufgabe 1372: Lösung von Henri Carnal (2019). In: „Elemente der Mathematik", Vol. 74, 40 – 42.

[3] Wirth, K. (2018): Determining the metric and the symmetry group of finite point sets in space with an application to cyclohexane. Journal of Mathematical Chemistry, 56, 213 – 231.

[4] Wunderlich, W. (1965): Starre, kippende, wackelige und bewegliche Achtflache. „Elemente der Mathematik", Vol. 20, 25 – 32.

Link: https://de.wikipedia.org/wiki/Cyclohexan

The diagrams were created with Cabri 3D.

Special acknowledgement
The authors thank F. Siegrist and K. Wirth for the information on their found classification of regular skew hexagons shown in a diagram of examples of the six types with the names of the corresponding symmetry groups. Two of these examples, those only with axial symmetry, had not be known to the authors before.

Authors

Prof. Dr. Rolfdieter Frank, Departement 3, Mathematics ,
University Koblenz-Landau, Location Koblenz, Germany
frank@uni-koblenz.de

Prof. Dr. Heinz Schumann, Faculty II, Mathematics,
University of Education Weingarten, Germany
Email: schumann@ph-weingarten.de