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THREE ANALOGIES BETWEEN THE TRIANGLE CENTERS AND ITS APPLICATIONS

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ABSTRACT. In this paper we present ten triangle centers which determines a special configuration: three lines concurrent in centroid and several similar triangles. Applied these similarities and other properties of this configuration, we will calculate with minimum effort the length of 45 line segments determined by these ten triangle centers.

1. INTRODUCTION AND MOTIVATIONS

It is well known that the circumcenter (O), the triangle centroid (G) and the orthocenter (H) of a triangle are collinear (Euler line). The centroid divide the segment OH in rapport $2:1$. Also it is known that the center of nine-point circle (N) is the midpoint of segment OH. I find more six triangle centers which are in a rapport similar with the triangle centroid (G). Concretely these points are the incenter (I), the Gergonne point (Ge), the Nagel point (Na), the Mittenpunkt (M), the Spieker center (Sp) and the complement of Mittenpunkt (Mc). The triplet of points (I, G, Na) and (M, G, Ge) are also collinear. The midpoints of segments INa and MGe are the Spieker center (Sp) and the complement of Mittenpunkt (Mc), respectively. These three analogies we will use to calculate the measures of 45 segments determined by these ten triangle centers.

Beside the habitual geometric notations in this paper we will use the Conway triangle notations: $S = 2\Delta$, where Δ is the area of the reference triangle ABC and $S_\varphi = S\cot\varphi$. In special cases

$$S_\omega = \frac{1}{2}(a^2+b^2+c^2),$$
$$S_A = bc\cos A = \frac{1}{2}(-a^2+b^2+c^2) = S_\omega - a^2,$$
$$S_B = ca\cos B = \frac{1}{2}(a^2-b^2+c^2) = S_\omega - b^2,$$
$$S_C = ab\cos C = \frac{1}{2}(a^2+b^2-c^2) = S_\omega - c^2,$$

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where a, b, c are the sides lengths, A, B, C the corresponding angles and ω is the Brocard angle of $\triangle ABC$. We will need also for the following relations:

$$S_A + S_B + S_C = \frac{a^2 + b^2 + c^2}{2} = S_\omega,$$
$$S_{BC} + S_{CA} + S_{AB} = S^2,$$
$$a^2 S_A + b^2 S_B + c^2 S_C = 2S^2,$$
$$a(s-a) + (s-b)(s-c) = b(s-b) + (s-c)(s-a)$$
$$= c(s-c) + (s-a)(s-b)$$
$$= (s-b)(s-c) + (s-c)(s-a) + (s-a)(s-b) = r(r+4R),$$
$$a(s-a) + b(s-b) + c(s-c) = 2r(r+4R),$$
$$aS_A + bS_B + cS_C = 4sr(r+R) = 2S(r+R),$$
$$bc + ca + ab = s^2 + r^2 + 4Rr,$$
$$a^2 + b^2 + c^2 = 2S_\omega = 2(s^2 - r^2 - 4Rr),$$
$$a^3 + b^3 + c^3 = 2s(s^2 - 3r^2 - 6Rr),$$
$$S_A S_B S_C = (S_A + S_B + S_C - 4R^2)S^2 = (S_\omega - 4R^2)S^2,$$
$$a^3 S_A + b^3 S_B + c^3 S_C = 4sr^2(3s^2 - r^2 - 6Rr - 8R^2),$$
$$a^4 S_A + b^4 S_B + c^4 S_C = 16s^2r^2(s^2 - r^2 - 4Rr - 3R^2).$$

The barycentric and the absolute barycentric coordinates of the above mentioned ten points which appear in our article are [1]:

$$X(1) = (a : b : c) = \left(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s}\right) = I \text{ (incenter)};$$
$$X(2) = (1 : 1 : 1) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = G \text{ (triangle centroid)};$$
$$X(3) = (S^2 - S_{BC} : S^2 - S_{CA} : S^2 - S_{AB}) = (a^2 S_A : b^2 S_B : c^2 S_C)$$
$$= \left(\frac{a^2 S_A}{2S^2}, \frac{b^2 S_B}{2S^2}, \frac{c^2 S_C}{2S^2}\right) = O \text{ (circumcenter)};$$
$$X(4) = \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right) = (S_{BC} : S_{CA} : S_{AB}) = \left(\frac{S_{BC}}{S^2}, \frac{S_{CA}}{S^2}, \frac{S_{AB}}{S^2}\right)$$
$$= H \text{ (orthocenter)};$$
$$X(5) = (S^2 + S_{BC} : S^2 + S_{CA} : S^2 + S_{AB})$$
$$= \left(\frac{S^2 + S_{BC}}{4S^2}, \frac{S^2 + S_{CA}}{4S^2}, \frac{S^2 + S_{AB}}{4S^2}\right) = N \text{ (nine-point center)};$$
$$X(7) = (bc - S_A : ca - S_B : ab - S_C)$$
$$= ((s-b)(s-c) : (s-c)(s-a) : (s-a)(s-b))$$
$$= \left(\frac{(s-b)(s-c)}{r(r+4R)}, \frac{(s-c)(s-a)}{r(r+4R)}, \frac{(s-a)(s-b)}{r(r+4R)}\right) = Ge \text{ (Gergonne point)};$$

$$\begin{aligned}
X(8) &= (bc + S_A : ca + S_B : ab + S_C) = (s-a : s-b : s-c) \\
&= \left(\frac{s-a}{s}, \frac{s-b}{s}, \frac{s-c}{s}\right) = Na \text{ (Nagel point)}; \\
X(9) &= (a(s-a) : b(s-b) : c(s-c)) \\
&= \left(\frac{a(s-a)}{2r(r+4R)}, \frac{b(s-b)}{2r(r+4R)}, \frac{c(s-c)}{2r(r+4R)}\right) = M \text{ (Mittenpunkt)}; \\
X(10) &= (b+c : c+a : a+b) = (2s-a : 2s-b : 2s-c) \\
&= \left(\frac{2s-a}{4s}, \frac{2s-b}{4s}, \frac{2s-c}{4s}\right) = Sp \text{ (Spieker center)}; \\
X(142) &= \left(ab + ac - (b-c)^2 : bc + ba - (c-a)^2 : ca + cb - (a-b)^2\right) \\
&= (a(s-a) + 2(s-b)(s-c) : b(s-b) + 2(s-c)(s-a) : \dots) \\
&= \left(\frac{a(s-a)+2(s-b)(s-c)}{4r(r+4R)}, \frac{b(s-b)+2(s-c)(s-a)}{4r(r+4R)}, \dots\right) = Mc
\end{aligned}$$

(complement of Mittenpunkt).

2. Collinear points

Theorem 2.1. *The circumcenter (O), the triangle centroid (G), the orthocenter (H) and the nine-point center (N) are collinear $(Euler\ line)$ and*

$$(a) \quad OH^2 = -2S_\omega + 9R^2 = -2s^2 + 2r^2 + 8Rr + 9R^2,$$
$$(b) \quad OH = \frac{3}{2}GH = 2ON = 2HN = 3GO = 6GN.$$

Proof. The triplet of points (O, G, H) are collinear if and only if

$$\begin{vmatrix} a^2S_A & b^2S_B & c^2S_C \\ 1 & 1 & 1 \\ S_{BC} & S_{CA} & S_{AB} \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} a^2S_A + S_{BC} & b^2S_B + S_{CA} & c^2S_C + S_{AB} \\ 1 & 1 & 1 \\ S_{BC} & S_{CA} & S_{AB} \end{vmatrix} = 0 \Leftrightarrow$$
$$\Leftrightarrow \begin{vmatrix} S^2 & S^2 & S^2 \\ 1 & 1 & 1 \\ S_{BC} & S_{CA} & S_{AB} \end{vmatrix} = 0. \text{ True.}$$

The nine-point center is the midpoint of segment OH (Figure 1). Indeed

$$x_O + x_H = \frac{a^2S_A}{2S^2} + \frac{S_{BC}}{S^2} = \frac{a^2S_A + 2S_{BC}}{2S^2} = \frac{S^2 + S_{BC}}{2S^2} = 2x_N.$$

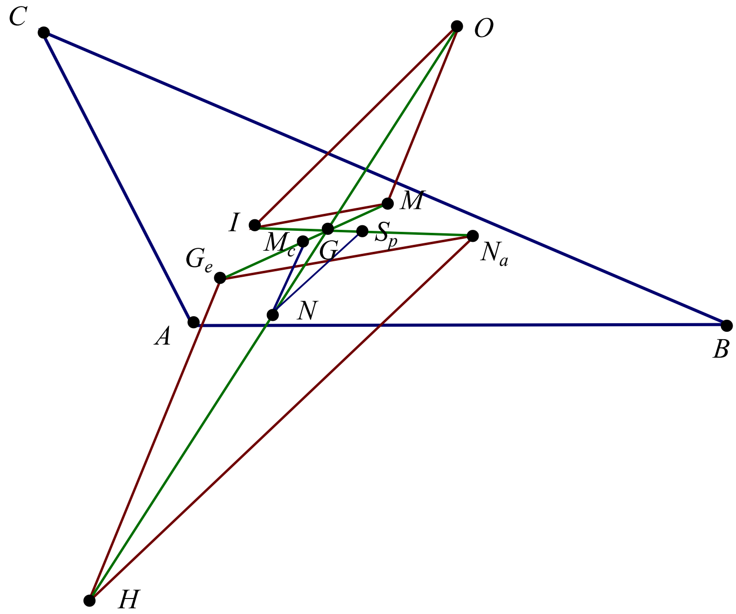

Figure 1.

The length of segment OH is

$$
\begin{aligned}
OH^2 &= \left(\frac{a^2S_A}{2S^2} - \frac{S_{BC}}{S^2}\right)^2 S_A + \left(\frac{b^2S_B}{2S^2} - \frac{S_{CA}}{S^2}\right) S_B + \left(\frac{c^2S_C}{2S^2} - \frac{S_{AB}}{S^2}\right)^2 S_C \\
&= \left(\frac{a^2S_A - 2S_{BC}}{2S^2}\right)^2 S_A + \left(\frac{b^2S_B - 2S_{CA}}{2S^2}\right)^2 S_B + \left(\frac{c^2S_C - 2S_{AB}}{2S^2}\right)^2 S_C \\
&= \left(\frac{S^2 - 3S_{BC}}{2S^2}\right)^2 S_A + \left(\frac{S^2 - 3S_{CA}}{2S^2}\right)^2 S_B + \left(\frac{S^2 - 3S_{AB}}{2S^2}\right)^2 S_C \\
&= \frac{9}{4}\left[\left(\frac{1}{3} - \frac{S_{BC}}{S^2}\right)^2 S_A + \left(\frac{1}{3} - \frac{S_{CA}}{S^2}\right)^2 S_B + \left(\frac{1}{3} - \frac{S_{AB}}{S^2}\right)^2 S_C\right] \Leftrightarrow OH^2 = \frac{9}{4} GH^2 \\
&= \frac{9}{4}\left[\frac{1}{9}(S_A + S_B + S_C) - \frac{2S_AS_BS_C}{S^2} + \frac{S_AS_BS_C(S_{BC} + S_{CA} + S_{AB})}{S^4}\right] \\
&= \frac{9}{4}\left[\frac{1}{9} S_\omega - \frac{S_AS_BS_C}{S^2}\right] = \frac{9}{4}\left[\frac{1}{9} S_\omega - \frac{(S_\omega - 4R^2)S^2}{S^2}\right] \\
&= \frac{9}{4}\left[-\frac{8}{9} S_\omega + 4R^2\right] = -2S_\omega + 9R^2.
\end{aligned}
$$

Consequently:

$$
\begin{aligned}
OH = \frac{3}{2} GH \Leftrightarrow 2OH = 3(OH - GO) \\
\Leftrightarrow OH = 3GO \Leftrightarrow OH = 3\left(\frac{OH}{2} - GN\right) \Leftrightarrow OH = 6GN.
\end{aligned}
$$

□

Theorem 2.2. *The incenter (I), the triangle centroid (G), the Nagel point (Na) and the Spieker center (Sp) are collinear $(Nagel\ line)$ and*

$$\begin{aligned} (a) &\quad (INa)^2 = S_\omega + 6r^2 - 12Rr = s^2 + 5r^2 - 16Rr, \\ (b) &\quad INa = \frac{3}{2} GNa = 2ISp = 2NaSp = 3GI = 6GSp. \end{aligned}$$

Proof. The triplet of points (I, G, Na) are collinear if and only if

$$\begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ s-a & s-b & s-c \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} s & s & s \\ 1 & 1 & 1 \\ s-a & s-b & s-c \end{vmatrix} = 0. \text{ True.}$$

The Spieker center is the midpoint of segment INa (Figure 1). Indeed

$$x_I + x_{Na} = \frac{a}{2s} + \frac{s-a}{s} = \frac{2s-a}{2s} = 2x_{Sp}.$$

The length of segment INa is

$$\begin{aligned} (INa)^2 &= \left(\frac{s-a}{s} - \frac{a}{2s}\right)^2 S_A + \left(\frac{s-b}{s} - \frac{b}{2s}\right)^2 S_B + \left(\frac{s-c}{s} - \frac{c}{2s}\right)^2 S_C \\ &= \left(\frac{2s-3a}{2s}\right)^2 SA + \left(\frac{2s-3b}{2s}\right)^2 S_B + \left(\frac{2s-3c}{2s}\right)^2 S_C \\ &= 9\left[\left(\frac{1}{3} - \frac{a}{2s}\right)^2 S_A + \left(\frac{1}{3} - \frac{b}{2s}\right)^2 S_B + \left(\frac{1}{3} - \frac{c}{2s}\right)^2 S_C\right] \\ &\qquad \Leftrightarrow (INa)^2 = 9GI^2 \\ &= 9\left[\frac{1}{9}(S_A + S_B + S_C) - \frac{aS_A + bS_B + cS_C}{3s} + \frac{a^2S_A + b^2S_B + c^2S_C}{4s^2}\right] \\ &= S_\omega - 3\frac{4sr(r+R)}{s} + 9\frac{2S^2}{4s^2} = S_\omega - 12r^2 - 12Rr + 18r^2 \\ &= S_\omega + 6r^2 - 12Rr = s^2 + 5r^2 - 16Rr. \end{aligned}$$

Consequently:

$$\begin{aligned} INa = 3GI &\Leftrightarrow INa = 3(INa - GNa) \Leftrightarrow INa = \frac{3}{2} GNa \\ &\Leftrightarrow 2INa = 3\left(\frac{INa}{2} + GSp\right) \Leftrightarrow INa = 6GSp. \end{aligned}$$

□

Theorem 2.3. *The Mittenpunkt (M), the triangle centroid (G), the Gergonne point (Ge) and the complement of the Mittenpunkt (Mc) are collinear and*

$$\begin{aligned} (a) &\quad GM^2 = \frac{1}{9} S_\omega - \frac{2s^2(r^2+2R^2)}{3(r+4R)^2} = -\frac{1}{9}(r^2+4Rr) + \frac{s^2(-5r^2+8Rr+4R^2)}{9(r+4R)^2}; \\ (b) &\quad MGe = \frac{3}{2} GGe = 2MMc = 2GeMc = 3GM = 6GMc. \end{aligned}$$

Proof. The triplet of points (M, G, Ge) are collinear if and only if

$$\begin{vmatrix} a(s-a) & b(s-b) & c(s-c) \\ 1 & 1 & 1 \\ (s-b)(s-c) & (s-c)(s-a) & (s-a)(s-b) \end{vmatrix} = 0 \Leftrightarrow$$

$$\begin{vmatrix} a(s-a)+(s-b)(s-c) & b(s-b)+(s-c)(s-a) & c(s-c)+(s-a)(s-b) \\ 1 & 1 & 1 \\ (s-b)(s-c) & (s-c)(s-a) & (s-a)(s-b) \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} r(r+4R) & r(r+4R) & r(r+4R) \\ 1 & 1 & 1 \\ (s-b)(s-c) & (s-c)(s-a) & (s-a)(s-b) \end{vmatrix} = 0. \text{ True.}$$

The complement of the Mittenpunkt is the midpoint of segment MGe (Figure 1). Indeed

$$x_M + x_{Ge} = \frac{a(s-a)}{2r(r+4R)} + \frac{(s-b)(s-c)}{r(r+4R)} = \frac{a(s-a)+2(s-b)(s-c)}{2r(r+4R)} = 2x_{Mc}.$$

Now we calculate the length of segment GM:

$$\begin{aligned} GM^2 &= \left(\frac{1}{3} - \frac{a(s-a)}{2r(r+4R)}\right)^2 S_A + \left(\frac{1}{3} - \frac{b(s-b)}{2r(r+4R)}\right)^2 S_B + \left(\frac{1}{3} - \frac{c(s-c)}{2r(r+4R)}\right)^2 S_C \\ &= \frac{1}{9}(S_A + S_B + S_C) - \frac{1}{3r(r+4R)}[a(s-a)S_A + b(s-b)S_B + c(s-c)S_C] \\ &\quad + \frac{1}{4r^2(r+4R)^2}[a^2(s-a)^2 S_A + b^2(s-b)^2 S_B + c^2(s-c)^2 S_C] \\ &= \frac{1}{9} S_\omega - \frac{1}{3r(r+4R)}[4s^2 r(r+R) - 8s^2 r^2] \\ &\quad + \frac{1}{4r^2(r+4R)^2}[8s^4 r^2 - 8s^2 r^2(3s^2 - r^2 - 6Rr - 8R^2) + 16 s^2 r^2 (s^2 - r^2 - 4Rr - 3R^2)] \\ &= \frac{1}{9} S_\omega - \frac{4s^2}{3(r+4R)}(R-r) - \frac{2s^2}{(r+4R)^2}(r^2 + 2Rr - 2R^2) = \frac{1}{9} S_\omega - \frac{2s^2(r^2+2R^2)}{3(r+4R)^2} \\ &= \frac{1}{9}(s^2 - r^2 - 4Rr) - \frac{2s^2(r^2+2R^2)}{3(r+4R)^2} \\ &= -\frac{1}{9}(r^2 + 4Rr) + \frac{s^2(r^2 + 8Rr + 16R^2 - 6r^2 - 12R^2)}{9(r+4R)^2} \\ &= -\frac{1}{9}(r^2 + 4Rr) + \frac{s^2(-5r^2 + 8Rr + 4R^2)}{9(r+4R)^2}. \end{aligned}$$

Between the segment MGe and GGe exist the following relation:

$$(MGe)^2 = \left(\frac{a(s-a)}{2r(r+4R)} - \frac{(s-b)(s-c)}{r(r+4R)}\right)^2 S_A$$
$$+\left(\frac{b(s-b)}{2r(r+4R)} - \frac{(s-c)(s-a)}{r(r+4R)}\right)^2 S_B + \left(\frac{c(s-c)}{2r(r+4R)} - \frac{(s-a)(s-b)}{r(r+4R)}\right)^2 S_C$$
$$= \left(\frac{a(s-a)-2(s-b)(s-c)}{2r(4+4R)}\right)^2 S_A + \left(\frac{b(s-b)-2(s-c)(s-a)}{2r(r+4R)}\right)^2 S_B$$
$$+\left(\frac{c(s-c)-2(s-a)(s-b)}{2r(r+4R)}\right)^2 S_C$$
$$= \frac{1}{4}\left[\left(1-\frac{3(s-b)(s-c)}{r(r+4R)}\right)^2 S_A + \left(1-\frac{3(s-c)(s-a)}{r(r+4R)}\right)^2 S_B + \left(1-\frac{3(s-a)(s-b)}{r(r+4R)}\right)^2 S_C\right]$$
$$= \frac{9}{4}\left[\left(\frac{1}{3}-\frac{(s-b)(s-c)}{r(r+4R)}\right)^2 S_A + \left(\frac{1}{3}-\frac{(s-c)(s-a)}{r(r+4R)}\right)^2 S_B + \left(\frac{1}{3}-\frac{(s-a)(s-b)}{r(r+4R)}\right)^2 S_C\right]$$
$$= \frac{9}{4}(GGe)^2.$$

Consequently:

$$MGe = \frac{3}{2}GGe \Leftrightarrow 2MGe = 3(MGe - GM) \Leftrightarrow MGe = 3GM$$
$$\Leftrightarrow MGe = 3\left(\frac{MGe}{2} - GMc\right) \Leftrightarrow MGe = 6GMc.$$

□

3. Distances of the circumcenter

With the formula

$$OP^2 = R^2 - \frac{a^2vw + b^2wu + c^2uv}{(u+v+w)^2},$$

where O is the circumcenter and $P = (u : v : w)$ is an arbitrary point, we obtain immediate the distances of the circumcenter to the incenter, the Gergonne point, the Nagel point, the Mittenpunkt and the Spieker center:

$$OI^2 = R^2 - \frac{abc(a+b+c)}{(a+b+c)^2} = R^2 - \frac{abc}{a+b+c} = R^2 - \frac{4Rsr}{2s} = R^2 - 2Rr;$$
$$(OGe)^2 = R^2 - \frac{(s-a)(s-b)(s-c)}{r^2(r+4R)^2}[a^2(s-a) + b^2(s-b) + c^2(s-c)]$$
$$= R^2 - \frac{sr^2}{r^2(r+4R)^2}[s(a^2+b^2+c^2) - (a^3+b^3+c^3)]$$
$$= R^2 - \frac{s}{(r+4R)^2}[2s(s^2-r^2-4Rr) - 2s(s^2-3r^2-6Rr)]$$
$$= R^2 - \frac{4s^2(r^2+Rr)}{(r+4R)^2};$$

$$\begin{aligned}
(ONa)^2 &= R^2 - \frac{1}{s^2}[a^2(s-b)(s-c) + b^2(s-c)(s-a) + c^2(s-a)(s-b)] \\
&= R^2 - \frac{1}{s^2}[a^2(-s^2 + as + bc) + b^2(-s^2 + bs + ca) + c^2(-s^2 + cs + ab)] \\
&= R^2 - \frac{1}{s^2}[-s^2(a^2 + b^2 + c^2) + s(a^3 + b^3 + c^3) + abc(a + b + c)] \\
&= R^2 - \frac{1}{s^2}[-2s^2(s^2 - r^2 - 4Rr) + 2s^2(s^2 - 3r^2 - 6Rr) + 2s \cdot 4Rsr] \\
&= R^2 - 2(-s^2 + r^2 + 4Rr + s^2 - 3r^2 - 6Rr + 4Rr) \\
&= R^2 - 4Rr + 4r^2 = (R - 2r)^2; \\
OM^2 &= R^2 - \frac{abc}{4r^2(r + 4R)^2}[a(s-b)(s-c) + b(s-c)(s-a) + c(s-a)(s-b)] \\
&= R^2 - \frac{4Rsr}{4r^2(r + 4R)^2}[2s^3 - a(b + c)s - b(c + a)s - c(a + b)s + 3abc] \\
&= R^2 - \frac{Rs}{r(r + 4R)^2}[2s^3 - 2s(bc + ca + ab) + 3abc] \\
&= R^2 - \frac{Rs}{r(r + 4R)^2}[2s^3 - 2s(s^2 + r^2 + 4Rr) + 12Rsr] \\
&= R^2 - \frac{Rs}{r(r + 4R)^2}[-2sr(r + 4R) + 12Rsr] \\
&= R^2 - \frac{2Rs^2}{(r + 4R)^2}[-(r + 4R) + 6R] = R^2 - \frac{2s^2(2R^2 - Rr)}{(r + 4R)^2}; \\
(OSp)^2 &= R^2 - \frac{1}{16s^2}[a^2(2s-b)(2s-c) + b^2(2s-c)(2s-a) + c^2(2s-a)(2s-b)] \\
&= R^2 - \frac{1}{16s^2}[a^2(2sa + bc) + b^2(2sb + ca) + c^2(2sc + ab)] \\
&= R^2 - \frac{1}{16s^2}[2s(a^3 + b^3 + c^3) + abc(a + b + c)] \\
&= R^2 - \frac{1}{16s^2}[4s^2(s^2 - 3r^2 - 6Rr) + 4Rsr \cdot 2s] \\
&= R^2 - \frac{1}{4}(s^2 - 3r^2 - 4Rr).
\end{aligned}$$

Since the coordinates of the point Mc are excessively complicated it is easier to calculate the length of segment OMc with the median's theorem.

Theorem 3.1. *The measure of line segment OMc is:*

$$(OMc)^2 = \frac{1}{4}(r + 2R)^2 - \frac{3s^2(r + 2R)^2}{4(r + 4R)^2}.$$

Proof. We apply the median's theorem for the triangle $OMGe$:

$$\begin{aligned}
4(OMc)^2 &= 2[OM^2 + (OGe)^2] - (MGe)^2 = 2[OM^2 + (OGe)^2] - 9GM^2 \\
&= 2\left[R^2 - \frac{2s^2(2R^2 - Rr)}{(r + 4R)^2} + R^2 - \frac{4s^2(r^2 + Rr)}{(r + 4R)^2}\right] - S_\omega + \frac{6s^2(r^2 + 2R^2)}{(r + 4R)^2}
\end{aligned}$$

$$= 2\left[2R^2 - \frac{2s^2(2R^2+Rr+2r^2)}{(r+4R)^2}\right] - S_\omega + \frac{6s^2(r^2+2R^2)}{(r+4R)^2}$$
$$= -S_\omega + 4R^2 + \frac{2s^2(3r^2+6R^2-4R^2-2Rr-4r^2)}{(r+4R)^2}$$
$$= -S_\omega + 4R^2 - \frac{2s^2(r^2+2Rr-2R^2)}{(r+4R)^2}$$
$$= -(s^2-r^2-4Rr) + 4R^2 - \frac{2s^2(r^2+2Rr-2R^2)}{(r+4R)^2}$$
$$= (r+2R)^2 - \frac{s^2(r^2+8Rr+16R^2+2r^2+4Rr-4R^2)}{(r+4R)^2}$$
$$= (r+2R)^2 - \frac{s^2(3r^2+12Rr+12R^2)}{(r+4R)^2} = (r+2R)^2 - \frac{3s^2(r+2R)^2}{(r+4R)^2}$$
$$\Leftrightarrow (OMc)^2 = \frac{1}{4}(r+2R)^2 - \frac{3s^2(r+2R)^2}{4(r+4R)^2}.$$

□

4. Main results

The ten triangle centers given in Introduction determine 45 line segments. From the three analogies we have already the length of 18 segments, plus also 6 distances of circumcenter. Our intention is to determine the length of the remaining segments, using similarities, the median's theorem and the Stewart's theorem [2].

Theorem 4.1. *The triangles $HNaGe$, OIM and $NSpMc$ are similar and*

$$HNa = 2OI = 4NSp,\ NaGe = 2IM = 4SpMc,\ GeH = 2MO = 4McN.$$

Proof. From the above analogies we have:

$$\frac{GH}{GO} = 2 = \frac{GNa}{GI} \quad \text{and} \quad \frac{GH}{GN} = 4 = \frac{GNa}{GSp}.$$

Since $OGI\measuredangle \equiv HGNa\measuredangle$, then the triangles $GHNa$, GOI and $GNSp$ are similar, i.e. $HNa \| OI \| NSp$ and $HNa = 2OI = 4NSp$ (Figure 1).
Similarly we obtain that $NaGe\|IM\|SpMc$ and $NaGe = 2IM = 4SpMc$, respectively $GeH\|MO\|McN$ and $GeH = 2MO = 4McN$. □

Theorem 4.2. *The measure of line segment IM is:*

$$IM^2 = -4Rr + \frac{4s^2(R^2+Rr)}{(r+4R)^2}.$$

Proof. We have:

$$IM^2 = \left(\frac{a}{2s} - \frac{a(s-a)}{2r(r+4R)}\right)^2 S_A + \left(\frac{b}{2s} - \frac{b(s-b)}{2r(r+4R)}\right)^2 S_B + \left(\frac{c}{2s} - \frac{c(s-c)}{2r(r+4R)}\right)^2 S_C =$$

$$= \frac{1}{4}\left[\left(\frac{1}{s}-\frac{s-a}{r(r+4R)}\right)^2 a^2S_A+\left(\frac{1}{s}-\frac{s-b}{r(r+4R)}\right)^2 b^2S_B+\left(\frac{1}{s}-\frac{s-c}{r(r+4R)}\right)^2 c^2S_C\right]$$

$$= \frac{1}{4s^2}\left(a^2S_A+b^2S_B+c^2S_C\right)-\frac{1}{2sr(r+4R)}\left[(s-a)a^2S_A+(s-b)b^2S_B+(s-c)c^2S_C\right]$$

$$+\frac{1}{4r^2(r+4R)^2}\left[a^2(s-a)^2S_A+b^2(s-b)^2S_B+c^2(s-c)^2S_C\right]$$

$$= 2r^2-\frac{1}{2sr(r+4R)}\left[8s^3r^2-4sr^2(3s^2-r^2-6Rr-8R^2)\right]$$

$$+\frac{1}{4r^2(r+4R)^2}\left[8s^4r^2-8s^2r^2(3s^2-r^2-6Rr-8R^2)+16s^2r^2(s^2-r^2-4Rr-3R^2)\right]$$

$$= 2r^2-\frac{2r}{r+4R}\left(-s^2+r^2+6Rr+8R^2\right)-\frac{2s^2}{(r+4R)^2}\left(r^2+2Rr-2R^2\right)$$

$$= \frac{2r}{r+4R}\left(s^2-2Rr-8R^2\right)-\frac{2s^2}{(r+4R)^2}\left(r^2+2Rr-2R^2\right)=-4Rr+\frac{4s^2(R^2+Rr)}{(r+4R)^2}.$$

□

Theorem 4.3.

(a) $HM \| OMc$ *and* $HM = 2OMc$; (b) $HI \| OSp$ *and* $HI = 2OSp$;
(c) $ONa \| IN$ *and* $ONa = 2IN$; (d) $OGe \| MN$ *and* $OGe = 2MN$;
(e) $IGe \| MSp$ *and* $IGe = 2MSp$; (f) $MNa \| IMc$ *and* $MNa = 2IMc$.

Proof. (a) From the above analogies we have: $\dfrac{GH}{GO} = 2 = \dfrac{GM}{GMc}$.
Since $OGMc\measuredangle \equiv HGM\measuredangle$, then the triangles GHM and $GOMc$ are similar, i.e. $HM \| OMc$ and $HM = 2OMc$ (Figure 2).

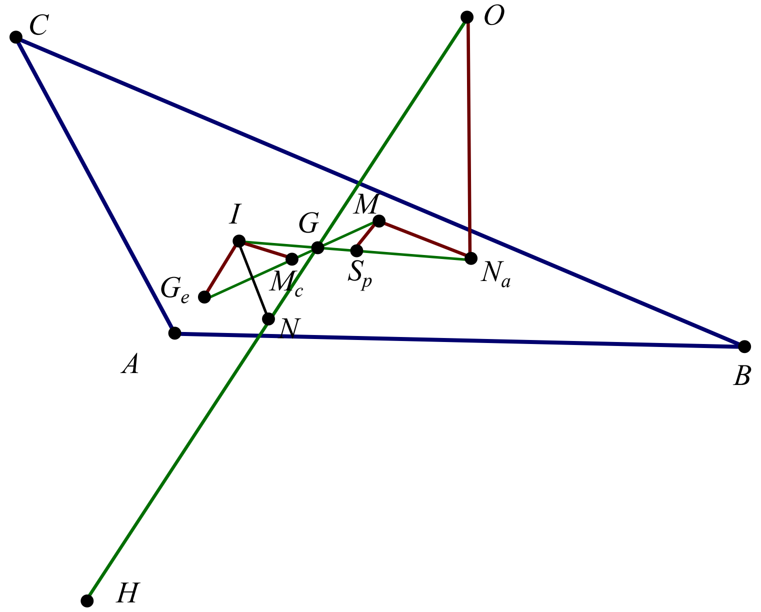

Figure 2.

It is easy to prove similarly the other properties. □

Theorem 4.4. *The measure of line segment IGe is:*

$$(IGe)^2 = r^2 - \frac{3s^2r^2}{(r+4R)^2}.$$

Proof. We use the following form of Stewart's theorem for the triangle ABC and for the cevian AD [2]:

$$AD^2 = \lambda AC^2 + (1-\lambda)AB^2 - \lambda(1-\lambda)BC^2, \text{ where } \lambda = \frac{BD}{BC}.$$

We apply the Stewart's theorem for the triangle $IMGe$ and for the cevian IG:

$$IG^2 = \lambda(IGe)^2 + (1-\lambda)IM^2 - \lambda(1-\lambda)(MGe)^2, \text{ where } \lambda = \frac{MG}{MGe} = \frac{1}{3}.$$

Therefore

$$\begin{aligned}
IG^2 = \frac{1}{3}(IGe)^2 + \frac{2}{3}IM^2 - \frac{2}{9}(MGe)^2 &\Leftrightarrow 9IG^2 = 3(IGe)^2 + 6IM^2 - 2(MGe)^2 \\
\Leftrightarrow 3(IGe)^2 = 9IG^2 + 2(MGe)^2 - 6IM^2 &\Leftrightarrow 3(IGe)^2 = (INa)^2 + 18GM^2 - 6IM^2 \\
\Leftrightarrow 3(IGe)^2 &= S_\omega + 6r^2 - 12Rr \\
+18\left(\frac{1}{9}S_\omega - \frac{2s^2(r^2+2R^2)}{3(r+4R)^2}\right) &- 6\left(-4Rr + \frac{4s^2(R^2+Rr)}{(r+4R)^2}\right) \\
\Leftrightarrow 3(IGe)^2 = 3S_\omega + 6r^2 + 12Rr &- \frac{12s^2(r^2+2R^2)}{(r+4R)^2} - \frac{24s^2(R^2+Rr)}{(r+4R)^2} \\
\Leftrightarrow (IGe)^2 &= S_\omega + 2r^2 + 4Rr - \frac{4s^2(r^2+2Rr+4R^2)}{(r+4R)^2} \\
&= s^2 - r^2 - 4Rr + 2r^2 + 4Rr - \frac{4s^2(r^2+2Rr+4R^2)}{(r+4R)^2} \\
&= s^2 + r^2 - \frac{4s^2(r^2+2Rr+4R^2)}{(r+4R)^2} = r^2 - \frac{3s^2r^2}{(r+4R)^2}.
\end{aligned}$$

□

Theorem 4.5. *The measure of line segment IMc is:* $(IMc)^2 = \frac{1}{4}(3r^2 - 4Rr) - \frac{s^2(r^2-4R^2)}{4(r+4R)^2}.$

Proof. We apply the median's theorem for the triangle $IMGe$:

$$\begin{aligned}
4(IMc)^2 &= 2[IM^2 + (IGe)^2] - (MGe)^2 = 2[IM^2 + (IGe)^2] - 9GM^2 \\
&= 2\left[-4Rr + \frac{4s^2(R^2+Rr)}{(r+4R)^2} + r^2 - \frac{3s^2r^2}{(r+4R)^2}\right] - S_\omega + \frac{6s^2(r^2+2R^2)}{(r+4R)^2} \\
&= -S_\omega + 2r^2 - 8Rr + \frac{2s^2(4R^2+4Rr-3r^2+3r^2+6R^2)}{(r+4R)^2}
\end{aligned}$$

$$= -(s^2 - r^2 - 4Rr) + 2r^2 - 8Rr + \frac{2s^2(10R^2 + 4Rr)}{(r+4R)^2}$$
$$= 3r^2 - 4Rr + \frac{s^2(20R^2 + 8Rr - r^2 - 8Rr - 16R^2)}{(r+4R)^2}$$
$$= 3r^2 - 4Rr - \frac{s^2(r^2 - 4R^2)}{(r+4R)^2}$$
$$\Leftrightarrow (IMc)^2 = \frac{1}{4}(3r^2 - 4Rr) - \frac{s^2(r^2-4R^2)}{4(r+4R)^2}.$$

□

Now we will calculate the measures of remaining 6 segments: GeN, NaN, HSp, $GeSp$, HMc, $NaMc$.

Theorem 4.6. *The measure of line segment GeN is:*

$$(GeN)^2 = \frac{1}{4}(R^2 - 2r^2 - 8Rr) - \frac{3s^2(r^2 - 4Rr)}{2(r+4R)^2}.$$

Proof. We apply the median's theorem for the triangle $GeOH$:

$$4(GeN)^2 = 2[(OGe)^2 + (HGe)^2] - OH^2 = 2[(OGe)^2 + 4OM^2] - OH^2$$
$$= 2\left[R^2 - \frac{4s^2(r^2+Rr)}{(r+4R)^2} + 4R^2 - \frac{8s^2(2R^2-Rr)}{(r+4R)^2}\right] - (-2S_\omega + 9R^2)$$
$$= 2S_\omega + R^2 - \frac{8s^2(r^2 - Rr + 4R^2)}{(r+4R)^2}$$
$$= 2(s^2 - r^2 - 4Rr) + R^2 - \frac{8s^2(r^2 - Rr + 4R^2)}{(r+4R)^2}$$
$$= R^2 - 2r^2 - 8Rr - \frac{2s^2(4r^2 - 4Rr + 16R^2 - r^2 - 8Rr - 16R^2)}{(r+4R)^2}$$
$$= R^2 - 2r^2 - 8Rr - \frac{2s^2(3r^2 - 12Rr)}{(r+4R)^2}$$
$$\Leftrightarrow (GeN)^2 = \frac{1}{4}(R^2 - 2r^2 - 8Rr) - \frac{3s^2(r^2-4Rr)}{2(r+4R)^2}.$$

□

Theorem 4.7. *The measure of line segment NaN is:* $(NaN)^2 = \dfrac{1}{4}(2s^2 + 6r^2 - 32Rr + R^2)$.

Proof. We apply the median's theorem for the triangle $NaOH$:

$$4(NaN)^2 = 2[(ONa)^2 + (HNa)^2] - OH^2 = 2[(ONa)^2 + 4OI^2] - OH^2$$
$$= 2[(R-2r)^2 + 4(R^2 - 2Rr)] - (-2S_\omega + 9R^2)$$
$$= 2S_\omega + 8r^2 - 24Rr + R^2 = 2(s^2 - r^2 - 4Rr) + 8r^2 - 24Rr + R^2$$
$$= 2s^2 + 6r^2 - 32Rr + R^2 \Leftrightarrow (NaN)^2 = \frac{1}{4}(2s^2 + 6r^2 - 32Rr + R^2).$$

□

Theorem 4.8. *The measure of line segment HSp is:*

$$(HSp)^2 = \frac{1}{4}(-3s^2 + r^2 + 8Rr + 16R^2).$$

Proof. We apply the median's theorem for the triangle $HINa$:

$$4(HSp)^2 = 2[HI^2 + (HNa)^2] - (INa)^2 = 2[4(OSp)^2 + 4OI^2] - (INa)^2$$
$$= 2(-s^2 + 3r^2 + 4Rr + 4R^2 + 4R^2 - 8Rr) - (s^2 + 5r^2 - 16Rr)$$
$$= 2(-s^2 + 3r^2 - 4Rr + 8R^2) - (s^2 + 5r^2 - 16Rr)$$
$$= -3s^2 + r^2 + 8Rr + 16R^2 \Leftrightarrow (HSp)^2 = \frac{1}{4}(-3s^2 + r^2 + 8Rr + 16R^2).$$

□

Theorem 4.9. *The measure of line segment $GeSp$ is:*

$$(GeSp)^2 = -\frac{1}{4}(3r^2 + 16Rr) + \frac{s^2(-r^2 + 2Rr)}{(r + 4R)^2)}.$$

Proof. We apply the median's theorem for the triangle $GeINa$:

$$4(GeSp)^2 = 2[(IGe)^2 + (GeNa)^2] - (INa)^2 = 2[(IGe)^2 + 4IM^2] - (INa)^2$$
$$= 2\left[r^2 - \frac{3s^2r^2}{(r + 4R)^2} - 16Rr + \frac{16s^2(R^2 + Rr)}{(r + 4R)^2}\right] - (s^2 + 5r^2 - 16Rr)$$
$$= -3r^2 - 16Rr - \frac{s^2(3r^2 - 16R^2 - 16Rr + r^2 + 8Rr + 16R^2)}{(r + 4R)^2}$$
$$= -3r^2 - 16Rr - \frac{s^2(4r^2 - 8Rr)}{(r + 4R)^2} = -3r^2 - 16Rr + \frac{4s^2(-r^2 + 2Rr)}{(r + 4R)^2}$$
$$\Leftrightarrow (GeSp)^2 = -\frac{1}{4}(3r^2 + 16Rr) + \frac{s^2(-r^2 + 2Rr)}{(r + 4R)^2}.$$

□

Theorem 4.10. *The measure of line segment HMc is:*

$$(HMc)^2 = \frac{1}{4}(3r^2 + 12Rr + 16R^2) - \frac{s^2(r^2 + 16Rr + 60R^2)}{4(r + 4R)^2}.$$

Proof. We apply the median's theorem for the triangle $HMGe$:

$$4(HMc)^2 = 2[HM^2 + (HGe)^2] - (MGe)^2 = 2[4(OMc)^2 + 4OM^2] - 9GM^2$$
$$= 2\left[(r+2R)^2 - \frac{3s^2(r+2R)^2}{(r + 4R)^2} + 4R^2 - \frac{8s^2(2R^2 - Rr)}{(r + 4R)^2}\right] - S_\omega + \frac{6s^2(r^2 + 2R^2)}{(r + 4R)^2}$$
$$= -S_\omega + 2(r+2R)^2 + 8R^2 - \frac{2s^2(3r^2 + 12Rr + 12R^2 + 16R^2 - 8Rr - 3r^2 - 6R^2)}{(r + 4R)^2}$$
$$= -(s^2 - r^2 - 4Rr) + 2(r + 2R)^2 + 8R^2 - \frac{2s^2(4Rr + 22R^2)}{(r + 4R)^2}$$
$$= 3r^2 + 12Rr + 16R^2 - \frac{s^2(r^2 + 8Rr + 16R^2 + 8Rr + 44R^2)}{(r + 4R)^2}$$

$$= 3r^2 + 12Rr + 16R^2 - \frac{s^2(r^2 + 16Rr + 60R^2)}{(r + 4R)^2}$$

$$\Leftrightarrow (HMc)^2 = \frac{1}{4}(3r^2 + 12Rr + 16R^2) - \frac{s^2(r^2 + 16Rr + 60R^2)}{4(r + 4R)^2}.$$

□

Theorem 4.11. *The measure of line segment NaMc is:*

$$(NaMc)^2 = \frac{1}{4}(7r^2 - 36Rr) + \frac{3s^2(r^2 + 8Rr + 12R^2)}{4(r + 4R)^2}.$$

Proof. We apply the median's theorem for the triangle $NaMMc$:

$$4(NaMc)^2 = 2[NaM^2 + (NaGe)^2] - (MGe)^2 = 2[4(IMc)^2 + 4IM^2] - 9GM^2$$

$$= 2\left[3r^2 - 4Rr - \frac{s^2(r^2 - 4R^2)}{(r + 4R)^2} - 16Rr + \frac{16s^2(R^2 + Rr)}{(r + 4R)^2}\right] - S_\omega + \frac{6s^2(r^2 + 2R^2)}{(r + 4R)^2}$$

$$= -S_\omega + 6r^2 - 40Rr + \frac{s^2(-2r^2 + 8R^2 + 32R^2 + 32Rr + 6r^2 + 12R^2)}{(r + 4R)^2}$$

$$= -(s^2 - r^2 - 4Rr) + 6r^2 - 40Rr + \frac{s^2(4r^2 + 32Rr + 52R^2)}{(r + 4R)^2}$$

$$= 7r^2 - 36Rr + \frac{3s^2(r^2 + 8Rr + 12R^2)}{(r + 4R)^2}$$

$$\Leftrightarrow (NaMc)^2 = \frac{1}{4}(7r^2 - 36Rr) + \frac{3s^2(r^2 + 8Rr + 12R^2)}{4(r + 4R)^2}.$$

5. Recapitulation

$$IG^2 = \frac{1}{9}(INa)^2 = \frac{1}{9}(s^2 + 5r^2 - 16Rr); \tag{1}$$

$$IO^2 = R^2 - 2Rr; \tag{2}$$

$$IH^2 = 4(OSp)^2 = -s^2 + 3r^2 + 4Rr + 4R^2; \tag{3}$$

$$IN^2 = \frac{1}{4}(ONa)^2 = \frac{1}{4}(R - 2r)^2 = \left(\frac{R}{2} - r\right)^2; \tag{4}$$

$$(IGe)^2 = r^2 - \frac{3s^2r^2}{(r + 4R)^2}; \tag{5}$$

$$(INa)^2 = S_\omega + 6r^2 - 12Rr = s^2 + 5r^2 - 16Rr; \tag{6}$$

$$IM^2 = -4Rr + \frac{4s^2(R^2 + Rr)}{(r + 4R)^2}; \tag{7}$$

$$(ISp)^2 = \frac{1}{4}(INa)^2 = \frac{1}{4}(s^2 + 5r^2 - 16Rr); \tag{8}$$

$$(IMc)^2 = \frac{1}{4}(3r^2 - 4Rr) - \frac{s^2(r^2 - 4R^2)}{4(r + 4R)^2}; \tag{9}$$

$$GO^2 = \frac{1}{9}\,OH^2 = \frac{1}{9}\,(-2s^2 + 2r^2 + 8Rr + 9R^2); \tag{10}$$

$$GH^2 = \frac{4}{9}\,OH^2 = \frac{4}{9}\,(-2s^2 + 2r^2 + 8Rr + 9R^2); \tag{11}$$

$$GN^2 = \frac{1}{36}\,OH^2 = \frac{1}{36}\,(-2s^2 + 2r^2 + 8Rr + 9R^2); \tag{12}$$

$$(GGe)^2 = 4GM^2 = \frac{4}{9}\,S_\omega - \frac{8s^2(r^2 + 2R^2)}{3(r + 4R)^2}\,; \tag{13}$$

$$(GNa)^2 = \frac{4}{9}\,(INa)^2 = \frac{4}{9}\,(s^2 + 5r^2 - 16Rr); \tag{14}$$

$$GM^2 = -\frac{1}{9}\,(r^2 + 4Rr) + \frac{s^2(-5r^2 + 8Rr + 4R^2)}{9(r + 4R)^2}\,; \tag{15}$$

$$(GSp)^2 = \frac{1}{36}\,(INa)^2 = \frac{1}{36}\,(s^2 + 5r^2 - 16Rr); \tag{16}$$

$$(GMc)^2 = \frac{1}{4}\,GM^2 = \frac{1}{36}\,S_\omega - \frac{s^2(r^2 + 2R^2)}{6(r + 4R)^2}\,; \tag{17}$$

$$OH^2 = -2S_\omega + 9R^2 = -2s^2 + 2r^2 + 8Rr + 9R^2; \tag{18}$$

$$ON^2 = \frac{1}{4}\,OH^2 = \frac{1}{4}\,(-2s^2 + 2r^2 + 8Rr + 9R^2); \tag{19}$$

$$(OGe)^2 = R^2 - \frac{4s^2(r^2 + Rr)}{(r + 4R)^2}\,; \tag{20}$$

$$(ONa)^2 = (R - 2r)^2; \tag{21}$$

$$OM^2 = R^2 - \frac{2s^2(2R^2 - Rr)}{(r + 4R)^2}\,; \tag{22}$$

$$(OSp)^2 = R^2 - \frac{1}{4}\,(s^2 - 3r^2 - 4Rr); \tag{23}$$

$$(OMc)^2 = \frac{1}{4}\,(r + 2R)^2 - \frac{3s^2(r + 2R)^2}{4(r + 4R)^2}\,; \tag{24}$$

$$HN^2 = \frac{1}{4}\,OH^2 = \frac{1}{4}\,(-2s^2 + 2r^2 + 8Rr + 9R^2); \tag{25}$$

$$(HGe)^2 = 4OM^2 = 4R^2 - \frac{8s^2(2R^2 - Rr)}{(r + 4R)^2}\,; \tag{26}$$

$$(HNa)^2 = 4IO^2 = 4(R^2 - 2Rr); \tag{27}$$

$$HM^2 = 4(OMc)^2 = (r + 2R)^2 - \frac{3s^2(r + 2R)^2}{(r + 4R)^2}\,; \tag{28}$$

$$(HSp)^2 = \frac{1}{4}\,(-3s^2 + r^2 + 8Rr + 16R^2); \tag{29}$$

$$(HMc)^2 = \frac{1}{4}\,(3r^2 + 12Rr + 16R^2) - \frac{s^2(r^2 + 16Rr + 60R^2)}{4(r + 4R)^2}\,; \tag{30}$$

$$(NGe)^2 = \frac{1}{4}\,(R^2 - 2r^2 - 8Rr) - \frac{3s^2(r^2 - 4Rr)}{2(r + 4R)^2}\,; \tag{31}$$

$$(NNa)^2 = \frac{1}{4}\,(2s^2 + 6r^2 - 32Rr + R^2); \tag{32}$$

$$NM^2 = \frac{1}{4}\,(OGe)^2 = \frac{1}{4}\,R^2 - \frac{s^2(r^2 + Rr)}{r + 4R)^2}\,; \tag{33}$$

$$(NSp)^2 = \frac{1}{4}\,IO^2 = \frac{1}{4}\,(R^2 - 2Rr); \tag{34}$$

$$(NMc)^2 = \frac{1}{4}\,OM^2 = \frac{1}{4}\,R^2 - \frac{s^2(2R^2 - Rr)}{2(r + 4R)^2}\,; \tag{35}$$

$$(GeNa)^2 = 4IM^2 = -16Rr + \frac{16s^2(R^2 + Rr)}{(r + 4R)^2}\,; \tag{36}$$

$$(GeM)^2 = 9GM^2 = S_\omega - \frac{6s^2(r^2 + 2R^2)}{(r + 4R)^2}\,; \tag{37}$$

$$(GeSp)^2 = -\frac{1}{4}\,(3r^2 + 16Rr) + \frac{s^2(-r^2 + 2Rr)}{(r + 4R)^2}\,; \tag{38}$$

$$(GeMc)^2 = \frac{9}{4}\,GM^2 = \frac{1}{4}\,S_\omega - \frac{3s^2(r^2 + 2R^2)}{2(r + 4R)^2}\,; \tag{39}$$

$$(NaM)^2 = 4(IMc)^2 = 3r^2 - 4Rr - \frac{s^2(r^2 - 4R^2)}{(r + 4R)^2}\,; \tag{40}$$

$$(NaSp)^2 = \frac{1}{4}\,(INa)^2 = \frac{1}{4}\,(s^2 + 5r^2 - 16Rr); \tag{41}$$

$$(NaMc)^2 = \frac{1}{4}\,(7r^2 - 36Rr) + \frac{3s^2(r^2 + 8Rr + 12R^2)}{4(r + 4R)^2}\,; \tag{42}$$

$$(MSp)^2 = \frac{1}{4}\,(IGe)^2 = \frac{r^2}{4} - \frac{3s^2r^2}{4(r + 4R)^2}\,; \tag{43}$$

$$(MMc)^2 = \frac{9}{4}\,GM^2 = \frac{1}{4}\,S_\omega - \frac{3s^2(r^2 + 2R^2)}{2(r + 4R)^2}\,; \tag{44}$$

$$(SpMc)^2 = \frac{1}{4}\,IM^2 = -Rr + \frac{s^2(R^2 + Rr)}{(r + 4R)^2}\,. \tag{45}$$

Other distance formulas was found in [3] and [4].

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