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ON GENERALIZED BERWALD METRICS

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ABSTRACT. In this paper, we prove that every Finsler metric of almost isotropic Berwald metric is an isotropic Berwald metric. Then, we extend Peyghan-Tayebi's result and prove that every non-trivial generalized Berwald manifold of dimension $n \geq 3$ is a Randers manifold.

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1. INTRODUCTION

Let (M, F) be a Finsler manifold. In local coordinates, a curve $\sigma = \sigma(t)$ is called a geodesic if and only if its coordinates $(\sigma^i(t))$ satisfy $\ddot{\sigma}^i + 2\,G^i(\dot{\sigma}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. F is called a Berwald metric if G^i are quadratic in $y \in T_xM$ for any $x \in M$. As a generalization of Berwald metrics, Bácsó-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry [3].
Other than Douglas metrics, the class of Weyl metrics and the class of generalized Douglas-Weyl metrics are some of important projectively invariant tensors in Finsler geometry [2][7]. In 1921, Weyl introduced a projective invariant for Riemannian metrics. Then Douglas extended Weyl's projective invariant to Finsler metrics. It is the celebrated Beltrami's theorem that a Riemannian metric has vanishing projective Weyl curvature if and only if it is of constant sectional curvature. Then, Szabó proved that Weyl metrics are exactly Finsler metrics of scalar flag curvature. Finsler metrics of scalar flag curvature form a rich and important class of Finsler metrics including Riemannian metrics of constant sectional curvature [1]. A Finsler metric F is called a generalized Douglas-Weyl metric if its Douglas curvature satisfies

$$D^i{}_{jkl|m}y^m = T_{jkl}y^i \tag{1.1}$$

for some tensor T_{jkl}, where $D^i{}_{jkl|m}$ denotes the horizontal covariant derivatives of Douglas curvature $D^i{}_{jkl}$ with respect to the Berwald connection of F. For a manifold M, let $\mathcal{G}DW(M)$ denotes the class of generalized Douglas-Weyl metrics. In [2], Bácsó-Papp showed that $\mathcal{G}DW(M)$ is closed under projective changes. Then, Najafi-Shen-Tayebi characterized generalized Douglas-Weyl Randers metrics [7]. For other progress about the Finslerian projective invariants, see [13] and [14].

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[2]**Keywords**: Douglas curvature, projective motion, isotropic Berwald metric.

A Finsler metric F is called of almost isotropic Berwald curvature if

$$B^i_{jkl} = c\{F_{jk}\delta^i_l + F_{kl}\delta^i_j + F_{lj}\delta^i_k + F_{jkl}y^i\}, \tag{1.2}$$

where $c = c(x, y)$ is a scalar function on TM. We show that a Finsler metric with almost isotropic Berwald curvature has isotropic Berwald curvature.

Theorem 1.1. *Let F be a Finsler metric being an almost isotropic Berwald metric. Then, F is an isotropic Berwald metric.*

A Finsler metric F is said to be a generalized Berwald metric if its Berwald curvature is in the following form

$$B^i_{jkl} = (\mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk})y^i + \lambda(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}), \tag{1.3}$$

where $\mu_i = \mu_i(x, y)$ and $\lambda = \lambda(x, y)$ are homogeneous functions on TM of degrees -2 and -1 with respect to y, respectively [10]. Every Finslerian surface is a generalized Berwald manifold. A non-Berwaldian generalized Berwald manifold is called a non-trivial generalized Berwald manifold. In [15] and [10], A. Tayebi and E. Peyghan proved that if an n-dimensional generalized Berwald manifold (M, F) with $n \geq 3$ satisfies one of the two following conditions:
(a) F is of non-zero scalar flag curvature or
(b) F is an isotropic Berwald metric,
then, F is a Randers metric. Here, we prove that conditions (a) and (b) actually are not necessary and can be omitted.

Theorem 1.2. *Let (M, F) be an n-dimensional non-trivial generalized Berwald manifold and $n > 2$. Then, F is a Randers metric.*

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M.
A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]\Big|_{s,t=0}, \quad u, v \in T_xM.$$

For every $x \in M$, we denote Minkowski norm on T_xM induced by F with $F|_{T_xM}$. Cartan torsion describes the non-Euclidean feature of F_x, which is defined as $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0}, \quad u, v, w \in T_xM.$$

The Cartan torsion is the family of $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$. The significant of Cartan torsion is $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y : T_xM \to \mathbb{R}$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Diecke's theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

Let (M,F) be a Finsler manifold. For $y \in T_xM_0$, define the Matsumoto torsion $M_y : T_xM \times T_xM \times T_xM \longrightarrow \mathbb{R}$ by $M_y(u,v,w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\left\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\right\},$$

and $h_{ij} := FF_{y^iy^j}$ is the angular metric of F. A Finsler metric F is said to be C-reducible metric if the Matsumoto torsion of F vanishes.

Given a Finsler manifold (M,F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where

$$G^i := \frac{1}{4}g^{il}\Big\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\Big\}, \ \ y \in T_xM.$$

$\mathbf{G}$ is called the spray associated to (M,F). In local coordinates, a curve $\sigma(t)$ is a geodesic of F if and only if its coordinates $(\sigma^i(t))$ satisfy $\ddot{\sigma}^i + 2G^i(\dot{\sigma}) = 0$.

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \times T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u,v) := E_{jk}(y)u^jv^k$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j\partial y^k\partial y^l}, \quad E_{jk} := \frac{1}{2}B^m_{jkm}.$$

The tensors $\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature, respectively. Then F is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = 0$ and $\mathbf{E} = 0$, respectively.
A Finsler metric F is said to be isotropic Berwald metric if its Berwald curvature is in the following form

$$B^i{}_{jkl} = c\{F_{y^jy^k}\delta^i{}_l + F_{y^ky^l}\delta^i{}_j + F_{y^ly^j}\delta^i{}_k + F_{y^jy^ky^l}y^i\}, \tag{2.1}$$

where $c = c(x)$ is a scalar function on M.

Define $\mathbf{D}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{D}_y(u,v,w) := D^i{}_{jkl}(y)u^iv^jw^k\frac{\partial}{\partial x^i}|_x$ where

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk,l}y^i\}.$$

We call $\mathbf{D} := \{\mathbf{D}_y\}_{y\in TM_0}$ the Douglas curvature. A Finsler metric F with $\mathbf{D} = 0$ is called a Douglas metric. The notion of Douglas metrics was proposed by B*á*cs*ó*-Matsumoto as a generalization of Berwald metrics [3]. The Douglas tensor $\mathbf{D}$ is a projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + Py^i$, where $P = P(x,y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is the same as that of $\bar{F}$.

Let $F = \alpha + \beta$ be a Randers metric on a manifold M, where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. We have following Lemma

Lemma 2.1. *([5]) A positive-definite Finsler metric F on a manifold of dimension $n \geqslant 3$ is a Randers metric if and only if $M_y = 0$, $\forall y \in TM_0$.*

Lemma 2.2. *([15]) Suppose that the Cartan tensor of a Finsler metric F is in the form $C_{ijk} = B_i h_{jk} + B_j h_{ik} + B_k h_{ij}$ with $y^i B_i = 0$. Then F is a C-reducible metric.*

3. Proof of Theorem 1.1

Let us consider an n-dimensional non-trivial generalized Berwald manifold (M, F) with $n > 2$ which is not a Berwlad metric.

Lemma 3.1. *Let F be a Finsler metric satisfying*

$$D^i_{jkl} = T_{jkl} y^i, \tag{3.1}$$

where T_{jkl} is a Finslerian tensor on M. Then F is a Douglas metric.

Proof. Taking vertical derivative from (3.1) with respect to y^s, we get

$$\begin{aligned}\frac{\partial {D^i}_{jkl}}{\partial y^s} &= \frac{\partial {B^i}_{jkl}}{\partial y^s} - \frac{2}{n+1}\Big\{\frac{\partial E_{jk}}{\partial y^s}\delta^i_l + \frac{\partial E_{kl}}{\partial y^s}\delta^i_j + \frac{\partial E_{lj}}{\partial y^s}\delta^i_k + \frac{\partial^2 E_{jk}}{\partial y^s \partial y^l} y^i + \frac{\partial E_{jk}}{\partial y^l}\delta^i_s\Big\}\\ &= \frac{\partial T_{jkl}}{\partial y^s} y^i + T_{jkl}\delta^i_s. \end{aligned}\tag{3.2}$$

Contracting i and s in (3.2) and using the relations

$$\frac{1}{2}\frac{\partial {B^s}_{jkl}}{\partial y^s} = \frac{\partial E_{jk}}{\partial y^l} = \frac{\partial E_{kl}}{\partial y^j} = \frac{\partial E_{lj}}{\partial y^k} \tag{3.3}$$

we get

$$0 = \frac{\partial {D^s}_{jkl}}{\partial y^s} = (n-2) T_{jkl}. \tag{3.4}$$

Therefore, for $n > 2$, we get $T_{jkl} = 0$. This completes the proof. □

Now, we give some applications of Lemma 3.1. First, we consider almost isotropic Berwald metrics. Here, we prove that in the relation (1.2) the function $c = c(x, y)$ actually depends only on position and consequently almost isotropic Berwald metrics are isotropic Berwald metrics.

Lemma 3.2. *Let F be an almost isotropic Berwald metric. Then the Douglas tensor of F is given by*

$$D^i_{jkl} = -c_l h_{jk} F^{-1} y^i. \tag{3.5}$$

Proof. By (1.2), we get

$$E_{jk} = \frac{n+1}{2} c(x, y) F_{jk}. \tag{3.6}$$

As it is known, the Douglas curvature is given by

$$D^i_{jkl} = B^i_{jkl} - \frac{2}{n+1}\Big\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + \frac{\partial E_{jk}}{\partial y^l} y^i\Big\}. \tag{3.7}$$

Plugging (1.2) and (3.6) into (3.7), we get (3.5). □

By Lemma 3.1, we get the following.

Proof of Theorem 1.1: Let F be an almost isotropic Berwald metric. By Lemma 3.2, the Douglas tensor of F is given by (3.5). Hence F satisfies (3.1). By Theorem 3.1, F is a Douglas metric, and consequently c depends only on x. Therefore, the Berwald tensor of F is given by

$$B^i_{jkl} = c(x)\Big\{F_{jk}\delta^i_l + F_{kl}\delta^i_j + F_{lj}\delta^i_k + F_{jkl}y^i\Big\}. \tag{3.8}$$

It means that F is an isotropic Berwald metric. □

4. Proof of Theorem 1.2

In order to prove Theorem 1.2, let us first show that the coefficient λ in (1.3) is in a special form.

Lemma 4.1. *Let (M, F) be an n-dimensional non-trivial generalized Berwald manifold and $n > 2$. Then, there exists scalar function $c = c(x)$ on M such that $\lambda = cF^{-1}$.*

Proof. It follows from (1.3) that $E_{jk} = \frac{n+1}{2}\lambda h_{jk}$. Now, (3.3) implies that

$$(\lambda_k F + \lambda F_k)h_{jt} = (\lambda_j F + \lambda F_j)h_{kt}. \tag{4.1}$$

Contracting (4.1) with g^{jt} and using $\lambda_y y^t = -\lambda$ yield

$$(n-2)(\lambda_k F + \lambda F_k) = (n-2)\frac{\partial}{\partial y^k}(\lambda F) = 0. \tag{4.2}$$

This completes the proof. □

Proof of Theorem 1.2: It is easy to see that F satisfies (3.1) with

$$T_{jkl} = \mu_j h_{kl} + \mu_k h_{jl} + (\mu_l - \lambda F^{-1}F_l - \lambda_l)h_{jk} - 2\lambda C_{jkl}. \tag{4.3}$$

Lemma 4.1 implies that $\lambda F^{-1}F_l + \lambda_l = 0$. Thus, (4.3) reduces to

$$T_{jkl} = \mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk} - 2\lambda C_{jkl}. \tag{4.4}$$

If $\lambda = 0$, then Theorem 3.1 implies

$$\mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk} = 0$$

and consequently, $\mu_j = 0$. This is a contradiction. Thus, we have $\lambda \neq 0$. Now, Lemma 3.1 implies that

$$C_{jkl} = \frac{1}{2\lambda}\{\mu_j h_{kl} + \mu_k h_{jl} + \mu_l h_{jk}\}. \tag{4.5}$$

By Lemmas 2.1 and 2.2, we get the proof. □

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