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BARROW'S INEQUALITY AND COMPLEX NUMBERS

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ABSTRACT. We present a proof of the Barrow's inequality using dot and cross products of complex numbers.

1. INTRODUCTION

The Barrow's inequality, which is stronger than and implies Erdös-Mordell inequality, is presented and proved in [1]. The object of this short note is to slightly modify the proof of the Erdös-Mordell inequality presented in [2] to get a proof of the Barrow's inequality. It is another application of the dot and cross products of complex numbers.

2. THE SETTING

Let ΔABC be a triangle and O a point in it. Consider a coordinate system with O as it origin and A is on the positive direction of the X-axis, B is in the first or second quadrant, and C is in the third or fourth quadrant. We associate to the vertices A, B, and C their corresponding complex numbers

$$\begin{array}{lll} A = |A|e^{i\theta_A} & , & \theta_A = 0, \\ B = |B|e^{i\theta_B} & , & \theta_B \in (0,\pi), \\ C = |C|e^{i\theta_C} & , & \theta_C \in [\pi, 2\pi), \quad \text{and} \quad \theta_C - \theta_B \in (0,\pi). \end{array} \tag{2.1}$$

The conditions (2.1) on the angles imply that O is in the triangle. We also associate to the foot of the angle bissectors of $\angle BOC$, $\angle COA$, and $\angle AOB$, their corresponding complex numbers : P for the side joining the vertices B and C, Q for C and A, and R for A and B. With this notation, the Barrow's inequality is

$$2(|P| + |Q| + |R|) \leq |A| + |B| + |C|. \tag{2.2}$$

Barrow's inequality strengthens the Erdös Mordell inequality which has an identical form except that $|P|$, $|Q|$, and $|R|$ are the three distances of O from the triangle's sides.

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3. Dot and cross products of complex numbers

Let z_j $(j = 1, 2)$ be two complex numbers. Using the cartesian form $z_j = x_j + iy_j$, we have

$$\bar{z}_1 z_2 = [x_1 x_2 + y_1 y_2] + i\,[x_1 y_2 - x_2 y_1]\,.$$

Let the length of z be defined by $|z_j| = \sqrt{\bar{z}z}$. Using the polar form, we have

$$z_j = |z_j|\, e^{i\theta_j} = |z_j|\,[\cos(\theta_j) + i\sin(\theta_j)]\,,$$

and we get

$$\bar{z}_1 z_2 = |z_1|\,|z_2|\,e^{i(\theta_2-\theta_1)} = |z_1|\,|z_2|\,[\cos(\theta_2-\theta_1) + i\sin(\theta_2-\theta_1)]\,.$$

The dot and cross products of complex numbers are two real numbers defined by the dot and cross products of their corresponding vectors in the plane, or in the space with a null third component. We set

$$\bar{z}_1 z_2 = z_1 \odot z_2 + i\, z_1 \otimes z_2, \tag{3.1}$$

so they are respectively defined by

$$z_1 \odot z_2 = \mathrm{Re}\{\bar{z}_1 z_2\} = x_1 x_2 + y_1 y_2 = |z_1|\,|z_2| \cos(\theta_2 - \theta_1), \tag{3.2}$$

and

$$z_1 \otimes z_2 = \mathrm{Im}\{\bar{z}_1 z_2\} = x_1 y_2 - x_2 y_1 = |z_1|\,|z_2| \sin(\theta_2 - \theta_1), \tag{3.3}$$

where $\mathrm{Re}\{z\} = x$ and $\mathrm{Im}\{z\} = y$ stands respectively for the real and the imaginary parts of $z = x + iy$.
Considering (3.1) and $(\bar{z}_1 z_2)^2 = \overline{z_1^2} z_2^2$, we obtain

$$z_1^2 \odot z_2^2 = (z_1 \odot z_2)^2 - (z_1 \otimes z_2)^2\,, \tag{3.4}$$

and

$$z_1^2 \otimes z_2^2 = 2\,(z_1 \odot z_2)\,(z_1 \otimes z_2)\,. \tag{3.5}$$

4. Area of a triangles

Let ΔOFG be a triangle with vertices O (the origin of the axes), and two other vertices noted F and G, F on the positive direction of the X-axis and G in the first or second quadrant. Let H be the foot of the bissector of FOG on the side joining F and G. Let $F'' = \frac{|F|}{|G|}G$, such that $|F''| = |F|$ and $\angle HOF'' = \angle HOG = \angle HOF$. The triangle $\Delta OHF''$ is symmetric to the triangle ΔOHF with respect to the line passing throught the points O and H, so

$$\mathrm{Area}(\Delta OHF'') = \mathrm{Area}(\Delta OHF).$$

Consequently we have

$$\mathrm{Area}(\Delta OFG) = \mathrm{Area}(\Delta OHF) + \mathrm{Area}(\Delta OHG) = \mathrm{Area}(\Delta OHF'') + \mathrm{Area}(\Delta OHG).$$

Considering the area of those triangles, we have

$$\mathrm{Area}(\Delta OFG) = \frac{1}{2} F \otimes G,$$

$$\mathrm{Area}(\Delta OHF'') = \frac{1}{2} H \otimes F'' = \frac{|F|}{2|G|}\, H \otimes G,$$

and

$$\text{Area}(\Delta OHG) = \frac{1}{2} H \otimes G.$$

So

$$F \otimes G = \left[\frac{|F| + |G|}{|G|}\right] H \otimes G.$$

If $F = f^2$ and $G = g^2$, using (3.5) we have

$$F \otimes G = f^2 \otimes g^2 = 2\,(f \odot g)\,(f \otimes g)\,.$$

Then

$$2\,(f \odot g)\,(f \otimes g) = \frac{|f|^2 + |g|^2}{|g|^2}\, H \otimes G,$$

and, since $\angle fOg = \angle HOG$, we get

$$2\,|f \odot g|\,|f|\,|g| = \frac{|f|^2 + |g|^2}{|g|^2}\, |H|\,|G| = \left[|f|^2 + |g|^2\right] |H|\,.$$

It follows that

$$|H| = 2|f \odot g| \frac{|f||g|}{|f|^2 + |g|^2|} \leq |f \odot g| \tag{4.1}$$

because $2|f||g| \leq |f|^2 + |g|^2|$, with equality holding iff $|f| = |g|$.

5. Proof of the inequality

We consider that $A = a^2$, $B = b^2$, and $C = c^2$, where

$$\begin{array}{llll} a = |a|e^{i\theta_a} & , \ \theta_a = 0, & \text{and} & 2\theta_a = \theta_A, \\ b = |b|e^{i\theta_b} & , \ \theta_b \in (0, \pi/2), & \text{and} & 2\theta_b = \theta_B, \\ c = |c|e^{i\theta_c} & , \ \theta_c \in [\pi/2, \pi), & \text{and} & 2\theta_c = \theta_C. \end{array} \tag{5.1}$$

It follows from (4.1) and (5.1) that

$$|A| + |B| + |C| - 2|P| - 2|Q| - 2|R| \geq |a|^2 + |b|^2 + |c|^2 - 2|b \odot c| - 2|c \odot a| - 2|a \odot b|$$

with equality iff $|a| = |b| = |c|$. Because $a \odot b \geq 0$, $b \odot c \geq 0$, and $c \odot a \leq 0$, we obtain

$$\begin{aligned} & |a|^2 + |b|^2 + |c|^2 - 2|b \odot c| - 2|c \odot a| - 2|a \odot b| \\ &\quad = a \odot a + b \odot b + c \odot c - 2b \odot c + 2c \odot a - 2a \odot b \\ &\quad = (a - b + c) \odot (a - b + c) \\ &\quad = |a - b + c|^2 \\ &\quad \geq 0, \end{aligned}$$

with equality iff $a - b + c = 0$. Hence (2.2) holds.

Equality holds if and only if $|a| = |b| = |c|$ and $a - b + c = 0$. In this case, using the dot product successively with a, b, and c we obtain the following system of equations

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \odot b \\ b \odot c \\ c \odot a \end{bmatrix} = \begin{bmatrix} |a|^2 \\ |b|^2 \\ |c|^2 \end{bmatrix},$$

and its solution is

$$\begin{bmatrix} a \odot b \\ b \odot c \\ c \odot a \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} |a|^2 \\ |b|^2 \\ |c|^2 \end{bmatrix}.$$

If $|a| = |b| = |c| = d$, we get

$$\begin{bmatrix} a \odot b \\ b \odot c \\ c \odot a \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} d^2,$$

from which we conclude that $\theta_b = \pi/3$ and $\theta_c = 2\pi/3$. Then, in case of equality, ΔABC is an equilateral triangle.

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