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ON CONCURRENCE OF NINE EULER LINES ON THE MORLEY'S CONFIGURATION

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ABSTRACT. We establish the concurrence of nine Euler lines on the configuration of Morley's trisector theorem with proof using computer algebra.

Euler proved in 1765 that the centroid, orthocenter, and circumcenter of a given triangle are collinear [1]. There are a few directions that Euler line has been studied. For one, it is known that Euler line passes through other interesting triangle centers, such as the nine point center, the de Longchamps point, the Schiffler point, the Exeter point, and the Gosssard perspector [7]. For the direction that is most relevant to our work, it concerns system of triangles with concurrent Euler lines. It is known that for a triangle ABC and its two Fermat points F_1 and F_2, the Euler lines of the 10 triangles with vertices chosen from A, B, C, F_1, and F_2 are concurrent at the centroid of the triangle ABC [2]. Furthermore, the Euler lines of the four triangles formed by an orthocentric system are concurrent at the nine-point center common to all of the triangles [3]. On the other hand, Morley's trisector theorem was discovered by Anglo-American mathematician Frank Morley in 1899. Since its discovery, there are many proofs, some of which are very technical. Recent proofs include the algebraic proof by Alain Connes (1998, 2004, [4], [5]), John Conway's elementary geometry proof [6] and a very recent one by the second author [8]. Our following result states that nine Euler lines in an extended Morley's configuration are concurrent (see Figure 1).

Theorem 1. *Let ABC be a triangle with Morley triangle XYZ. Let K_a, K_b and K_c be circumcenters of triangles AYZ, BZX, and CXY respectively. Let L_a, L_b, and L_c be circumcenters of triangles XBC, YCA, and ZAB respectively. Then the Euler lines of the eight triangles*

$$\triangle K_aK_bK_c,\ \triangle L_aL_bL_c,\ \triangle K_aL_bL_c,\ \triangle K_bL_cL_a,$$

$$\triangle K_cL_aL_b,\ \triangle L_aK_bK_c,\ \triangle L_bK_cK_a,\ \triangle L_cK_aK_b$$

are concurrent at circumcenter of triangle ABC. Thus with Euler line of ABC, we have nine concurrent Euler lines.

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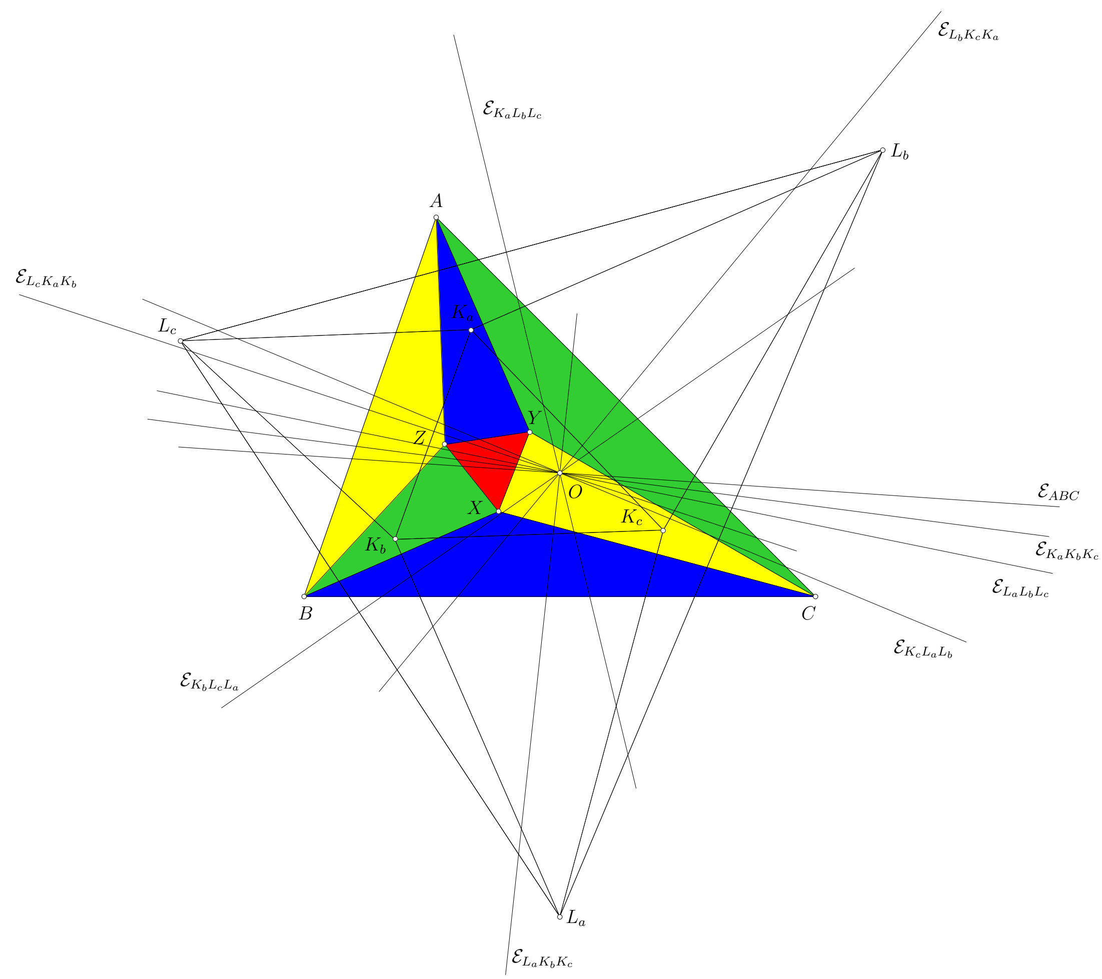

FIGURE 1. Extended Morley's configuration

Proof. Without loss of generality, denote B and C by

$$B = (0,0), C = (1,0), \angle B = 3\beta, \angle C = 3\gamma.$$

Then line AB and line AC have slope $\tan(3\beta)$ and $-\tan(3\gamma)$, respectively. For convenience, denote

$$\tan(\beta) = s, \tan(\gamma) = t.$$

It follows that

$$\tan(3\beta) = \frac{3\tan(\beta) - \tan^3(\beta)}{1 - 3\tan^2(\beta)} = \frac{3s - s^3}{1 - 3s^2} =: m_{ab}$$

and

$$-\tan(3\gamma) = -\frac{3\tan(\gamma) - \tan^3(\gamma)}{1 - 3\tan^2(\gamma)} = -\frac{3t - t^3}{1 - 3t^2} =: m_{ac}.$$

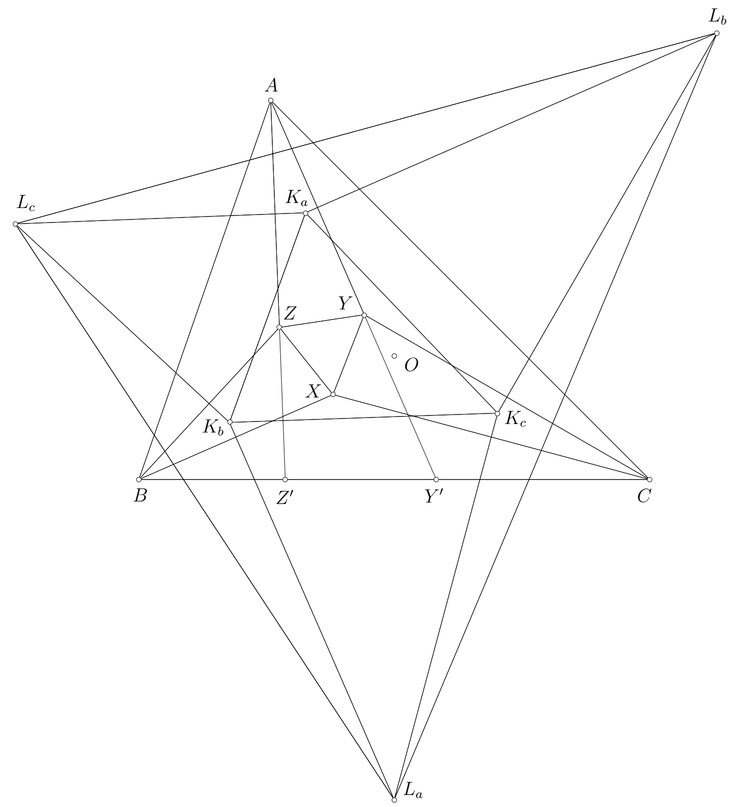

Figure 2. Proof of extended Morley's configuration

Solving the equations for AB and AC as follows, one gets the coordinates for $A = (a, b)$:

$$y = m_{ab}x, \qquad y = m_{ac}(x-1), \qquad A = (a, b).$$

To solve for the Morley triangle XYZ, we need to determine the equations for the lines BZ, BX, CY, CX, AZ and AY. The first four are straightforward. For the last two, we apply some geometry: Let AZ intersect BC at Z' and AY intersect BC at Y' (see Figure 2). Then

$$\angle AZ'C = \angle ABC + \angle BAZ' = 3\beta + \frac{\pi}{3} - (\beta+\gamma) = 2\beta + \left(\frac{\pi}{3} - \gamma\right)$$

and

$$\angle AY'B = \angle BCA + \angle CAY' = 3\gamma + \frac{\pi}{3} - (\beta+\gamma) = 2\gamma + \left(\frac{\pi}{3} - \beta\right).$$

Denoting the slope of BZ as m_{bz}, the slope of $AZ = AZ'$ as m_{az}, etc., we find that

$$m_{bz} = \tan(2\beta) = \frac{2\tan(\beta)}{1-\tan^2(\beta)} = \frac{2s}{1-s^2}$$

$$m_{bx} = \tan(\beta) = s, \quad m_{cx} = -\tan(\gamma) = -t$$

$$m_{cy} = -\tan(2\gamma) = -\frac{2\tan(\gamma)}{1-\tan^2(\gamma)} = -\frac{2t}{1-t^2}$$

$$m_{az} = \tan\left(2\beta + \left(\frac{\pi}{3} - \gamma\right)\right) = \frac{\tan(2\beta) + \tan(\frac{\pi}{3} - \gamma)}{1 - \tan(2\beta)\tan(\frac{\pi}{3} - \gamma)},$$

where $\tan(2\beta) = 2s/(1 - s^2)$ and

$$\tan\left(\frac{\pi}{3} - \gamma\right) = \frac{\tan(\pi/3) - \tan(\gamma)}{1 + \tan(\pi/3)\tan(\gamma)} = \frac{\sqrt{3} - t}{1 + \sqrt{3}t}.$$

Similarly,

$$m_{ay} = -\tan\left(2\gamma + \left(\frac{\pi}{3} - \beta\right)\right) = -\frac{\tan(2\gamma) + \tan(\frac{\pi}{3} - \beta)}{1 - \tan(2\gamma)\tan(\frac{\pi}{3} - \beta)},$$

where $\tan(2\gamma) = 2t/(1 - t^2)$ and

$$\tan\left(\frac{\pi}{3} - \beta\right) = \frac{\tan(\pi/3) - \tan(\beta)}{1 + \tan(\pi/3)\tan(\beta)} = \frac{\sqrt{3} - s}{1 + \sqrt{3}s}.$$

Now we can write down all the linear equations in order to solve for X, Y and Z:

$$\begin{gathered} L_{bz} : y = m_{bz}x \\ L_{bx} : y = m_{bx}x \\ L_{cy} : y = m_{cy}(x - 1) \\ L_{cx} : y = m_{cx}(x - 1) \\ L_{az} : y - b = m_{az}(x - a) \\ L_{ay} : y - b = m_{ay}(x - a), \end{gathered}$$

where L_{bz} is the equation for line BZ, etc.

Once we have all these equations, it is straightforward to solve for X, Y and Z, and then all the circumcenters, and Euler lines. Note that solving these and checking for concurrency is easy by a computer algebra system such as SAGE [9]. For completeness, we record the results as follows.

$$K_a = \left(\frac{t(4s^3t - s^2t^2 - 5s^2 - 4st - t^2 + 3)}{(3s^2t^2 - s^2 - 8st - t^2 + 3)(s + t)}, -\frac{st(s^2t^2 + s^2 + 8st + t^2 - 7)}{(3s^2t^2 - s^2 - 8st - t^2 + 3)(s + t)}\right)$$

$$K_b = \left(\frac{p_1(s, t)}{q_1(s, t)}, \frac{p_2(s, t)}{q_2(s, t)}\right),$$

where

$$p_1(s, t) = t(3s^4t^4 + 2\sqrt{3}s^4t^3 + 8\sqrt{3}s^3t^4 - 8s^4t^2 - 16s^3t^3 + 10s^2t^4 - 6\sqrt{3}s^4t - 16\sqrt{3}s^3t^2 - 28\sqrt{3}s^2t^3$$
$$-3s^4 + 16s^3t + 40s^2t^2 - 16st^3 - t^4 + 8\sqrt{3}s^3 + 20\sqrt{3}s^2t + 32\sqrt{3}st^2 + 2\sqrt{3}t^3 - 18s^2 - 48st - 6\sqrt{3}t + 9)$$

$$q_1(s, t) = 2(3\sqrt{3}s^4t^4 + 3\sqrt{3}s^3t^5 - 6s^4t^3 - 3s^3t^4 + 3s^2t^5 - 4\sqrt{3}s^4t^2 - 18\sqrt{3}s^3t^3 - 15\sqrt{3}s^2t^4 - \sqrt{3}st^5$$
$$+2s^4t + 26s^3t^2 + 18s^2t^3 - 7st^4 - t^5 + \sqrt{3}s^4 + 11\sqrt{3}s^3t + 30\sqrt{3}s^2t^2 + 22\sqrt{3}st^3 + 2\sqrt{3}t^4 - 3s^3$$
$$-33s^2t - 30st^2 - 3\sqrt{3}s^2 - 9\sqrt{3}st - 6\sqrt{3}t^2 + 9s + 9t)$$

$$p_2(s, t) = t(3\sqrt{3}s^4t^4 - 6s^4t^3 - 4\sqrt{3}s^4t^2 - 16\sqrt{3}s^3t^3 - 6\sqrt{3}s^2t^4 + 2s^4t + 32s^3t^2 + 4s^2t^3 - 8st^4 + \sqrt{3}s^4$$
$$+16\sqrt{3}s^3t + 32\sqrt{3}s^2t^2 + 16\sqrt{3}st^3 - \sqrt{3}t^4 - 44s^2t - 16st^2 + 10t^3 - 10\sqrt{3}s^2 - 16\sqrt{3}st$$

$$-12\sqrt{3}t^2+24s+18t-3\sqrt{3})$$

$q_2(s,t)=q_1(s,t)$

$$K_c=\left(\frac{r_1(s,t)}{s_1(s,t)},\frac{r_2(s,t)}{s_2(s,t)}\right),$$

where

$r_1(s,t)=-(3s^5t^4+2\sqrt{3}s^5t^3-4\sqrt{3}s^4t^4+4s^5t^2-10s^4t^3+4s^3t^4+2\sqrt{3}s^5t+2\sqrt{3}s^4t^2+20\sqrt{3}s^3t^3$
$+2\sqrt{3}s^2t^4+s^5-2s^4t+4s^3t^2-36s^2t^3-7st^4-2\sqrt{3}s^4-12\sqrt{3}s^3t-40\sqrt{3}s^2t^2-14\sqrt{3}st^3-2\sqrt{3}t^4$
$+12s^2t+48st^2+6t^3+6\sqrt{3}s^2+18\sqrt{3}st+6\sqrt{3}t^2-9s-18t)$

$s_1(s,t)=2(3\sqrt{3}s^5t^3+3\sqrt{3}s^4t^4+3s^5t^2-3s^4t^3-6s^3t^4-\sqrt{3}s^5t-15\sqrt{3}s^4t^2-18\sqrt{3}s^3t^3-4\sqrt{3}s^2t^4-s^5$
$-7s^4t+18s^3t^2+26s^2t^3+2st^4+2\sqrt{3}s^4+22\sqrt{3}s^3t+30\sqrt{3}s^2t^2+11\sqrt{3}st^3+\sqrt{3}t^4-30s^2t-33st^2-3t^3$
$-6\sqrt{3}s^2-9\sqrt{3}st-3\sqrt{3}t^2+9s+9t)$

$r_2(s,t)=s(3\sqrt{3}s^4t^4-6s^3t^4-6\sqrt{3}s^4t^2-16\sqrt{3}s^3t^3-4\sqrt{3}s^2t^4-8s^4t+4s^3t^2+32s^2t^3+2st^4-\sqrt{3}s^4$
$+16\sqrt{3}s^3t+32\sqrt{3}s^2t^2+16\sqrt{3}st^3+\sqrt{3}t^4+10s^3-16s^2t-44st^2-12\sqrt{3}s^2$
$-16\sqrt{3}st-10\sqrt{3}t^2+18s+24t-3\sqrt{3})$

$s_2(s,t)=s_1(s,t)$

$$L_a=\left(\frac{1}{2},\frac{st-1}{2(s+t)}\right)$$

$$L_b=\left(\frac{l_1(s,t)}{m_1(s,t)},\frac{l_2(s,t)}{m_2(s,t)}\right),$$

where

$l_1(s,t)=-(3s^7t^4-2\sqrt{3}s^7t^3-10\sqrt{3}s^6t^4-12s^7t^2+10s^6t^3+37s^5t^4-2\sqrt{3}s^7t+42\sqrt{3}s^6t^2+18\sqrt{3}s^5t^3$
$-18\sqrt{3}s^4t^4+s^7+34s^6t-156s^5t^2-154s^4t^3-7s^3t^4-4\sqrt{3}s^6-78\sqrt{3}s^5t+54\sqrt{3}s^4t^2+106\sqrt{3}s^3t^3$
$+18\sqrt{3}s^2t^4+15s^5+270s^4t+156s^3t^2-18s^2t^3-9st^4-150\sqrt{3}s^3t-162\sqrt{3}s^2t^2-42\sqrt{3}st^3-6\sqrt{3}t^4$
$-45s^3+54s^2t+108st^2+18t^3+36\sqrt{3}s^2+54\sqrt{3}st+18\sqrt{3}t^2-27s-54t)$

$m_1(s,t)=2(3\sqrt{3}s^7t^3+3\sqrt{3}s^6t^4+3s^7t^2-21s^6t^3-24s^5t^4-\sqrt{3}s^7t-21\sqrt{3}s^6t^2-3\sqrt{3}s^5t^3+17\sqrt{3}s^4t^4$
$-s^7-s^6t+117s^5t^2+125s^4t^3+8s^3t^4+4\sqrt{3}s^6+33\sqrt{3}s^5t-51\sqrt{3}s^4t^2-95\sqrt{3}s^3t^3-15\sqrt{3}s^2t^4-15s^5$
$-183s^4t-159s^3t^2+9s^2t^3+117\sqrt{3}s^3t+153\sqrt{3}s^2t^2+39\sqrt{3}st^3+3\sqrt{3}t^4+45s^3-27s^2t-81st^2-9t^3$
$-36\sqrt{3}s^2-45\sqrt{3}st-9\sqrt{3}t^2+27s+27t)$

$l_2(s,t)=s(\sqrt{3}s^4t^4-8s^4t^3-8s^3t^4-6\sqrt{3}s^4t^2+8\sqrt{3}s^3t^3+6\sqrt{3}s^2t^4+48s^3t^2+24s^2t^3+\sqrt{3}s^4+8\sqrt{3}s^3t$
$-36\sqrt{3}s^2t^2-24\sqrt{3}st^3-3\sqrt{3}t^4-8s^3-56s^2t+6\sqrt{3}s^2+40\sqrt{3}st+18\sqrt{3}t^2-24t-3\sqrt{3})(s^2-3)$

$m_2(s,t)=m_1(s,t)$

$$L_c = \left(\frac{u_1(s,t)}{v_1(s,t)}, \frac{u_2(s,t)}{v_2(s,t)}\right),$$

where

$$u_1(s,t) = t(3s^4t^6-4\sqrt{3}s^4t^5+4\sqrt{3}s^3t^6-11s^4t^4-32s^3t^5-6s^2t^6+16\sqrt{3}s^4t^3+12\sqrt{3}s^3t^4-4\sqrt{3}st^6 +9s^4t^2+96s^3t^3+78s^2t^4+32st^5-t^6-12\sqrt{3}s^4t-84\sqrt{3}s^3t^2-48\sqrt{3}s^2t^3-12\sqrt{3}st^4+4\sqrt{3}t^5-9s^4 -162s^2t^2-96st^3-15t^4+36\sqrt{3}s^3+144\sqrt{3}s^2t+84\sqrt{3}st^2-54s^2+45t^2-36\sqrt{3}s-36\sqrt{3}t+27)$$

$$v_1(s,t) = 2(3\sqrt{3}s^4t^6+3\sqrt{3}s^3t^7-24s^4t^5-21s^3t^6+3s^2t^7+17\sqrt{3}s^4t^4-3\sqrt{3}s^3t^5-21\sqrt{3}s^2t^6-\sqrt{3}st^7 +8s^4t^3+125s^3t^4+117s^2t^5-st^6-t^7-15\sqrt{3}s^4t^2-95\sqrt{3}s^3t^3-51\sqrt{3}s^2t^4+33\sqrt{3}st^5+4\sqrt{3}t^6+9s^3t^2 -159s^2t^3-183st^4-15t^5+3\sqrt{3}s^4+39\sqrt{3}s^3t+153\sqrt{3}s^2t^2+117\sqrt{3}st^3-9s^3-81s^2t-27st^2+45t^3 -9\sqrt{3}s^2-45\sqrt{3}st-36\sqrt{3}t^2+27s+27t)$$

$$u_2(s,t) = t(\sqrt{3}s^4t^6-8s^4t^5-8s^3t^6+3\sqrt{3}s^4t^4+8\sqrt{3}s^3t^5-6\sqrt{3}s^2t^6+24s^4t^3+48s^3t^4+48s^2t^5-21\sqrt{3}s^4t^2 -48\sqrt{3}s^3t^3-18\sqrt{3}s^2t^4+8\sqrt{3}st^5+\sqrt{3}t^6-72s^3t^2-144s^2t^3-56st^4-8t^5+9\sqrt{3}s^4+72\sqrt{3}s^3t+126\sqrt{3}s^2t^2 +16\sqrt{3}st^3+3\sqrt{3}t^4+144st^2+24t^3-54\sqrt{3}s^2-120\sqrt{3}st-21\sqrt{3}t^2+72s+9\sqrt{3})$$
$$v_2(s,t) = v_1(s,t).$$

Finally the circumcenter of ABC is

$$O = \left(\frac{1}{2}, \frac{(s^2t^2-3s^2-8st-3t^2+1)(st-1)}{2(3s^2t^2-s^2-8st-t^2+3)(s+t)}\right),$$

and all the Euler lines concur at this point.

□

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