Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.9, (2020), Issue 2, pp.140-147

DUAL CONNECTIONS ON A HILBERT BUNDLE OF GENERALIZED STATISTICAL MANIFOLDS

AMOUR T. GBAGUIDI AMOUSSOU AND CARLOS OGOUYANDJOU

ABSTRACT. In this paper, we propose a Hilbert bundle of generalized statistical manifold with the introduction of a family of α-connections. The expressions of parallel transport for the exponential and mixture connections are derived, and we prove that those two connections are dual and curvature-free.

1. INTRODUCTION

Information geometry uses differential geometry techniques to study probability distributions viewed as points of statistical manifolds. Traditionnally, connections are key tools of differential geometry and they have been widely investigated for manifolds considered in information geometry. Amari[2] defined a family of α-connections related to the Fisher information metric on a statistical manifold. Later, α-connections related to the Fisher information metric on a Hilbert bundle of statistical manifold have been proposed by Amari and Kumon[3]. Crasmareanu et al.[4] extend the notion of statistical structure from Riemannian geometry to the general framework of path spaces endowed with a nonlinear connection and a generalized metric. Vigelis et al.[11] introduced a generalized statistical manifold by considering a new metric and α-connections based on φ-function, which generalize Fisher information metric and its α-connections. In a generalized statistical manifold, the metric and connections are defined in terms of $\phi^{-1} \circ p$ where ϕ is a φ-function and p is a probability density function. In the case of Fisher information metric, the φ-function is the exponential map. Recently, Gbaguidi et al.[7] proposed a family of α-connections on a Hilbert bundle of generalized statistical manifold but the proposed (1)-connection and (-1)-connection are not necessarily dual. In this paper, we construct a Hilbert bundle of generalized statistical manifold and we define a family of α-connections on this Hilbert bundle. We derive the α-parallel transport associated with α-connection for $\alpha = \pm 1$. We prove that our proposed exponential connection and mixture connection are curvature free and we establish their the duality result.

The rest of the paper is organized as follows. In section 2, the relevant results on generalized statistical manifold are recalled. Section 3 is devoted to the main results: the proposed Hilbert bundle of generalized statistical manifold is constructed as well as our

2010 *Mathematics Subject Classification.* Primary 53C05; Secondary 53C21; 55R10.
Key words and phrases. Riemanian manifold; Generalized statistical manifold; Hilbert bundle; α-connection; Dual connection; Parallel transport.

new α-connections and the properties of the exponential and mixture connections are established.

2. SOME NOTIONS ON GENERALIZED STATISTICAL MANIFOLD

Here, we briefly summarize some useful results on generalized statistical manifolds (see[11, 6, 5]).
Let (χ, Σ, μ) be a measure space and set
$P_\mu = \{p \in L^0 : p > 0, \int_\chi p(x;\theta)d\mu(x) = 1\}$ where L^0 denotes the set of all real-valued, measurable functions on χ. [11] A function $\phi : \mathbb{R} \to (0, \infty)$ is said to be a ϕ-function if :

(i) ϕ is convex,
(ii) $\lim_{x\to-\infty} \phi(x) = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$,
(iii) There exists a measurable function $u_0 : \chi \to (0, \infty)$ such that for all measurable function $c : \chi \to \mathbb{R}$ satisfying $\int_\chi \phi(c(x))d\mu(x) = 1$, we have $\int_\chi \phi(c(x) + \lambda u_0(x))d\mu(x) < \infty$, for all $\lambda > 0$.

The function defined by $\phi(x) = \exp(ax + b), a \in (0; \infty), b \in \mathbb{R}, \quad \forall x \in \mathbb{R}$ is a φ-function with $u_0 = \mathbf{1}_\mathbb{R}$ (indicator function of $\mathbb{R}$) and μ is the Lebesgue measure.
The Kaniadakis'κ-exponential (see[5]) $\exp_\kappa : \mathbb{R} \to (0, \infty)$ is defined by

- for $\kappa = 0$, $\exp_\kappa$ is the usual exponential map,
- for $\kappa \in [-1, 0[\cup]0, 1]$, $\exp_\kappa(x) = (\kappa x + \sqrt{1 + \kappa^2 x^2})^{1/\kappa}$.

Let $u_0 : \chi \to (0, \infty)$ be any measurable function for which $\int_\chi \exp_\kappa(u_0)d\mu < +\infty$. The Kaniadakis'$\kappa$-exponential is another example of φ-function.
φ-functions are useful to define generalized statistical manifold (see [11]).

Let ϕ be a invertible and smooth φ-function with u_0 a fixed function satisfied the condition (iii). A generalized statistical manifold with respect to ϕ is a family of probability distributions

$$M = \{p(\cdot;\theta) : \theta \in \Theta\} \subset P_\mu$$

such that:

(1) Θ is an open and connected set in $\mathbb{R}^n$.
(2) Each $p(.,\theta)$ is given in terms of $\theta \in \Theta$ by a one to one mapping.
(3) Every function $p(x;\cdot)$ is smooth for all x and the operations of integration with respect to μ and differentiation with respect to θ^i (i.e. $\partial/\partial\theta^i$) are always commutative.
(4) The support of $p(\cdot,\theta)$ does not depend on θ for all $\theta \in \Theta$.
(5) The matrix $g = (g_{ij})$, which is defined by

$$g_{ij} = -\,\mathrm{E}'_\theta \left[\frac{\partial^2 f_\theta}{\partial\theta^i \partial\theta^j}\right] \tag{2.1}$$

is positive definite at each $\theta \in \Theta$, where $f_\theta(\cdot) = \phi^{-1}(p(\cdot;\theta))$ and

$$\mathrm{E}'_\theta [\cdot] = \frac{\int_\chi (\cdot)\phi^{(1)}(f_\theta)d\mu}{\int_\chi u_0\phi^{(1)}(f_\theta)d\mu}. \tag{2.2}$$

g_{ij} is invariant under reparametrization. When ϕ is the usual exponential function and $u_0 = 1$, g is the Fisher information matrix. Given a generalized statistical manifold, the following lemma holds.

Lemma 2.1. [11] *For $i, j \in \{1, 2, \cdots\}$ and $\theta \in M$,*

$$\mathrm{E}'_\theta\left[\frac{\partial f_\theta}{\partial \theta^i}\right] = 0 \quad \text{and} \quad g_{ij} = \mathrm{E}''_\theta\left[\frac{\partial f_\theta}{\partial \theta^i}\frac{\partial f_\theta}{\partial \theta^j}\right],$$

where

$$\mathrm{E}''_\theta[\cdot] = \frac{\int_\chi (\cdot)\phi^{(2)}(f_\theta)d\mu}{\int_\chi u_0\phi^{(1)}(f_\theta)d\mu}. \tag{2.3}$$

Set θ a point of the manifold M. The tangent space $\top_\theta M$ at θ is spanned by the tangent vector $\partial_i = \partial/\partial\theta^i$ of the i-th coordinate curve θ^i $i = 1, \cdots, n$ at θ. Hence any tangent vector $A \in \top_\theta M$ can be written as $A = A^i\partial_i$, where A^i $i = 1, \cdots, n$ are the components of vector A and Einstein's summation convention is used throughout the paper. The inner product of two basis vectors ∂_i and ∂_j induces a metric in $\top_\theta M$. Representing $\partial_i \in \top_\theta M$ by $\partial_i f_\theta$, the metric is defined by

$$g_{ij}(\theta) = g(\partial_i, \partial_j) = \langle \partial_i f_\theta, \partial_j f_\theta\rangle = \mathrm{E}''_\theta\left[\partial_i f_\theta \partial_j f_\theta\right], \forall\theta \in M. \tag{2.4}$$

3. Main results

3.1. **The Hilbert bundle of generalized statistical manifold.** Let Y denote the set of smooth an μ-integrable functions r defined from χ to $\mathbb{R}$. Consider a smooth and bijective φ-function $\phi : \mathbb{R} \to (0, \infty)$ such that $\forall x \in (0, \infty)$, $\phi^{(2)}(x) \neq 0$ and a generalized statistical manifold $M = \{p(\cdot;\theta); \theta = (\theta^1, \cdots, \theta^n) \in \Theta \subseteq \mathbb{R}^n\}$ endowed with the Riemannian metric g defined by (2.4) and parametrized by $\theta = (\theta^1, \cdots, \theta^n)$. For all $\theta \in M$, we define the linear space

$$H_\theta = \{r \in \mathrm{Y} : \mathrm{E}''_\theta[r] = 0, \mathrm{E}''_\theta[r^2] < +\infty\} \tag{3.1}$$

where E''_θ is defined by relation (2.3). In the remainder, we assume that $\mathrm{E}''_\theta[u_0] \neq 0$, $\mathrm{E}''_\theta\left[(\partial_i f_\theta)^2\right] < \infty$, $\mathrm{E}''_\theta[\partial_i f_\theta] = 0$ and for all $(\theta, \theta') \in M^2$, $r \in H_\theta$,

$$E''_\theta[r^2] < +\infty \Longrightarrow \mathrm{E}''_\theta\left[r^2\left(\frac{\phi^{(2)}(f_\theta)}{\phi^{(2)}(f_{\theta'})}\right)^2\right] < +\infty. \tag{3.2}$$

For each $\theta \in M$ and $r, s \in H_\theta$ we set

$$\langle r, s\rangle_\theta := \mathrm{E}''_\theta[rs]. \tag{3.3}$$

Similarly to Proposition 3.1 in Gbaguidi et al.[7], we obtain the following result.

Proposition 3.1. *For all $\theta \in M$, $\langle\cdot,\cdot\rangle_\theta$ is an inner product and $(H_\theta, \langle\cdot,\cdot\rangle_\theta)$ is a Hilbert space.*

When $\theta \neq \theta'$, H_θ and $H_{\theta'}$ are different Hilbert spaces. Hence for neighboring points θ and θ' in M, a one-to-one correspondence can be established between H_θ and $H_{\theta'}$. When the correspondence is affine, it is called an affine connection. Assume that a vector $r \in H_\theta$ at θ corresponds to $r + dr \in H_{\theta+d\theta}$ at a neighboring points $\theta + d\theta$, where d denotes infinitesimally small change.

Lemma 3.1. *For all $\theta \in M$ and $r \in H_\theta$,*

$$\mathrm{E}''_\theta[dr] = -\,\mathrm{E}'''_\theta[\partial_i f_\theta r]d\theta^i + o\left(\|d\theta\|\right). \tag{3.4}$$

Proof. Let $\theta \in M$ and $r \in H_\theta$. Then $r + dr \in H_{\theta+d\theta}$. Let $\Phi_x(\theta) = \phi^{(2)}(f_\theta(x))$, $x \in \chi$. The function Φ_x is differentiable on Θ. Then by Taylor expansion of the function $\theta \mapsto \Phi_x(\theta + d\theta)$ we obtain

$$\Phi_x(\theta + d\theta) \;=\; \Phi_x(\theta) + d_\theta\Phi_x(d\theta) + o(\|d\theta\|) \tag{3.5}$$

where $d_\theta\Phi_x$ denotes the differential of Φ_x at θ. Thus

$$\begin{aligned}
r + dr \in H_{\theta+d\theta} \;&\Rightarrow\; \mathrm{E}''_{\theta+d\theta}[r + dr] = 0\\
&\Rightarrow\; \int_\chi [r(x) + dr(x)]\Phi_x(\theta + d\theta)d\mu(x) = 0\\
&\Rightarrow\; \int_\chi [r(x) + dr(x)]\left(\Phi_x(\theta) + d_\theta\Phi_x(d\theta) + o(\|d\theta\|)\right)d\mu(x) = 0\\
&\Rightarrow\; \mathrm{E}''_\theta[dr] + \mathrm{E}'''_\theta[r\partial_i f_\theta]d\theta^i\\
&\qquad + o(\|d\theta\|)\int_\chi [r(x) + dr(x)]d\mu(x) = 0.
\end{aligned} \tag{3.6}$$

Using $\int_\chi |r(x) + dr(x)|d\mu(x) < \infty$ and (3.6) we deduce $\mathrm{E}''_\theta[dr] = -\,\mathrm{E}'''_\theta[\partial_i f_\theta r]d\theta^i + o\left(\|d\theta\|\right)$. ☐

Since $\mathrm{E}''_\theta[\partial_i f_\theta] = 0$ and $\mathrm{E}''_\theta[(\partial_i f_\theta)^2] < +\infty$, the tangent vectors $\partial_i f_\theta(x)$ belong to H_θ. Hence the tangent space $\top_\theta M$ spanned by those tangent vectors is a linear subspace of H_θ, and the inner product defined in $\top_\theta M$ is compatible with that in H_θ. We have:

$$H_\theta = \top_\theta M \oplus N_\theta$$

where N_θ is the orthogonal complement of $\top_\theta M$ in H_θ. We define the Hilbert-bundle with base space M as

$$H = \cup_{\theta\in M} H_\theta. \tag{3.7}$$

3.2. **α-connection on the Hilbert bundle.** The theory of connection on Hilbert bundle has been investigated in many publications such as [8, 9]. Set $r \in H_\theta$. Differentiating the identity $\mathrm{E}'_\theta[r] = 0$ with respect to θ yields $\mathrm{E}''_\theta[\partial_i r] = -\,\mathrm{E}'''_\theta[\partial_i f_\theta r]$, $\mathrm{E}''[r] = \mathrm{E}''_\theta[\partial_i f_\theta] = 0$. Set

$$dr = \frac{1+\alpha}{2}\,\mathrm{E}''_\theta[\partial_i r]\frac{1}{\mathrm{E}''_\theta[1]}d\theta^i - \frac{1-\alpha}{2}\left[\frac{\phi^{(3)}(f_\theta)}{\phi^{(2)}(f_\theta)}\partial_i f_\theta r - \mathrm{E}''_\theta[u_0\partial_i f_\theta]r\right]d\theta^i. \tag{3.8}$$

We have

$$\begin{aligned}
\mathrm{E}''_\theta[dr] \;&=\; \mathrm{E}''\left[\frac{1+\alpha}{2}\,\mathrm{E}''_\theta[\partial_i r]\frac{1}{\mathrm{E}''_\theta[1]}d\theta^i - \frac{1-\alpha}{2}\frac{\phi^{(3)}(f_\theta)}{\phi^{(2)}(f_\theta)}\partial_i f_\theta r d\theta^i\right]\\
&=\; \frac{1+\alpha}{2}\,\mathrm{E}''_\theta[\partial_i r]d\theta^i - \frac{1-\alpha}{2}\,\mathrm{E}'\left[\frac{\phi^{(3)}(f_\theta)}{\phi^{(2)}(f_\theta)}\partial_i f_\theta r d\theta^i\right]\\
&=\; \frac{1+\alpha}{2}\,\mathrm{E}'_\theta[\partial_i r]d\theta^i - \frac{1-\alpha}{2}\,\mathrm{E}''\left[\partial_i f_\theta r\right]d\theta^i\\
&=\; -\,\mathrm{E}'''\left[\partial_i f_\theta r\right]d\theta^i.
\end{aligned}$$

The α-connection is given by the following α-covariant derivative $\bar{\nabla}^{(\alpha)}$. Let r be a vector field, which attaches a vector $r(.,\theta)$ to every point $\theta \in M$. Then, the rate of the intrinsic change of the vector $r(.,\theta)$ as θ changes in the direction ∂_i is given by the α-covariant derivative :

$$\begin{aligned} \bar{\nabla}^{(\alpha)}_{\partial_i} r &= \partial_i r - \frac{1+\alpha}{2}\, \mathrm{E}''_\theta[\partial_i r] \frac{1}{\mathrm{E}''_\theta[1]} \\ &\quad + \frac{1-\alpha}{2}\left[\frac{\phi^{(3)}(f_\theta)}{\phi^{(2)}(f_\theta)}\partial_i f_\theta r - \mathrm{E}''_\theta[u_0 \partial_i f_\theta] r\right]. \end{aligned} \tag{3.9}$$

This is a generalization of the α-connection studied by Amari[1].
We have $\mathrm{E}''_\theta[\bar{\nabla}^{(\alpha)}_{\partial_i} r] = 0$ because of the identity $\partial_i \mathrm{E}''_\theta[r] = \mathrm{E}''_\theta[\partial_i r] + \mathrm{E}'''_\theta[\partial_i f_\theta r]$. The α-covariant derivative in the direction $A = A^i \partial_i \in \top_\theta M$ is given by $\bar{\nabla}^{(\alpha)}_A r = A^i \bar{\nabla}^{(\alpha)}_{\partial_i} r$. The 1-connection is called the exponential connection and the -1-connection is called the mixture connection. For each point $\theta \in M$, the tangent space $\top_\theta M$ is a subset of the Hilbert space H_θ. Hence the tangent bundle of M,

$$\top M = \cup_{\theta \in M} \top_\theta M,$$

is a subset of H. We can define an affine connection in $\top M$ by introducing an affine correspondence between $\top_\theta M$ and $\top_{\theta'} M$ for neighboring points θ and θ'. An affine connection given such that $r \in H_\theta$ corresponds to $r + dr \in H_{\theta+d\theta}$, induces an affine connection in $\top_\theta M$ such that $r \in \top_\theta M \subset H_\theta$ corresponds to the orthogonal projection of $r + dr \in H_{\theta+d\theta}$ onto $\top_{\theta+d\theta} M$.

3.3. Parallel Transport on the Hilbert bundle.

Definition 3.1. *Let $c = \{c(t), t \in [0,1]\}$ be a curve in M. A vector field $r(\cdot,t) \in H_{c(t)}$ defined along the curve is said to be α-parallel, when*

$$\bar{\nabla}^{(\alpha)}_{\dot{c}} r = \dot{r} - \frac{1+\alpha}{2}\, \mathrm{E}''_c[\dot{r}] \frac{1}{\mathrm{E}''_c[1]} + \frac{1-\alpha}{2}\left[\frac{\phi^{(3)}(f_c)}{\phi^{(2)}(f_c)}\dot{f}_c r - \mathrm{E}''_c[u_0 \dot{f}_c] r\right] = 0, \tag{3.10}$$

where $\dot{r}$ denotes $\partial r/\partial t$, etc.

Definition 3.2. *A vector $r_1(\cdot) \in H_\theta$ is the α-parallel transport of $r_0(\cdot) \in H_{\theta_0}$ along a curve $c = \{c(t), t \in [0,1]\}$ connecting $\theta_0 = c(0)$ and $\theta_1 = c(1)$, when $r_0(\cdot) = r(\cdot,0)$ and $r_1(\cdot) = r(\cdot,1)$ in the solution $r(\cdot,\cdot)$ of (3.10).*

The parallel transport of a vector $r(\cdot)$ from θ to θ' in general depends on the curve $c = \{c(t), t \in [0,1]\}$ along which the parallel takes place.

Theorem 3.1. *(see* [10]*) For an affine connection, parallel transport is independent of the path if and only if the curvature tensor vanishes.*

Now, we derive explicit expressions for the e- and m-parallel transport operators from H_θ to $H_{\theta'}$, for $(\theta,\theta') \in M^2$.

Theorem 3.2. *Let $^{(e)}\pi_\theta^{\theta'}$ and $^{(m)}\pi_\theta^{\theta'}$ be the e- and m-parallel transport operators from H_θ to $H_{\theta'}$. Then*

$$^{(e)}\pi_\theta^{\theta'} r(x) = r(x) - \mathrm{E}''_{\theta'}[r]\frac{1}{\mathrm{E}''_{\theta'}[1]}, \tag{3.11}$$

$$^{(m)}\pi_\theta^{\theta'} r(x) = \frac{r(x)\phi^{(2)}(f_\theta(x))\int_\chi u_0\phi^{(1)}(f_{\theta'})d\mu}{\phi^{(2)}(f_{\theta'}(x))\int_\chi u_0\phi^{(1)}(f_\theta)d\mu}. \tag{3.12}$$

Proof. Let $c = \{c(t), t \in [0,1]\}$ be a curve connecting two points $\theta = c(0)$ and $\theta' = c(1)$. Let $r^{(\alpha)}(x,t)$ be an α-parallel transport vector defined along the curve c. Then, it satisfies (3.10). When $\alpha = 1$, it reduces to

$$\frac{\dot r^{(e)}(x,t)}{u_0(x)} = \mathrm{E}''_{c(t)}\left[\dot r^{(e)}(\cdot,t)\right].$$

Since the right-hand side

$$\mathrm{E}'_{c(t)}\left[\dot r^{(e)}(\cdot,t)\right] = \frac{\int_\chi \dot r^{(e)}(x,t)\phi^{(1)}\left(f_{c(t)}(x)\right)d\mu(x)}{\int_\chi u_0(x)\phi^{(1)}\left(f_{c(t)}(x)\right)d\mu(x)}$$

of (3.12) does not depend on x, its solution (with the initial condition $r(x) = r^{(e)}(x,0)$) is given by

$$r^{(e)}(x,t) = r(x) + a(t)\frac{1}{\mathrm{E}''_{c(t)}[1]}$$

where

$$a(t) = -\,\mathrm{E}''_{c(t)}[r].$$

Then

$$^{(e)}\pi_\theta^{\theta'} r(x) = r(x) - \mathrm{E}''_{\theta'}[r]\frac{1}{\mathrm{E}''_{\theta'}[1]}.$$

When $\alpha = -1$, (3.10) reduces to

$$\dot r^{(m)}(x,t) + \left\{\frac{\phi^{(3)}(f_{c(t)}(x))}{\phi^{(2)}(f_{c(t)}(x))}\dot f_{c(t)}(x) - \mathrm{E}''_{c(t)}[u_0\dot f_{c(t)}]\right\} r^{(m)}(x,t) = 0.$$

The solution is

$$r^{(m)}(x,t) = k(x)\frac{\int_\chi u_0\phi^{(1)}(f_{c(t)}(x))d\mu}{\phi^{(2)}(f_{c(t)}(x))}$$

where $k(x) = \frac{r(x)\phi^{(2)}(f_\theta(x))}{\int_\chi u_0\phi^{(1)}(f_\theta(x))d\mu(x)}$. Then

$$^{(m)}\pi_\theta^{\theta'} r(x) = \frac{r(x)\phi^{(2)}(f_\theta(x))\int_\chi u_0\phi^{(1)}(f_{\theta'})d\mu}{\phi^{(2)}(f_{\theta'}(x))\int_\chi u_0\phi^{(1)}(f_\theta)d\mu}.$$

□

Remark 3.1. *Using the assumption (3.2) the m-parallel transport operator from H_θ to $H_{\theta'}$ is well defined. Let $c = \{c(t), t \in [0,1]\}$ be a curve on M, we have*

$$\bar{\nabla}^{(e)}_{\dot{c}(t)} r(x,t) = \lim_{h\to 0} \frac{{}^{(e)}\pi^{c(t)}_{c(t+h)} r(x,t+h) - r(x,t)}{h} \in H_{c(t)},$$

$$\bar{\nabla}^{(m)}_{\dot{c}(t)} r(x,t) = \lim_{h\to 0} \frac{{}^{(m)}\pi^{c(t)}_{c(t+h)} r(x,t+h) - r(x,t)}{h} \in H_{c(t)},$$

$$\bar{\nabla}^{(\alpha)}_{\dot{c}(t)} r(x,t) = \frac{1+\alpha}{2}\bar{\nabla}^{(e)}_{\dot{c}(t)} r(x,t) + \frac{1-\alpha}{2}\bar{\nabla}^{(m)}_{\dot{c}(t)} r(x,t) \in H_{c(t)}.$$

The two previous theorems imply the following result.

Corollary 3.1. *The exponential $\bar{\nabla}^{(1)}$ and mixture connections $\bar{\nabla}^{(-1)}$ are curvature free.*

Theorem 3.3. *The exponential $\bar{\nabla}^{(1)}$ and mixture connections $\bar{\nabla}^{(-1)}$ are mutualy dual in sense that for all $(\theta, \theta') \in M^2$, $(r,s) \in H^2_\theta$,*

$$\left\langle {}^{(e)}\pi^{\theta'}_{\theta} r, {}^{(m)}\pi^{\theta'}_{\theta} s \right\rangle_{\theta'} = \langle r, s\rangle_\theta.$$

Proof. Let $(\theta, \theta') \in M^2$, $(r,s) \in H^2_\theta$,

$$\begin{aligned}
\left\langle {}^{(e)}\pi^{\theta'}_{\theta} r, {}^{(m)}\pi^{\theta'}_{\theta} s \right\rangle_{\theta'} &= \left\langle r - \mathrm{E}''_{\theta'}[r]\frac{1}{\mathrm{E}''_{\theta'}[1]}, \frac{s\phi^{(2)}(f_\theta)\int_\chi u_0\phi^{(1)}(f_{\theta'})d\mu}{\phi^{(2)}(f_{\theta'})\int_\chi u_0\phi^{(1)}(f_\theta)d\mu} \right\rangle_{\theta'} \\
&= \left\langle r, \frac{s\phi^{(2)}(f_\theta)}{\phi^{(2)}(f_{\theta'})} \right\rangle_{\theta'} \frac{\int_\chi u_0\phi^{(1)}(f_{\theta'})d\mu}{\int_\chi u_0\phi^{(1)}(f_\theta)d\mu} \\
&\quad - \frac{\mathrm{E}''_{\theta'}[r]}{\mathrm{E}''_{\theta'}[1]} \left\langle 1, \frac{s\phi^{(2)}(f_\theta)}{\phi^{(2)}(f_{\theta'})} \right\rangle_{\theta'} \frac{\int_\chi u_0\phi^{(1)}(f_{\theta'})d\mu}{\int_\chi u_0\phi^{(1)}(f_\theta)d\mu} \\
&= \mathrm{E}''_\theta[rs] - \frac{\mathrm{E}''_{\theta'}[r]}{\mathrm{E}''_{\theta'}[1]}\mathrm{E}''_\theta[s] \\
&= \mathrm{E}''_\theta[rs].
\end{aligned}$$

□

Acknowledgments. The authors would like to express their profound gratitude to the anonymous referees for their fruitful remarks and to the "Centre d'Excellence Africain en Sciences Mathématiques et Applications" (CEA-SMA) at the "Institut de Mathématiques et de Sciences Physiques" (IMSP) for their financial support.

References

[1] S. Amari, *Differential-Geometrical Methods in Statistics.* Lecture Note in Statist. Springer, New York, 28 (1985).

[2] S. Amari, *Information geometry and its applications.* Applied mathematical sciences series. Springer, Berlin/Heidelberg, 194(2016).

[3] S. Amari and M. Kumon *Estimation in the presence of infinitely many nuissance parameters-geometry of estimating functions.* The annals of statistics, Vol. 16, No. 3(1988): 1044-1068.

[4] Crasmareanu, Mircea; Hretcanu, Cristina-Elena *Statistical structures on metric path spaces.* Chin. Ann. Math., Ser. B 33, N. 6 (2012): 889-902.

[5] D. De Souza, R. F. Vigelis, and C. C. Cavalcante, *Geometry induced by a generalization of renyi divergence.* Entropy N. 18 (2016).
[6] H. Ishi, *Explicit formula of Koszul-Vinberg characteristic functions for a wide class of regular convex cones.* Entropy N. 18 (2016).
[7] A. Gbaguidi Amoussou, F. Djibril Moussa, C. Ogouyandjou and M.A. Diop, *New connections on the fiber-bundle of generalized statistical manifolds.* Balkan Society of Geometers Proceedings, Geometry Balkan Press, (2019): 23-32.
[8] H. Gauchman, *Connection colligation on Hilbert bundles.* Integration equations and operator theory Vol. 6 (1983)
[9] S. Kobayashi, and K. Nomizu, *Foundations of differential geometry.* New york 1 (2) (1963).
[10] S. Norbert, *General Relativity, Graduate Texts in physics.* 2nd ed. Springer (2013).
[11] R.F. Vigelis, D. C. de Souza, and C.C. Cavalcante, *New metric and connections in statistical manifolds.* International conference on geometric sciences of information, Springer, Berlin, Vol 9389 (2015): 222-229.
[12] R. Walter, *Principles of mathematical analysis.* McGraw Hill, (1974).

INSTITUT DE MATHMATIQUES ET DE SCIENCES PHYSIQUES(IMSP)
UNIVERSIT D'ABOMEY-CALAVI(UAC), BÉNIN
01 B.P. : 613 PORTO-NOVO, RÉPUBLIQUE DU BÉNIN.
Email address: amour.gbaguidi@imsp-uac.org

INSTITUT DE MATHMATIQUES ET DE SCIENCES PHYSIQUES(IMSP)
UNIVERSIT D'ABOMEY-CALAVI(UAC), BÉNIN
01 B.P. : 613 PORTO-NOVO, RÉPUBLIQUE DU BÉNIN.
Email address: ogouyandjou@imsp-uac.org