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AN EXAMPLE OF SZABÓ PSEUDO-RIEMANNIAN METRIC

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ABSTRACT. In this paper we exhibit an example of Szabó pseudo-Riemannian metric of neutral signature obtained by the deformed Riemannian extension.

1. INTRODUCTION

Patterson and Walker [10], introduced the notion of Riemannian extensions and showed how a pseudo-Riemannian metric can be construct on the $2n$-dimensional cotangent bundle of an n-dimensional affine manifold. In [1], the authors generalize the Riemannian extension to the so-called deformed Riemannian extensions. In this paper, we construct an example of Szabó pseudo-Riemannian metric of neutral signature obtained by the deformed Riemannian extension [3].

Our paper is organized as follows. In the section 2, we recall some basic definitions and results on the classical Riemann extension and the deformed Riemannian extension developed in [1]. In section 3, we recall some results on affine Szabó manifolds. Finally in section 4, we construct an example of pseudo-Riemannian Szabó metric of signature $(2,2)$, using the deformed Riemannian extensions, whose Szabó operators are nilpotent.

Throughout this paper, all manifolds, tensors fields and connections are always assumed to be differentiable of class $\mathcal{C}^\infty$.

2. DEFORMED RIEMANNIAN EXTENSIONS

Let (M,∇) be an n-dimensional affine manifold and T^*M be its cotangent bundle and let $\pi : T^*M \to M$ be the natural projection defined by $\pi(p,\omega) = p \in M$ and $(p,\omega) \in T^*M$. A system of local coordinates $(U, u_i), i = 1,\ldots,n$ around $p \in M$ induces a system of local coordinates $(\pi^{-1}(U), u_i, u_{i'} = \omega_i), i' = n+i = n+1,\ldots,2n$ around $(p,\omega) \in T^*M$, where $u_{i'} = \omega_i$ are components of covectors ω in each cotangent space T_p^*M, $p \in U$ with respect to the natural coframe $\{du^i\}$.

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If we use the notation $\partial_i = \frac{\partial}{\partial u_i}$ and $\partial_{i'} = \frac{\partial}{\partial \omega_i}, i = i, \dots, n$ then at each point $(p, \omega) \in T^*M$, its follows that

$$\{(\partial_1)_{(p,\omega)}, \dots, (\partial_n)_{(p,\omega)}, (\partial_{1'})_{(p,\omega)}, \dots, (\partial_{n'})_{(p,\omega)}\},$$

is a basis for the tangent space $(T^*M)_{(p,\omega)}$.

For each vector field X on M, define a function $\iota X : T^*M \longrightarrow \mathbb{R}$ by

$$\iota X(p, \omega) = \omega(X_p).$$

This function is locally expressed by,

$$\iota X(u_i, u_{i'}) = u_{i'} X^i,$$

for all $X = X^i \partial_i$. Vector fields on T^*M are characterized by their actions on functions ιX. The complete lift X^C of a vector field X on M to T^*M is characterized by the identity

$$X^C(\iota Z) = \iota[X, Z], \quad \text{for all} \quad Z \in \Gamma(TM).$$

Moreover, since a $(0, s)$-tensor field on M is characterized by its evaluation on complete lifts of vector fields on M, for each tensor field T of type $(1, 1)$ on M, we define a 1-form ιT on T^*M which is characterized by the identity

$$\iota T(X^C) = \iota(TX).$$

For more details on the geometry of cotangent bundle, we refer to the book of Yano and Ishihara [12].

Let ∇ be a torsion free affine connection on an n-dimensional affine manifold M. The Riemannian extension g_∇ is the pseudo-Riemannian metric on T^*M of neutral signature (n, n) characterized by the identity

$$g_\nabla(X^C, Y^C) = -\iota(\nabla_X Y + \nabla_Y X).$$

In the locally induced coordinates $(u_i, u_{i'})$ on $\pi^{-1}(U) \subset T^*M$, the Riemannian extension is expressed by

$$g_\nabla = \begin{pmatrix} -2u_{k'} f_{ij}^k & \delta_i^j \\ \delta_i^j & 0 \end{pmatrix}, \tag{2.1}$$

with respect to $\{\partial_1, \dots, \partial_n, \partial_{1'}, \dots, \partial_{n'}\}(i, j, k = 1, \dots, n; k' = k+n)$, where f_{ij}^k are the coefficients of the torsion free affine connection ∇ with respect to (U, u_i) on M.

In [1], the authors introduced a deformation of the Riemannian extension above by means of a symmetric $(0, 2)$-tensor field ϕ on (M, ∇); more precisely, their consider the cotangent bundle T^*M equipped with the metric of neutral signature (n, n) given by

$$g_{(\nabla,\phi)} := g_\nabla + \pi^*\phi, \tag{2.2}$$

which is called the deformed Riemannian extension. Let f_{ij}^k be the coefficients of the torsion free affine connection ∇ and let ϕ_{ij} be the local components of the symmetric $(0,2)$-tensor field ϕ. In local coordinates the deformed Riemannian extension is given by:

$$g_{(\nabla,\phi)} = \begin{pmatrix} \phi_{ij}(u) - 2u_{k'} f_{ij}^k & \delta_i^j \\ \delta_i^j & 0 \end{pmatrix}, \tag{2.3}$$

with respect to $\{\partial_1, \ldots, \partial_n, \partial_{1'}, \ldots, \partial_{n'}\}, (i,j,k = 1,\ldots,n; k' = k+n)$.

The classical and deformed Riemannian extensions provide a link between the affine and pseudo-Riemannian geometries. Some properties of the torsion free affine connection ∇ can be investigated by means of the corresponding properties of the classical and deformed Riemannian extensions (see [2, 3, 4] for more details and references therein).

3. The affine Szabó manifolds

Let (M,∇) be an affine manifold and $X \in \Gamma(T_pM)$. We define the affine Szabó operator denoted $\mathcal{S}^\nabla(X)$ with respect to X and $p \in M$ by $\mathcal{S}^\nabla(X) : T_pM \to T_pM$ such that

$$\mathcal{S}^\nabla(X)Y = (\nabla_X \mathcal{R}^\nabla)(Y,X)X,$$

for any vector field Y. The affine Szabó operator satisfies $\mathcal{S}^\nabla(X)X = 0$ and $\mathcal{S}^\nabla(\beta X) = \beta^3 \mathcal{S}(X)$, for $\beta \in \mathbb{R}^*$ and $X \in T_pM$. If $Y = \partial_m$, for $m = 1,2,\ldots,n$ and $X = \sum_i \alpha_i \partial_i$, one gets

$$\mathcal{S}^\nabla(X)\partial_m = \sum_{i,j,k=1}^{n} \alpha_i \alpha_j \alpha_k (\nabla_i \mathcal{R}^\nabla)(\partial_m, \partial_j)\partial_k,$$

where $\nabla_i = \nabla_{\partial_i}$.

Definition 3.1. [5] *Let (M,∇) be a smooth affine manifold and $p \in M$. We says that (M,∇) is affine Szabó at $p \in M$ if the affine Szabó operator $\mathcal{S}^\nabla$ has the same characteristic polynomial for every vector field X on M. Also (M,∇) is called affine Szabó if (M,∇) is affine Szabó at each $p \in M$.*

Theorem 3.2. [5] *Let (M,∇) be an n-dimensional affine manifold and $p \in M$. Then (M,∇) is affine Szabó at $p \in M$ if and only if the characteristic polynomial of the affine Szabó operator $\mathcal{S}^\nabla$ is $P_\lambda[\mathcal{S}^\nabla(X)] = \lambda^n$, for every $X \in T_pM$.*

Let Σ be a surface endowed the torsion free affine connection ∇ given by

$$\nabla_{\partial_1}\partial_1 = f_1(u_1)\partial_2 \quad \text{and} \quad \nabla_{\partial_1}\partial_2 = f_2(u_1)\partial_2. \tag{3.1}$$

The non zero component of the curvature tensor $\mathcal{R}$ of the torsion free affine connection ∇ is

$$\mathcal{R}^\nabla(\partial_1,\partial_2)\partial_1 = a\partial_2 \tag{3.2}$$

where $a = \partial_1 f_2 + f_2^2$. The non zero component of the Ricci tensor Ricc is

$$\mathrm{Ric}(\partial_1,\partial_1) = -a \tag{3.3}$$

Let $X = \alpha_i \partial_i$, $i = 1, 2$ be a vector on M, then, the affine Szabó operator is given by

$$(\nabla_X \mathcal{R}^\nabla)(\partial_1, X)X = A\partial_2, \ \ (\nabla_X \mathcal{R}^\nabla)(\partial_2, X)X = B\partial_2,$$

where the coefficients A and B are given by

$$\begin{aligned} A &= \alpha_1^2 \alpha_2 \partial_1 a + \alpha_1 \alpha_2^2 \partial_2 a \\ B &= -\alpha_1^3 \partial_1 a - \alpha_1^2 \alpha_2 \partial_2 a. \end{aligned}$$

The matrix associated to $\mathcal{S}^\nabla(X)$ with respect to the basis $\{\partial_1, \partial_2\}$ is given by

$$(\mathcal{S}^\nabla(X)) = \begin{pmatrix} 0 & 0 \\ A & B \end{pmatrix}.$$

Its characteristic polynomial is given by $P_\lambda[\mathcal{S}^\nabla(X)] = \lambda^2 - \lambda B$. Since (M, ∇) is affine Szabó, by Theorem 3.2, 0 is the only eigenvalue of the affine Szabó operator $\mathcal{S}^\nabla(X)$. Therefore, $\mathrm{trace}(\mathcal{S}^\nabla(X)) = B = 0$. The latter implies that

$$\partial_1 a = 0, \quad \text{and} \quad \partial_2 a = 0. \tag{3.4}$$

The converse is obvious. We have the following:

Theorem 3.3. *Let (Σ, ∇) be an affine surface endowed the torsion free affine connection ∇ given by $\nabla_{\partial_1}\partial_1 = f_1(u_1)\partial_2$ and $\nabla_{\partial_1}\partial_2 = f_2(u_1)\partial_2$. Then (Σ, ∇) is affine Szabó if and only if: $\partial_1 a = 0$ and $\partial_2 a = 0$, where $a = \partial_1 f_2 + f_2^2$.*

Affine Szabó connections are of interest not only in affine geometry, but also in the study of pseudo-Riemannian Szabó metrics since they provide some nice examples without Riemannian analogue by means of the Riemannian extensions (see [6, 7] for more details).

4. THE DEFORMED RIEMANNIAN EXTENSIONS OF AN AFFINE SZABÓ MANIFOLD

A pseudo-Riemannian manifold (M, g) is said to be Szabó if the Szabó operators $\mathcal{S}(X) = (\nabla_X R)(\cdot, X)X$ has constant eigenvalues on the unit pseudo-sphere bundles $S^\pm(TM)$. Any Szabó manifold is locally symmetric in the Riemannian [11] and the Lorentzian [9] setting but the higher signature case supports examples with nilpotent Szabó operators (cf. [8] and the references therein). Next we will use the deformed Riemannian contruction to exhibit a four-dimensional Szabó metric.

Let $M = \mathbb{R}^2$ and ∇ be the torsion free connection defined by

$$\nabla_{\partial_1}\partial_1 = u_1\partial_2 \quad \text{and} \quad \nabla_{\partial_1}\partial_2 = \frac{1}{u_1}\partial_2. \tag{4.1}$$

The deformed Riemannian extension of the torsion free affine connection defined by (4.1) is the pseudo-Riemannian metric tensor on $\mathbb{R}^4$ of signature $(2,2)$ given by

$$\begin{aligned} g_{(\nabla,\phi)} &= \big(\phi_{11} - 2u_1u_4\big)du_1 \otimes du_1 + \phi_{22}du_2 \otimes du_2 \\ &\quad + \Big(\phi_{12} - 2\frac{u_4}{u_1}\Big)(du_1 \otimes du_2 + du_1 \otimes du_2) \\ &\quad + (du_1 \otimes du_3 + du_3 \otimes du_1 + du_2 \otimes du_4 + du_4 \otimes du_2). \end{aligned} \tag{4.2}$$

Then the Christoffel symbols of $g_{(\nabla,\phi)}$ are given by

$$\begin{aligned} \tilde\Gamma^2_{11} &= u_1, \quad \tilde\Gamma^2_{12} = \frac{1}{u_1} \\ \tilde\Gamma^3_{14} &= -u_1, \quad \tilde\Gamma^4_{14} = -\frac{1}{u_1}, \quad \tilde\Gamma^3_{24} = -\frac{1}{u_1} \\ \tilde\Gamma^3_{11} &= \frac{1}{2}\partial_1\phi_{11} - \phi_{12}u_1 + u_4, \quad \tilde\Gamma^4_{11} = \partial_1\phi_{12} - \frac{1}{2}\partial_2\phi_{11} - \phi_{22}u_1 \\ \tilde\Gamma^3_{12} &= \frac{1}{2}\partial_2\phi_{11} - \frac{\phi_{12}}{u_1} + 2\frac{u_4}{u_1^2}, \quad \tilde\Gamma^4_{12} = \frac{1}{2}\partial_1\phi_{22} - \frac{\phi_{22}}{u_1}, \\ \tilde\Gamma^3_{22} &= \partial_2\phi_{12} - \frac{1}{2}\partial_1\phi_{22}, \quad \tilde\Gamma^4_{22} = \frac{1}{2}\partial_2\phi_{22} \end{aligned}$$

and the others are zero. Then the only nonvanishing covariant derivatives are given by

$$\begin{aligned} \tilde\nabla_{\partial_1}\partial_1 &= u_1\partial_2 + \Big(\frac{1}{2}\partial_1\phi_{11} - \phi_{12}u_1 + u_4\Big)\partial_3 + \Big(\partial_1\phi_{12} - \frac{1}{2}\partial_2\phi_{11} - \phi_{22}u_1\Big)\partial_4 \\ \tilde\nabla_{\partial_1}\partial_2 &= \frac{1}{u_1}\partial_2 + \Big(\frac{1}{2}\partial_2\phi_{11} - \frac{\phi_{12}}{u_1} + 2\frac{u_4}{u_1^2}\Big)\partial_3 + \Big(\frac{1}{2}\partial_1\phi_{22} - \frac{\phi_{22}}{u_1}\Big)\partial_4 \\ \tilde\nabla_{\partial_1}\partial_4 &= -u_1\partial_3 - \frac{1}{u_1}\partial_4, \quad \tilde\nabla_{\partial_2}\partial_2 = \Big(\partial_2\phi_{12} - \frac{1}{2}\partial_1\phi_{22}\Big)\partial_3 + \Big(\frac{1}{2}\partial_2\phi_{22}\Big)\partial_4 \\ \tilde\nabla_{\partial_2}\partial_4 &= -\frac{1}{u_1}\partial_3. \end{aligned}$$

It follows that the only nonvanishing components of the curvature tensor of $(\mathbb{R}^4, g_{(\nabla,\phi)})$ are given by

$$\begin{aligned} R(\partial_1,\partial_2)\partial_1 &= \Big[\frac{1}{2}\partial_1^2\phi_{22} - \partial_2\partial_1\phi_{12} + \frac{1}{2}\partial_2^2\phi_{11} + \frac{\phi_{22}}{u_1^2} - \frac{1}{u_1}\partial_1\phi_{22} + \frac{1}{2}u_1\partial_2\phi_{22}\Big]\partial_4, \\ R(\partial_1,\partial_2)\partial_2 &= \Big[\partial_1\partial_2\phi_{12} - \frac{1}{2}\partial_1^2\phi_{22} - \frac{1}{2}\partial_2^2\phi_{11} - \frac{\phi_{22}}{u_1^2} - \frac{1}{2}u_1\partial_2\phi_{22} + \frac{1}{u_1}\partial_1\phi_{22}\Big]\partial_3, \end{aligned}$$

Put $R(\partial_1,\partial_2)\partial_1 = a_4\partial_4$ and $R(\partial_1,\partial_2)\partial_2 = b_3\partial_3$. Now, let $X = \sum_{i=1}^4 \alpha_i\partial_i$ a vector field on $\mathbb{R}^4$, then we have:

$$\begin{aligned} (\nabla_X R)(\partial_1, X)X &= S_{31}\partial_3 + S_{41}\partial_4 \\ (\nabla_X R)(\partial_2, X)X &= S_{32}\partial_3 + S_{42}\partial_4 \\ (\nabla_X R)(\partial_3, X)X &= 0 \quad \text{and} \quad (\nabla_X R)(\partial_4, X)X = 0 \end{aligned}$$

where

$$\begin{aligned} S_{31} &= \alpha_2^3\partial_2 b_3 + \alpha_1^2\alpha_2\Big[-b_3\Gamma_{11}^2 + a_4\Gamma_{14}^3\Big] + \alpha_1\alpha_2^2\Big[\partial_1 b_3 - 3b_3\Gamma_{12}^2 + a_4\Gamma_{24}^3\Big] \\ S_{32} &= \alpha_1^3\Big[a_4u_1 + b_3u_1\Big] + \alpha_1^2\alpha_2\Big[-\partial_1 b_3 + 3\frac{b_3}{u_1} + \frac{a_4}{u_1}\Big] - \alpha_1\alpha_2^2\partial_2 b_3 \\ S_{41} &= \alpha_1^2\alpha_2\Big[\partial_1 a_4 - a_4\Gamma_{12}^2 + a_4\Gamma_{14}^4\Big] + \alpha_1\alpha_2^2\partial_2 a_4 \\ S_{42} &= \alpha_1^3\Big[2\frac{a_4}{u_1} - \partial_1 a_4\Big] - \alpha_1^2\alpha_2\partial_2 a_4. \end{aligned}$$

Hence, the matrix associated to the Szabó operator with respect to the basis $\{\partial_1, \dots, \partial_4\}$ is given by

$$(\mathcal{S}(X)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ S_{31} & S_{32} & 0 & 0 \\ S_{41} & S_{42} & 0 & 0 \end{pmatrix} \tag{4.3}$$

It follows from the expression (4.3) of the Szabó operator $\mathcal{S}(X) = (\nabla_X R)$, where X is a nonnull vector field on $\mathbb{R}^4$, that its characteristic polynomial satisfies

$$P_\lambda(\mathcal{S}(X)) = \det(S(X) - \lambda \mathrm{Id}_4) = \lambda^4.$$

Thus all eigenvalues are zero. This proves that $(g_{\nabla,\phi})$ is Szabó. Hence we have the following

Theorem 4.1. *Let $M = \mathbb{R}^2$ and ∇ be the torsion free connection defined by $\nabla_{\partial_1}\partial_1 = u_1\partial_2$ and $\nabla_{\partial_1}\partial_2 = \frac{1}{u_1}\partial_2$. Then the pseudo-Riemannian metric $g_{(\nabla,\phi)}$ on the cotangent bundle T^*M of neutral signature $(2,2)$ defined by setting*

$$\begin{aligned} g_{\nabla,\phi} &= (\phi_{11} - 2u_1u_4)du_1 \otimes du_1 + \phi_{22}du_2 \otimes du_2 \\ &\quad + \Big(\phi_{12} - 2\frac{u_4}{u_1}\Big)(du_1 \otimes du_2 + du_1 \otimes du_2) \\ &\quad + (du_1 \otimes du_3 + du_3 \otimes du_1 + du_2 \otimes du_4 + du_4 \otimes du_2). \end{aligned}$$

is Szabó for any symmetric $(0,2)$-tensor field ϕ.

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