Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.9, (2020), Issue 2, pp.96-106

SOME METRIC PROPERTIES OF SEMI-REGULAR HEXAGONS AND A CONSTRUCTIVE TASK

NENAD STOJANOVIĆ

ABSTRACT. A simple polygon $\mathcal{A}_6 \equiv A_1A_2...A_6$ that has equal either all sides or all interior angles is called a semi-regular hexagon. In terms of this definition, we can distinguish between two types of semi-regular hexagons: equilateral (they have equal all sides and different interior angles) and equiangular (they have equal interior angles and different sides). Unlike regular polygons, one characteristic element is not enough to analyze the metric properties of semi-regular polygons, so the additional one is needed. To select this additional characteristic element, note that regular triangles $\triangle A_1A_3A_5, \triangle A_2A_4A_6$ can be inscribed to each semi-regular equilateral hexagon by joining the vertices. Let us use the mark $\delta = \angle(a, b_1)$ to mark the angle between side a of the semi-regular hexagon and side b_1 of inscribed regular triangle $\triangle A_1A_3A_5$ This parameter is present in every semi-regular hexagon, therefore, in interpreting the metric properties of the semi-regular hexagon in this paper, in addition to side a, we also use angle $\delta = \angle(a, b_1)$. The manner in which convexities, the possibility of construction, the calculation of the surface area and the problem of the inscribed circle and other metric properties depend on side a and angle $\delta = \angle(a, b_1)$ is elaborated in this paper. A formula has been designed to calculate the surface area of the semi-regular equilateral hexagon depending on parameters a and δ, as well as side a and radius r of the inscribed circle. The problem of the relation between the sides of inscribed triangles $\mathcal{P}_3^i; i = 1, 2$ is also pondered upon.

1. 1 INTRODUCTION

A simple polygon $\mathcal{A}_6 \equiv A_1A_2A_3A_4A_5A_6$ that has equal all sides and different interior angles is called a semi-regular hexagon [1]. In terms of this definition, we can distinguish between two types of semi-regular hexagons: equilateral hexagons (they have equal all sides and different interior angles) and equiangular hexagons (they have equal interior angles and different sides). We consider vertex $A_i, i = 1, ..., 6$ of a semi-regular equilateral hexagon to be in an even, or in an odd position, if index i is an even, or an odd number, respectively,(Figure1),[1],[2]. In this paper, we dealt with the metric properties of convex equilateral semi-regular hexagons. Unlike with regular polygons, the length of one of the sides is not sufficient for analyzing the metric properties of equilateral semi-regular polygons, so another characteristic element is needed,[3-5].

2010 *Mathematics Subject Classification.* Primary 51MO5; Secondary 11DO4.
Key words and phrases. semi-regular polygons,semi-regular hexagon, an inscribed and circumscribed circle, a ratio of surface areas of semi-regular polygons.

For the selection of this characteristic element of equilateral semi-regular hexagons, note that triangles $\triangle A_1A_3A_5, \triangle A_2A_4A_6$ can be inscribed to every semi-regular equilateral hexagon by joining odd, or even vertices.
Equilateral triangles $\mathcal{P}_3^1 \equiv \triangle A_1A_3A_5, \mathcal{P}_3^1 \equiv \triangle A_2A_4A_6$ are the inscribed triangles to semi-regular hexagon $\mathcal{A}_6$. Let us mark the sides of these triangles with b_1, and b_2, respectively, and their interior angles with $\gamma; \gamma = \frac{\pi}{3}$. Let us use the mark $\delta = \angle(a, b_i); i = 1, 2$ to mark the angle between side a of semi-regular hexagon $\mathcal{A}_6$ and the side of the inscribed regular triangle. For triangle $\mathcal{P}_3^1 \equiv \triangle A_1A_3A_5$ the following applies $\delta = \angle(a, b_1)$. Isosceles triangles $\triangle A_1A_2A_3$, $\triangle A_3A_4A_5$ and $\triangle A_5A_6A_1$ are triangles that border semi-regular hexagon $\mathcal{A}_6$ (Figure 1), [6].
In this way, besides side $A_iA_{i+1} = a; A_{i+1} = A_1, i = 1, 2, \ldots, 6$ of the semi-regular hexagon, we have also included the other characteristic element, i.e. angle $\delta = \angle(a, b), b_1 = b$ which is required in analyzing the metric properties. In this paper, the following marks are used:
- the side of semi-regular polygon $\mathcal{P}_6$ is marked with a,
- the number of sides of the "inscribed" regular polygon is marked with n
- the side of regular triangle $\mathcal{P}_3^i; i = 1, 2$ "inscribed" to semi-regular hexagon $\mathcal{A}_6 = A_1A_2...A_6$,
and constructed by joining its even vertices, or odd vertices is marked with $b_i, i \in \mathbb{N}, i = 1, 2$.
- the interior angles of the semi-regular hexagon at the vertices of "inscribed" regular triangle $\mathcal{P}_3^1 \equiv \triangle A_1A_3A_5$ are marked with α, and at the other vertices with β.

For the interior angles the following applies

$$\alpha = \frac{\pi}{3} + 2\delta, \beta = \pi - 2\delta = \frac{\pi}{3} + 2(\frac{\pi}{3} - \delta) = \frac{\pi}{3} + 2\varphi, \varphi = \frac{\pi}{3} - \delta. \tag{1.1}$$

The parameters which are used in the analysis of the metric properties of the semi-regular hexagon are the length of its side a and the value of angle $\delta = \angle(a, b_1)$ where b_1 indicates the side of the equilateral triangle inscribed to it and formed by joining the odd vertices. Mark $\mathcal{A}_6^{a,\delta}$ indicates that the semi-regular equilateral convex hexagon is determined by side a and angle δ,[5].

2. My Result

2.1. **The convexity of a semi-regular hexagon.** The following theorem shows the dependence of the convexity of a semi-regular equilateral hexagon on angle $\delta = \angle(a, b)$, where $b = b_1$ is the side of inscribed regular triangle $\mathcal{P}_3^1 \equiv \triangle A_1A_3A_5$.

Theorem 2.1. *Semi-regular equilateral hexagon $\mathcal{A}_6$ with interior angles defined by relations (1.1) is:*
1) convex for all values $\delta \in \langle 0; \frac{\pi}{3} \rangle$, (for $\delta = \frac{\pi}{6}$, $\mathcal{A}_6$ it is a regular hexagon),
2) non-convex for all values $\delta \in [\frac{\pi}{3}, \frac{\pi}{2} \rangle$ and
3) not defined for all values $\delta > \frac{\pi}{2}$.

Proof: By definition, a simple polygon is said to be convex, if all its interior angles are smaller than π(Figure 1),[2]. Furthermore, using this definition, it is easy to determine

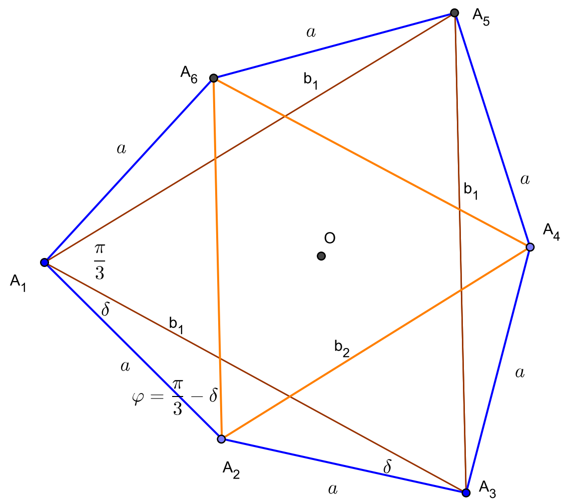

Figure 1. Basic elements of a semi-regular equilateral hexagon

the values of interior angles α, β for angle $0 < \varphi < \frac{\pi}{3}$.

1. For all $\delta \in \langle 0; \frac{\pi}{6} \rangle$ from inequality $0 < \delta < \frac{\pi}{6}$ we get that $\frac{\pi}{3} < \alpha < \frac{2\pi}{3}$, from which it follows that all values of the interior angles are equal to angle $\alpha < \pi$ and are in the interval $\langle \frac{\pi}{3}; \frac{2\pi}{3} \rangle$.
By a similar procedure, it is easy to show that the interior angles equal to angle β are also smaller than π. Indeed, from inequality $0 < \delta < \frac{\pi}{6}$ by a series of elementary transformations we find that $\frac{2\pi}{3} < \beta < \pi$, that is, all values of the interior angles equal to angle β are in interval $\langle \frac{2\pi}{3}; \pi \rangle$.
Since all the interior angles of the semi-regular equilateral hexagon are smaller than π, for all angle values $\delta \in \langle 0; \frac{\pi}{6} \rangle$ it is convex for the values of angle δ.
For $\delta = \frac{\pi}{6}$ the interior angles of the hexagon are equal, so for this value of angle δ, it is a convex regular hexagon.
2. If $\delta \in \langle \frac{\pi}{6}; \frac{\pi}{3}]$ then from inequality $\frac{\pi}{6} < \delta < \frac{\pi}{3}$ we find that the values of the interior angles of a semi-regular hexagon equal to angle α are in interval $\langle \frac{2\pi}{3}; \pi \rangle$. By a similar procedure, it is shown that the values of the interior angles equal to angle β are in interval $\langle \frac{\pi}{3}; \frac{2\pi}{3} \rangle$. Based on this, it follows that the semi-regular equilateral hexagon is also convex for all angle values $\delta \in \langle \frac{\pi}{6}; \frac{\pi}{3} \rangle$.
3. Let it be that $\delta \in [\frac{\pi}{3}; \frac{\pi}{2} \rangle$. Starting from inequality $\frac{\pi}{3} \leq \delta < \frac{\pi}{2}$ we find that for the interior angles equal to angle α the interval of values is $[\pi; \frac{4\pi}{3} \rangle$. Therefore, all the interior angles equal to angle α are greater than π, then the semi-regular equilateral hexagon is *non-convex* for all the values of angle $\delta \in [\frac{\pi}{3}; \frac{\pi}{2} \rangle$.
4. If $\delta \geq \frac{\pi}{2}$, all the interior angles that are equal to angle α are greater than or equal to angle π, while all the interior angles are equal to angle $\beta \leq 0$. So, for these values of angle δ, the semi-regular hexagon does not exist.

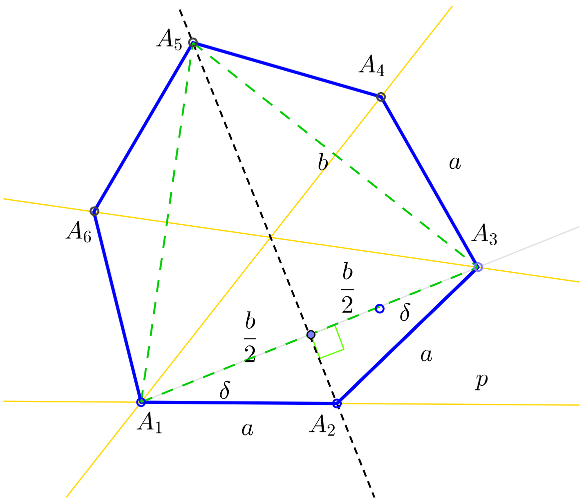

Figure 2. Semi-rectangular hexagon of side a and angle δ

2.2. **Surface area of a semi-regular hexagon.** The dependence of the surface of a convex equilateral semi-regular hexagon on the length of its side a and angle $\delta = \angle(a, b)$ closed with sides a and b of inscribed equilateral triangle $\mathcal{P}_3^1 \equiv \triangle A_1A_3A_5$ is shown in a Theorem

Theorem 2.2. *The surface area $\mathcal{A}(P_6)$, of a semi-regular equilateral convex hexagon of a side a and angle $\delta = \angle(a, b)$ is determined in a relation*

$$\mathcal{A}(P_6) = 2\sqrt{3} \cdot a^2 \cos\delta \cdot \cos(\frac{\pi}{3} - \delta). \tag{2.1}$$

Proof: Observe that the surface area of the semi-angular equilateral hexagon is equal to the sum of the surface areas of the isosceles triangles constructed above each side of regular triangle $\mathcal{P}_3^1 \equiv \triangle A_1A_3A_5$ which is inscribed to it (Figure 2). That is

$$\mathcal{A}(P_6) = 3 \cdot \mathcal{A}(P_1) + \mathcal{A}(P_2) \tag{2.2}$$

where $\mathcal{A}(P_1)$ is the surface area of the isosceles bordering triangle, and $\mathcal{A}(P_2)$ is the surface area of the inscribed equilateral triangle.
As the surface area of inscribed regular triangle $\triangle A_1A_3A_5$, where side b is $b = 2a \cdot \cos\delta$ is $\mathcal{A}(P_2) = \frac{b^2\sqrt{3}}{4}$, and with the surface area of isosceles triangle $\triangle A_1A_2A_3$, $\mathcal{A}(P_1) = a^2 \sin\delta \cos\delta$, we get that

$$\begin{aligned}
\mathcal{A}(P_6) &= 3 \cdot \mathcal{A}(P_1) + \mathcal{A}(P_2) \\
&= 3a^2 \sin\delta\cos\delta + a^2\sqrt{3}\cos^2\delta \\
&= \sqrt{3}a^2\cos\delta(\sqrt{3}\sin\delta + \cos\delta).
\end{aligned}$$

If we notice that $\sqrt{3}\sin\delta + \cos\delta = \tan\frac{\pi}{3}\sin\delta + \cos\delta = 2\cos(\frac{\pi}{3} - \delta)$ we find that the surface area of the semi-regular equilateral hexagon is

$$\mathcal{A}(P_6) = 2a^2\sqrt{3}\cos\delta\cos(\frac{\pi}{3} - \delta).$$

Corollary 2.1. *For angle $\delta = \frac{\pi}{6}$ the hexagon is regular, and its surface is*

$$\mathcal{A}(P_6) = 2a^2\sqrt{3}\cos\frac{\pi}{6}\cos(\frac{\pi}{3} - \frac{\pi}{6}) = \frac{3\sqrt{3}}{2}\cdot a^2$$

which is consistent with the formula for the surface area of the regular hexagon, which is obtained if in$\mathcal{A}(P_n) = \frac{1}{4}na^2\cot\frac{\alpha}{2}$ we add that $n = 6$ and $\alpha = \frac{\pi}{3}$, wherein $\alpha = \frac{2\pi}{n}$ is the central angle of the regular n-side polygon.

The surface area of the semi-regular hexagon can be calculated also if the length of side b of the inscribed regular triangle and angle δ are known, i.e. it applies that

Corollary 2.2. *The surface area of the semi-regular hexagon, with side b of the inscribed regular triangle and angle δ is calculated by the following formula*

$$\mathcal{A}\left(P_6\right) = b^2\frac{\sqrt{3}}{2}\cdot\frac{\cos\left(\frac{\pi}{3}-\delta\right)}{\cos}\delta \tag{2.3}$$

where b is the side of the inscribed regular triangle and angle δ=$\angle(a,b)$.

Proof. From enequality $b = 2a\cos\delta$ we express side a and use the formula for the surface area of the semi-regular hexagon

$$\begin{aligned}\mathcal{A}(P_6) &= 2(\frac{b}{2\cos\delta})^2\sqrt{3}\cos\delta\cos(\frac{\pi}{3}-\delta)\\ &= b^2\frac{\sqrt{3}}{2}\cdot\frac{\cos(\frac{\pi}{3}-\delta)}{\cos\delta}.\end{aligned}$$

2.3. **The radius of the inscribed circle.** A circle can be inscribed, but not circumscribed to an equilateral semi-regular hexagon. Let us analyze the ratio of the radius of the inscribed circle (apothem), side a and angle δ, as well as the calculation of the surface area depending on the radius of the semi-regular equilateral hexagon.
The following theorem shows the relation between the radius of the inscribed circle and parameters a and δ of the semi-regular hexagon.

Theorem 2.3. *The radius of the circle inscribed to the semi-regular equilateral hexagon with side a and angle $\delta \in \langle 0; \frac{\pi}{3}\rangle, \delta \neq \frac{\pi}{6}$ is expressed by the following*

$$r = \frac{2a}{\sqrt{3}}\cdot\cos\delta\cos\left(\frac{\pi}{3}-\delta\right) \tag{2.4}$$

Proof: Note triangle A_1A_2O (Figure 3). The following applies for the angles of this triangle $\angle OA_1A_2 = \frac{\alpha}{2} = \frac{\pi}{6} + \delta$ and $\angle A_1A_2O = \frac{\beta}{2} = \frac{\pi}{2} - \delta$. From the special right triangle OA_1K it follows that $\tan\frac{\alpha}{2} = \frac{r}{x}$, and from there we have the following:

$$r = x\cdot\tan\frac{\alpha}{2} \tag{2.5}$$

and $x = d(A_1, K)$. Similarly, from special right triangle OKA_2, it follows that $\tan\frac{\beta}{2} = \frac{r}{a-x}$,and from that we get

$$r = (a-x)\tan\frac{\beta}{2}, \tag{2.6}$$

and $d(K, A_2) = a - x$.

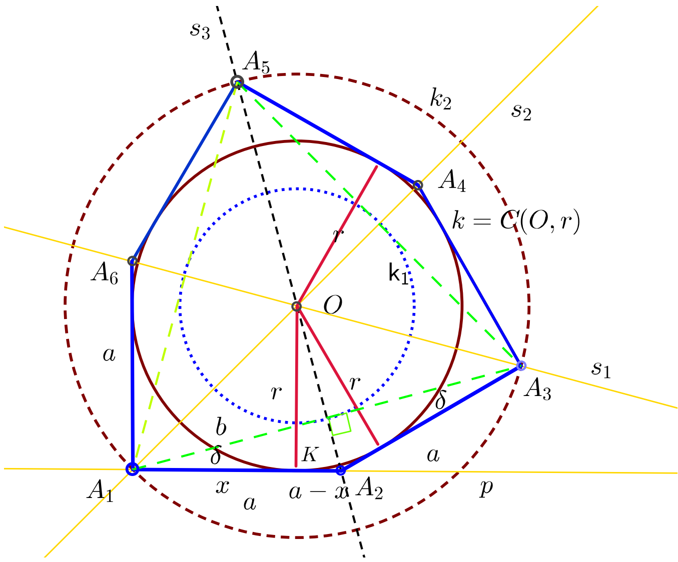

Figure 3. Radius of the inscribed circle.

If we equate (2.5) and (2.6), and after completing the elemental transformations we find that

$$x = \frac{a \cdot \tan\frac{\beta}{2}}{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}. \tag{2.7}$$

If we put this into equality (2.5) we get that

$$r = \frac{a \cdot \tan\frac{\beta}{2}\tan\frac{\alpha}{2}}{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}, \tag{2.8}$$

i.e.

$$r = \frac{a \cdot \tan(\frac{\pi}{6} + \delta)\tan(\frac{\pi}{2} - \delta)}{\tan(\frac{\pi}{2} - \delta) + \tan(\frac{\pi}{6} + \delta)}. \tag{2.9}$$

Since

$$\tan(\frac{\pi}{2} - \delta) = \frac{\sin(\frac{\pi}{2} - \delta)}{\cos(\frac{\pi}{2} - \delta)} = \cot\delta, \quad \tan(\frac{\pi}{6} + \delta) = \frac{\cot\delta + \sqrt{3}}{\sqrt{3}\cot\delta - 1}$$

we find that

$$r = \frac{a\cot\delta(\cot\delta + \sqrt{3})}{\sqrt{3}(1 + \cot^2\delta)} = ... = \frac{2a}{\sqrt{3}} cos\delta cos(\frac{\pi}{3} - \delta).$$

Proposition 2.1. *The dependence of the radius of the circle inscribed on side b of the inscribed special right triangle and angle δ is given in the following formula:*

$$r = \frac{1}{\sqrt{3}} \cdot b\cos(\frac{\pi}{3} - \delta) \tag{2.10}$$

Proof. It is easy to show that the equality is valid using equality $b = 2a\cos\delta$ and equality $r = \frac{2a}{\sqrt{3}} \cdot \cos\delta\cos(\frac{\pi}{3} - \delta)$ in which we find that equality is valid.

The manner of calculating the surface area of the semi-regular equilateral hexagon, if the length of the radius of inscribed circle r and angle δ are given, is seen in the following Theorem

Theorem 2.4. *The surface area of the semi-regular equilateral hexagon of the radius of inscribed circle r and angle* $\delta \in \langle 0, \frac{\pi}{3}\rangle$ *and* $\delta \neq \frac{\pi}{6}$ *is calculated by the following formula*

$$\mathcal{A}(P_6) = \frac{3\sqrt{3}}{2}\frac{r^2}{\cos\delta\cos(\frac{\pi}{3}-\delta)}. \tag{2.11}$$

Proof: If from equality $r = \frac{2a}{\sqrt{3}}\cos\delta\cdot\cos(\frac{\pi}{3}-\delta)$ we express side a using r and δ, we find that

$$a = \frac{\sqrt{3}\cdot r}{2\cos\delta\cos(\frac{\pi}{3}-\delta)}. \tag{2.12}$$

If we substitute this equality in the formula for the surface area of semi-regular hexagon

$$\mathcal{A}(P_6) = 2a^2\sqrt{3}\cos\delta\cos(\frac{\pi}{3}-\delta)$$

after calculation we find that

$$\mathcal{A}(P_6) = \frac{3\sqrt{3}}{2}\frac{r^2}{\cos\delta\cos(\frac{\pi}{3}-\delta)}.$$

Corollary 2.3. *If* $\delta = \frac{\pi}{6}$ *the surface area of the semi-regular hexagon is equal to*

$$\begin{aligned}\mathcal{A}(P_6) &= \frac{3\sqrt{3}}{2}\frac{r^2}{\cos\delta\cos(\frac{\pi}{3}-\delta)}\\ &= \frac{3\sqrt{3}}{2}\frac{r^2}{\cos\frac{\pi}{6}\cos(\frac{\pi}{3}-\frac{\pi}{6})}\\ &= \frac{3\sqrt{3}}{2}\frac{r^2}{(\frac{\sqrt{3}}{2})^2} = 2\sqrt{3}r^2\end{aligned}$$

which coincides with the formula for the surface area of the regular hexagon if radius r of the inscribed circle is given.

Example 2.1. *If we note that* $\cos\delta\cos(\frac{\pi}{3}-\delta) = \frac{\sqrt{3}r}{2a}$ *then from the formula for the surface area of the semi-regular hexagon we find that the surface area of the semi-regular hexagon, with given side a and radius r of the inscribed circle is calculated by the following formula*

$$\mathcal{A}(P_6) = 2a^2\sqrt{3}\cos\delta\cos(\frac{\pi}{3}-\delta) = 2a^2\sqrt{3}a^2\frac{\sqrt{3}r}{2a} = 3ar. \tag{2.13}$$

2.4. **A constructive task on semi-regular hexagons.** The triangles inscribed to the equilateral convex hexagon by joining odd or even vertices generate a new equiangular convex hexagon, to which we can construct the inscribed equilateral triangles in the same way, and which then generate a new equilateral convex hexagon, etc. Between the semi-regular hexagons constructed in this way, a relation can be established, that is, the following theorem holds

Theorem 2.5. *The intersection points of the sides of regular triangles $\mathcal{P}_3^1 \equiv A_1A_3A_5$, $\mathcal{P}_3^2 \equiv A_2A_4A_6$ inscribed to equilateral semi-regular hexagon $\mathcal{A}_6 \equiv A_1A_2A_3A_4A_5A_6$ are the vertices of hexagon $\mathcal{B}_6 \equiv B_1B_2B_3B_4B_5B_6$, to which triangles $\mathcal{T}_3^1 \equiv B_1B_3B_5$, $\mathcal{T}_3^2 \equiv B_2B_4B_6$ are inscribed, and their intersection points of the sides are located at the vertices of hexagon $\mathcal{C}_6 \equiv C_1C_2C_3C_4C_5C_6$, and so on.(Figure 4). The following needs to be proven:*

1. *(1) that $\mathcal{B}_6$ is an equiangular semi-regular hexagon,*
2. *(2) that $\mathcal{C}_6$ is an equilateral semi-regular hexagon,*
3. *(3) that the ratio of the lengths of the smaller and the larger side of convex equiangular hexagon $\mathcal{B}_6$ inscribed to equilateral convex semi-regular hexagon $\mathcal{A}_6^{a,\delta}$ is in interval $(0, \frac{\sqrt{3}}{3})$ for all values of angle $\delta \in \left(0, \frac{\pi}{3}\right)$,*
4. *(4) that inequalities $1 < \frac{R_\mathcal{A}}{r_\mathcal{B}} \leq \frac{3}{2}$ apply for all values of angle $\delta \in \left(0, \frac{\pi}{3}\right)$, with $R_\mathcal{A}$ being the length of the radius of the circle inscribed to hexagon $\mathcal{A}_6$, and $r_\mathcal{B}$ being the length of the radius of the circle circumscribed to hexagon $\mathcal{B}_6$.*

Proof. 1.) Let $\mathcal{A}_6 \equiv A_1A_2A_3A_4A_5A_6$ (Figure 4.) be a semi-regular convex equilateral hexagon and $\mathcal{P}_3^1 \equiv A_1A_3A_5; \mathcal{P}_3^2 \equiv A_2A_4A_6$ be the regular triangles inscribed to it, the sides of which are marked with b_1, b_2 respectively.
Let $B_1B_2 = c_1; B_2B_3 = c_2$ be the sides of this hexagon. To prove that $\mathcal{B}_6$ is an equiangular semi-regular hexagon, it needs to be proven that all interior angles are equal and the sides are different.

Let us use mark $\delta = \angle(a, b_1)$ to mark the angle between side a of hexagon $\mathcal{A}_6$ and the side of equilateral triangle $\mathcal{P}_3^1 \equiv A_1A_3A_5$. Note that for the interior angles at vertices A_1, A_3, A_5 the following applies $\angle A_1 \cong \angle A_3 \cong \angle A_5 = \frac{\pi}{3} + 2\delta$, and that for the interior angles at vertices A_2, A_4, A_6 the following applies $\angle A_2 \cong \angle A_4 \cong \angle A_6 = \frac{\pi}{3} + 2(\frac{\pi}{3} - \delta)$.
Since $\mathcal{A}_6$ is an equilateral semi-regular hexagon, i.e. $A_1A_2 = A_2A_3 = ... = A_6A_1 = a$, it follows that triangles $\triangle A_1B_1A_2$; $\triangle A_2B_2A_3$;
$\triangle A_3B_3A_4$;$\triangle A_4B_4A_5$;$\triangle A_5B_5A_6$,; $\triangle A_6B_6A_1$ are congruent, hence their interior angles

$$\angle A_1B_1A_2 \equiv \angle A_2B_2A_3 \equiv \angle A_3B_3A_4 \equiv \angle A_4B_4A_5 \equiv \angle A_5B_5A_6 \equiv \angle A_6B_6A_1 = \frac{2\pi}{3} \quad (2.14)$$

are also congruent,(Figure 4).
Note that the interior angles of hexagon $\mathcal{B}_6$ are transverse to the angles from the previous equality, and that the following is applicable

$$\angle B_1 = \angle B_2 = \angle B_3 = \angle B_4 = \angle B_5 = \angle B_6 = \varphi = \frac{2\pi}{3} \quad (2.15)$$

It follows that $\mathcal{B}_6$ is the equiangular convex hexagon. Let us prove that the sides of hexagon $\mathcal{B}_6$ are different from each other.
Let us note triangles $\triangle B_1A_2B_2; \triangle B_3A_4B_4; \triangle B_5A_6B_6$. Let us observe now $\triangle B_1A_2B_2$. It has been shown that $\triangle A_1B_1A_2 \cong \triangle A_2B_2A_3$ and from that it follows that line segments $B_1A_2 \cong A_2B_2$ are congruent. Since the interior angle is at vertex $\angle A_2 = \frac{\pi}{3}$ and point S_1 is on the bisector of base B_1B_2 it follows that $\angle S_1$ is a right angle and $\triangle B_1A_2S_1$ is orthogonal to the angle at vertex $\angle B_1 = \frac{\pi}{3}$. Similarly, it turns out that $\angle B_2 = \frac{\pi}{3}$.
On the basis of this, it follows that $\triangle B_1A_2B_2$ is an equilateral triangle and $B_1B_2 = c_1$ is the side of hexagon $\mathcal{B}_6$ (Figure 4).

Similarly, triangles $\triangle B_3A_4B_4; \triangle B_5A_6B_6$ are proven to be equilateral, i.e. the following applies $\triangle B_1A_2B_2 \equiv \triangle B_3A_4B_4 \equiv \triangle B_5A_6B_6$.
Let us note now that $\triangle B_2A_3B_3$. From the already seen congruence
$\triangle A_2B_2B_1 \cong \triangle A_3B_3B_4$ it follows that $B_2A_3 \cong B_3A_3$. Since $\angle A_3 = \frac{\pi}{3}$ it follows that $\triangle B_2A_3B_3$ is an equilateral triangle and that side $B_1B_3 = c_2$ is the side of hexagon $\mathcal{B}_6$. In the same way it is shown that $\triangle B_4A_5B_5; \triangle B_6A_1B_1$ are mutually congruent equilateral triangles, with bases $B_2B_3 = B_4B_5 = B_6B_1 = c_2$. Equilateral triangles $\triangle B_1A_2B_2$ and $\triangle B_2A_3B_3$ are not congruent, i.e. the following applies $B_1B_2 = c_1 \neq B_1B_3 = c_2$, and therefore, based on this we can conclude that equiangular hexagon $\mathcal{B}_6$ does not have equal sides, i.e. it is an equiangular semi-regular hexagon.
2) Let us show that $\mathcal{C}_6 \equiv C_1C_2C_3C_4C_5C_6$ is an equilateral semi-regular hexagon. Note, for example, that, from the congruence of interior angles $\angle B_1 \cong \angle B_2$ and sides $B_6B_1 \cong B_2B_3 = c_2$ of equilangular hexagon $\mathcal{B}_6$ and $B_1B_2 \cong B_2B_1 = c_1$ the congruence of triangles $\triangle B_6B_1B_2 \cong \triangle B_1B_2B_3$ follows. From the congruence of these triangles, follows the congruence of sides $B_2B_4 \cong B_4B_6 \cong B_6B_2 = b$.
Similarly, the congruence of triangles $\triangle B_3B_4B_5 \cong \triangle B_4B_5B_6 \cong \triangle B_5B_6B_1 \cong \triangle B_6B_1B_2$ is shown. From the congruence of these triangles, the congruence of equilateral triangles $\triangle B_1B_3B_5; \triangle B_2B_4B_6$ follows. From the congruence of these triangles, the congruence of angles $\angle C_6B_1C_2 \cong \angle C_6B_2C_2 = \frac{\pi}{3}$ follows.
Furthermore, from the congruence of special right triangles $\triangle B_1C_1T_1 \cong \triangle B_2C_1T_1$ the congruence of sides $B_1C_1 \cong B_2C_2$ follows, and from the congruence of transverse angles $\angle C_6C_1B_1 \cong \angle C_2C_1B_2$ the congruence of triangles $\triangle B_1C_1C_6 \cong \triangle B_2C_1C_2$ follows. The congruence of these triangles implies the equality of sides $C_6C_1 \cong C_1C_2 = a_1$ of hexagon $\mathcal{C}_6$. A similar procedure shows that all sides of hexagon $\mathcal{C}_6$ are equal, that is, it is an equilateral hexagon. The difference of angles can be easily proven.

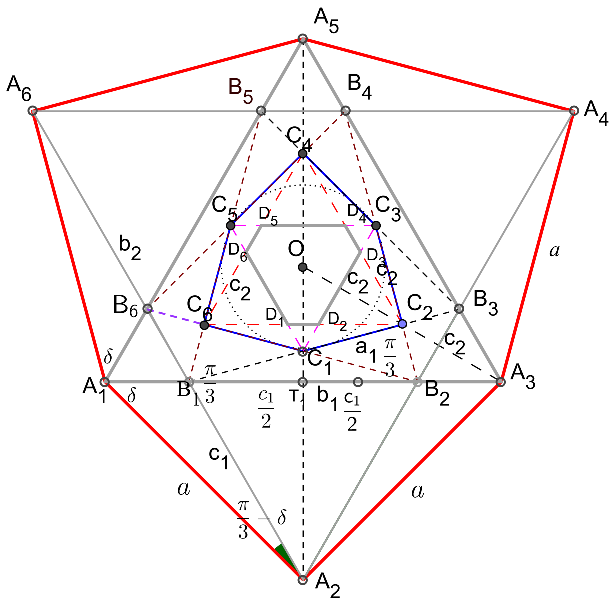

Figure 4. Relationship between the inscribed semi-regular hexagons

3). To determine the ratio of the sides of inscribed semi-regular hexagon $\mathcal{B}_6$, let us express the lengths of the sides relative to the length of side a of hexagon $\mathcal{A}_6$ and angle $\delta = \angle(a, b_1)$ (Figure 4).
Note equilateral triangle $\triangle B_1A_2B_2$. From special right triangle $\triangle A_1A_2T_1$ we find that side c_1 of hexagon $\mathcal{B}_6$ is given by the following equality

$$c_1 = \frac{2\sqrt{3}}{3}a\sin\delta. \tag{2.16}$$

And from isosceles triangle $\triangle A_6A_1A_2$ with base $A_2A_6 = A_2B_1 + B_1B_6 + B_6A_6 = 2c_1 + c_2$ and with the value of angles being $\angle A_2 \equiv \angle A_6 = \frac{\pi}{3} - \delta$ we find that side c_2 of hexagon $\mathcal{B}_6$ is given by the following equality

$$c_2 = \frac{2\sqrt{3}}{3}a\sin(\frac{\pi}{3} - \delta). \tag{2.17}$$

From equalities 2.16 and 2.17 we find that

$$\frac{c_1}{c_2} = \frac{\sin\delta}{\sin(\frac{\pi}{3} - \delta)} \tag{2.18}$$

Let us consider the dependence of the equality from the value of angle $\delta = \angle(a, b_1)$. Note that for $\delta = \frac{\pi}{6}$, $c_1 = c_2$ and in this case, hexagons $\mathcal{B}_6$ and $\mathcal{A}_6$ are regular hexagons. Since semi-regular hexagon $\mathcal{A}_6$ is convex for $\delta \in (0, \frac{\pi}{3})$ and $\delta \neq \frac{\pi}{6}$ we distinguish two cases:

(1) If $\delta \in (0, \frac{\pi}{6})$ then $\frac{\pi}{3} - \delta > \delta$ and $\sin(\frac{\pi}{3} - \delta) > \sin\delta$ i.e.

$$\sin\delta < \sin\left(\frac{\pi}{3} - \delta\right) \Leftrightarrow \frac{2\sqrt{3}}{3}a\sin\delta < \frac{2\sqrt{3}}{3}\sin(\frac{\pi}{3} - \delta) \Rightarrow c_1 < c_2.$$

Since for $0 < \delta < \frac{\pi}{6}$ it is $0 < \sin\delta < \frac{1}{2}$ and $\frac{1}{2} < \sin(\frac{\pi}{3} - \delta) < \frac{\sqrt{3}}{2}$ we find that the ratio of the smaller side,c_1, and the greater side, c_2, of the semi-regular hexagon is in interval $(0, \frac{\sqrt{3}}{3})$ i.e. the following inequalities are valid

$$0 < \frac{c_1}{c_2} = \frac{\frac{2\sqrt{3}}{3}a\sin\delta}{\frac{2\sqrt{3}}{3}a\sin(\frac{\pi}{3} - \delta)} = \frac{\sin\delta}{\sin(\frac{\pi}{3} - \delta)} < \frac{\sqrt{3}}{3}.$$

(2) In a similar procedure, we see that if $\delta \in (\frac{\pi}{6}, \frac{\pi}{3})$ then $\frac{\pi}{3} - \delta < \delta$ and $\sin(\frac{\pi}{3} - \delta) < \sin\delta$ and $c_1 > c_2$. So the ratio of the length of smaller side c_2 and larger side c_1 of semi-regular equiangular hexagon $\mathcal{B}_6$ is in interval $(0, \frac{\sqrt{3}}{3})$ i.e. the following inequalities apply

$$0 < \frac{c_1}{c_2} = \frac{\frac{2\sqrt{3}}{3}a}{\frac{2\sqrt{3}}{3}a}\frac{\sin(\frac{\pi}{3} - \delta)}{\sin\delta} < \frac{\sqrt{3}}{3}.$$

In both cases, the ratio of the length of the smaller and larger sides of isosceles hexagon $\mathcal{B}_6$ which is inscribed to equilateral hexagon $\mathcal{A}_6^{a,\delta}$ is in interval $(0, \frac{\sqrt{3}}{3})$.

4) Using the marks as in Figure 4, and the previously obtained results on the radius of the circle which is inscribed to the equilateral hexagon, we find that $R_{\mathcal{A}} = \frac{2\sqrt{3}}{3}a \cdot$

$\cos\delta\cos\left(\frac{\pi}{3}-\delta\right)$. Furthermore, from special right triangle $\triangle B_1S_1O$ there is $r_{\mathcal{B}}^2=(\frac{\sqrt{3}}{3}a\cos\delta)^2+(\frac{\sqrt{3}}{3}a\sin\delta)^2$, and after calculation we get $r_{\mathcal{B}}=\frac{a\sqrt{3}}{3}$, with a being the length of the side of semi-regular hexagon $\mathcal{A}_6^{a,\delta}$. Further on

$$\frac{R_{\mathcal{A}}}{r_{\mathcal{B}}}=2cos\delta cos\left(\frac{\pi}{3}-\delta\right)=\cos\Big(2\delta-\frac{\pi}{3}\Big)+\frac{1}{2}$$

.

To estimate the value of the ratio, note that for $\delta\in\left(0,\frac{\pi}{6}\right)$ from inequality $0<\delta<\frac{\pi}{6}$ it follows that $-\frac{\pi}{3}<2\delta-\frac{\pi}{3}<0$. From this we get $\frac{1}{2}<\cos(2\delta-\frac{\pi}{3})<1$ i.e. the following inequalities are true $1<\frac{R_{\mathcal{A}}}{r_{\mathcal{B}}}=\cos\left(2\delta-\frac{\pi}{3}\right)+\frac{1}{2}<\frac{3}{2}$.
By similar reasoning, for $\delta\in\left(\frac{\pi}{6},\frac{\pi}{3}\right)$ from inequality $\frac{\pi}{6}<\delta<\frac{\pi}{3}$ we get the following inequalities $0<2\delta-\frac{\pi}{3}<\frac{\pi}{3}$, and from this we get thatinequalty

$$1<\frac{R_{\mathcal{A}}}{r_{\mathcal{B}}}=\cos\left(2\delta-\frac{\pi}{3}\right)+\frac{1}{2}<\frac{3}{2}$$

is valid.
Note that for $\delta=\frac{\pi}{6}$ semi-regular hexagon $\mathcal{A}_6^{a,\delta}$ is regular, and then it is $\frac{R_{\mathcal{A}}}{r_{\mathcal{B}}}=\frac{3}{2}$.

2.5. **Conclusion.** By using characteristic elements, i.e. side a and angle between side a of the semi-regular hexagon and side b_1 of the regular triangle inscribed to the semi-regular hexagon, one can express in a very simple way many of its metric properties, such as: convexity, possibility of construction, surface area calculation, radius of the inscribed circle, and some other properties. Only one problem in the ratio between the sides of inscribed triangles $\mathcal{P}_3^i, i=1,2$ has been considered here, however, other relations may also be considered.

References

[1] Hincin, Ja, A., Aleksandrov, S.P., Markusevic, I. *Poligoni i Poliedri*, Enclikopedija elementarne matematike, Knjiga cetvrta, Moskva, 1963.
[2] Radojcic,M., *Elementarna Geometrija*, Naucna knjiga, Beograd, 1961.
[3] Stojanovic, N., Neka metricka svojstva uopstenih polupravilnih poligona, Disertacija, Sarajevo 2015.
[4] Stojanovic, N., *Inscribed circle of general semi-regular polygon and some of its features*, International Journal of Geometry, Vol.2., 2013, N_0.1, pp. 5 - 22 .
[5] N.Stojanovic,*Some metric properties of general semi-regular polygons*, Global Journal of Advanced Research on Classical and Modern Geometries,Vol.1, Issue 2,(2012) pp.39-56.
[6] N.Stojanovic,V. Govedarica, Jedan pristup analizi konveksnosti i racunanju povrsine jednakostranih polupravillnih poligona, II MKRS, Zbornik radova,Trebinje,(2013),pp.87-105.

University of Banja Luka
Faculty of Agriculture
Bosnia and Herzegovina.
Email address: nenad.stojanovic@agro.unibl.org