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A STUDY OF A FINSLER METRIC ARISING FROM LAPLACE TRANSFORM

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ABSTRACT. In this paper we introduce a new type of construction of (α, β)-metrics obtained from Laplace transform on Bessel functions. Some properties of this metrics are studied. The variational problem and the main scalar of this new metric will be studied also in this paper.

1. PRELIMINARIES

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) For each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s,t=0}, \ \ u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0}, \ \ u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Diecke Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$. There are many connections in Finsler geometry (see [24]). In this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by symbols "$|$" and "$,$" respectively. The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where $J_i := I_{i|s}y^s$. A Finsler metric is said to be weakly Landsbergian if $\mathbf{J} = 0$. For more details on Finsler metrics; Cartan torsion and Landsberg curvature please see [1] and [22].

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Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} g^{il} \Big[\frac{\partial^2(F^2)}{\partial x^k \partial y^l} y^k - \frac{\partial(F^2)}{\partial x^l}\Big], \quad y \in T_xM.$$

The $\mathbf{G}$ is called the spray associated to (M, F). In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y) u^j v^k$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2} B^m{}_{jkm}.$$

The $\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature, respectively. Then F is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = \mathbf{0}$ and $\mathbf{E} = \mathbf{0}$, respectively.
The S-curvature was introduced by Z. Shen in [26], in the following way:

Definition 1.1. ([26]) *Let V be an n-dimensional real vector space and F be a Minkowski norm on V. For a basis $\{e_i\}$ of V, let:*

$$\sigma_F = \frac{Vol\ (B^n)}{Vol\ \{y^i \in \mathbb{R}^n | F(y^i e_i) < 1\}}$$

where Vol represent the volume of a subset in the standard Euclidean space $\mathbb{R}^n$ and B^n is the open ball with radius 1. The quantity: $\tau(y) = ln \frac{\sqrt{det(g_{ij}(y))}}{\sigma_F}$, $y \in V - \{0\}$, is called distorsion of (V, F). Let (M, F) be a Finsler space and $\tau(x, y)$, the distorsion of the Minkowski norm F_x on T_xM. For $y \in T_xM - \{0\}$, let $\tau(t)$ be the geodesic with $\tau(0) = x$ and $\dot{\tau}(0) = y$. Then the quantity

$$S(x, y) = \frac{d}{dt} [\tau(\sigma(t), \dot{\sigma}(t))] |_{t=0}, \tag{1.1}$$

is called S-curvature of the Finsler space (M, F).

Remark 1.1. *A Finsler space (M, F) is said to have almost isotropic S-curvature if there exist a smooth function $c(x)$ on M and a closed 1-form η such that:*

$$S(x, y) = (n + 1) (c(x)F(y) + \eta(y)), \tag{1.2}$$

$x \in M$, $y \in T_xM$.

Remark 1.2. *If, in (2.2), we have $\eta = 0$, then (M, F) is said to have isotropic S-curvature. If $\eta = 0$ and $c(x)$ is constant, then (M, F) is said to have constant S-curvature.*

The S-curvature of an G-invariant homogeneous (α, β)-metric $F = \alpha\phi(s)$, can be expressed in the following way ([8]):

$$S = \left(2\Psi - \frac{f'(b)}{bf(b)}\right)(r_0 + s_0) - \alpha^{-1}\frac{\Phi}{2\Delta^2}\left(r_{00} - 2\alpha Q s_0\right) \tag{1.3}$$

where:

$$f(b)=\frac{\int_0^\pi(\sin t)^{n-2}T(b\cos t)dt}{\int_0^\pi(\sin t)^{n-2}dt};\ T(s)=\phi(\phi-s\phi')^{n-2}\left[(\phi-s\phi')+(b^2-s^2)\phi''\right];$$

$$Q=\frac{\phi'}{\phi-s\phi'};\ \ \Delta=1+sQ+(b^2+s^2)Q';\ \ \Psi=\frac{Q'}{2\Delta};$$

$$\Phi=-(Q-sQ')\left\{n\Delta+1+sQ\right\}-(b^2-s^2)(1+sQ)Q'' \tag{1.4}$$

$$r_{ij}=\frac{1}{2}\left(b_{i|j}+b_{j|i}\right);\ \ s_{ij}=\frac{1}{2}\left(b_{i|j}-b_{j|i}\right);$$

$$s_j=b^is_{ij};\ \ s^i_j=a^{il}s_{lj};\ \ s_0=s_iy^i;\ \ s^i_0=s^i_jy^i;\ \ r_{00}=r_{ij}y^iy^j;\ \ r_j=b^ir_{ij}.$$

The Busemann-Hausdorff volume form $dV_{BH}=\sigma_F(x)dx^1dx^2\cdots dx^n$, is defined by:

$$\sigma_F=\frac{Vol\,(w_n)}{Vol\left\{y^i\in\mathbb{R}|F(x,y^i\frac{\partial}{\partial x_i})<1\right\}}.$$

Then, the S-curvature is defined by:

$$S(y)=\frac{\partial G^i}{\partial y^i}(x,y)-y^i\frac{\partial}{\partial x^i}\left[ln\sigma_F(x)\right] \tag{1.5}$$

where $y=y^i\frac{\partial}{\partial x^i}|_x\in T_xM$. For more details please see [8].

Lemma 1.1. ([8]) *Let $F=\alpha\phi(s)$; $s=\frac{\beta}{\alpha}$, be a non-Riemann (α,β)-metric on a manifold M of dimension $n\geq 3$ and $\beta=\|\beta_x\|_\alpha$. Suppose that F is not a Finsler metric of Randers type. Then F is of isotropic S-curvature, $S=(n+1)cF$, if and only if one of the following holds:*

- *β satisfies: $r_{ij}=\epsilon(b^2a_{ij}-b_ib_j)$, $s_j=0$; where $\epsilon=\epsilon(x)$ is a scalar function and $\phi=\phi(s)$ satisfies: $\Phi=-2(n+1)k\frac{\phi\Delta^2}{b^2-s^2}$, with $k=const$. In this case, $S=(n+1)cF$, with $c=k\epsilon$.*
- *β satisfies $r_{ij}=0$; $s_j=0$. In this case, $S=0$.*

The Landsberg curvature is expressed in [27] and is given by:

$$L_{ijk}=\frac{-\rho}{6\alpha^5}\left\{h_ih_jC_k+h_jh_kC_i+h_ih_kC_j+3E_iT_{jk}+3E_jT_{ik}+3E_kT_{ij}\right\} \tag{1.6}$$

where:

$$h_i=\alpha b_i-s\overline{y}_i;\ \ T_{ij}=\alpha^2a_{ij}-\overline{y}_i\overline{y}_j$$

$$C_i=\left(X_4r_{00}+Y_4\alpha s_0\right)h_i+3\Lambda D_i$$

$$E_i=\left(X_6r_{00}+Y_6\alpha s_0\right)h_i+3\mu D_i$$

$$D_i=\alpha^2\left(s_{i0}+\Gamma r_{i0}+\Pi\alpha s_i\right)-\left(\Gamma r_{00}+\Pi\alpha s_0\right)\overline{y}_i$$

$$X_4=\frac{1}{2\Delta^2}\left\{-2\Delta Q'''+3(Q-sQ')Q''+3(b^2-s^2)(Q'')^2\right\} \tag{1.7}$$

$$X_6=\frac{1}{2\Delta^2}\left\{(Q-sQ')^2+2\left[2(s+b^2Q)-(b^2-s^2)(Q-sQ')\right]Q''\right\}$$

$$Y_4=-2QX_4+\frac{3Q'Q''}{\Delta}$$

$$Y_6 = -2QX_6 + \frac{(Q - sQ')Q'}{\Delta}$$

$$\Lambda = -Q''; \ \mu = -\frac{1}{3}(Q - sQ'); \ \Gamma = \frac{1}{\Delta}; \ \Pi = -\frac{Q}{\Delta}.$$

Remark 1.3. *The Landsberg curvature for an (α, β)-metric is given in* [29] *in the following way:*

$$J_i = \frac{-1}{2\alpha^4\Delta}\left(\frac{2\alpha^2}{b^2 - s^2}\left[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\right](r_0 + s_0)h_i + \right.$$
$$\frac{\alpha}{b^2 - s^2}\left[\Psi_1 + s\frac{\phi}{\Delta}\right](r_{00} - 2\alpha Q s_0)h_i + \alpha\left[-\alpha Q' s_0 h_i + \alpha Q(\alpha^2 s_i - \overline{y}_i s_0) + \right. \tag{1.8}$$
$$\left.\left.\alpha^2\Delta s_{i0} + \alpha^2(r_{i0} - 2\alpha Q s_0) - (r_{00} - 2\alpha Q s_0)\overline{y}_i\right]\frac{\Phi}{\Delta}\right)$$

where:

$$\Psi_1 = \sqrt{b^2 - s^2}\Delta^{\frac{1}{2}}\left[\frac{\sqrt{b^2 - s^2}}{\Delta^{\frac{3}{2}}}\right]'$$
$$h_i = ab_i - s\overline{y}_i; \ \overline{y}_i = a_{ij}y^j$$
$$\Phi = -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q''. \tag{1.9}$$

For more details please see [29].

Remark 1.4. *Acording to* [11]*, the S-curvature of the (α, β)-metric $F = \alpha\phi(s)$, can be computed as follows:*

$$S = \left\{Q' - 2\Psi Qs - 2[\Psi Q]'(b^2 - s^2) - 2(n+1)Q\Theta + 2\lambda\right\} s_0 + \tag{1.10}$$
$$2\left\{\Psi + \lambda\right\} r_0 + \alpha^{-1}\left\{(b^2 - s^2)\Psi' + (n+1)\Theta\right\} r_{00},$$

where $\lambda = -\frac{\mu'(b)}{2b\mu(b)}$ and

$$\mu(b) = \frac{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left[\int_0^\pi \frac{\sin^{n-2}\theta}{\phi^n(b\cos\theta)}\right]^{-1}. \tag{1.11}$$

Here, Γ represent the Euler function.

Remark 1.5. *The mean Cartan torsion of an (α, β)-metric is given by:*

$$I_i = \frac{1}{2}\frac{\partial}{\partial y^i}\left[(n+1)\frac{\phi}{\phi'} - (n-2)\frac{s\phi''}{\phi - s\phi'} - \frac{3s\phi'' - (b^2 - s^2)\phi'''}{(\phi - s\phi') + (b^2 - s^2)\phi''}\right] = \tag{1.12}$$
$$-\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i - sy_i).$$

For more details please see [29].

Another important result is the following one:

Lemma 1.2. *(*[1]*)Let F be an (α, β)-metric. Then F is locally Minkowskian if and only if α is flat and $b_{i|j} = 0$, (that is β parallel with respect to α, $r_{ij} = 0$; $s_{ij} = 0$).*

Next, we will present some remarks regarding the Lagrange spaces in Finsler geometry:

Definition 1.2. [12] *A Lagrange space is a pair $L^n = (M, L(x,y))$ formed by a smooth real, n-dimensional manifold M and a regular differentiable Lagrangian $L(x,y)$, for which the d-tensor field g_{ij} has constant signature over the manifold $\widetilde{TM}$.*

From [31] and [16], Finsler spaces endowed with (α, β)-metrics were applied succefully to the study of gravitational magnetic fields. Other important results from [12] are presented as follows:
Let $F^n = (M, F(x,y))$ be a Finsler space. It has an (α, β)-metric if the fundamental function can be expressed in the following form: $F(x,y) = \check{F}(\alpha(x,y), \beta(x,y))$, where $\check{F}$ is a differentiable function of two variables with: $\alpha^2(x,y) = a_{ij}(x)y^iy^j$; $\beta(x,y) = b_i(x)y^i$.
$a = a_{ij}(x)dx^idx^j$ is a pseudo-Riemannian metric on the base manifold M and $b_i(x)dx^i$ is the electromagnetic 1-form on M. As we know from , if we denote by $L^n = (M, L)$ a Lagrange space; the fundamental tensor $g_{ij}(x,y)$ of L^n is: $g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$ and this tensor can be written as follows for (α, β)-Lagrangians:

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_{-1}\left(b_i\mathcal{Y}_j + b_j\mathcal{Y}_i\right) + \rho_{-2}\mathcal{Y}_i\mathcal{Y}_j$$

where $b_i = \frac{\partial \beta}{\partial y^i}$; $\mathcal{Y}_i = a_{ij}y^j = \alpha\frac{\partial \alpha}{\partial y^i}$.
$\rho; \rho_0; \rho_{-1}; \rho_{-2}$ are invariants of the space L^n.
Here, $\rho; \rho_0; \rho_{-1}; \rho_{-2}$ are given by (see [12]):

$$\rho = \frac{1}{2\alpha}L_\alpha; \rho_0 = \frac{1}{2}L_{\beta\beta};$$
$$\rho_{-1} = \frac{1}{2\alpha}L_{\alpha\beta}; \rho_{-2} = \frac{1}{2\alpha^2}\left(L_{\alpha\alpha} - \frac{1}{\alpha}L_\alpha\right). \tag{1.13}$$

where $L_\alpha = \frac{\partial L}{\partial \alpha}$; $L_\beta = \frac{\partial L}{\partial \beta}$; $L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2}$; $L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2}$ and $L_{\alpha\beta} = \frac{\partial^2 L}{\partial \alpha \partial \beta}$.
Shimada and Sabău in [28], have proved that the system of covectors $\{b_i, \mathcal{Y}_i\}$ is independent. The following formulae holds (see [12]):

$$y_i = \frac{1}{2}\frac{\partial L}{\partial y^i} = \rho_1 b_i + \rho\mathcal{Y}_i; \rho_1 = \frac{1}{2}L_\beta;$$
$$\frac{\partial \rho_1}{\partial y^i} = \rho_0 b_i + \rho_{-1}\mathcal{Y}_i; \frac{\partial \rho}{\partial y^i} = \rho_{-1}b_i + \rho_{-2}\mathcal{Y}_i$$
$$\frac{\partial \rho_0}{\partial y^i} = r_{-1}b_i + r_{-2}\mathcal{Y}_i; \frac{\partial \rho_{-1}}{\partial y^i} = r_{-2}b_i + r_{-3}\mathcal{Y}_i \tag{1.14}$$
$$\frac{\partial \rho_{-2}}{\partial y^i} = r_{-3}b_i + r_{-4}\mathcal{Y}_i$$

with $r_{-1} = \frac{1}{2}L_{\beta\beta\beta}$; $r_{-2} = \frac{1}{2\alpha}L_{\beta\beta\beta}$; $r_{-3} = \frac{1}{2\alpha^2}\left(L_{\alpha\alpha\beta} - \frac{1}{\alpha}L_{\alpha\beta}\right)$ and
$r_{-4} = \frac{1}{2\alpha^3}\left(L_{\alpha\alpha\alpha} - \frac{3}{\alpha}L_{\alpha\alpha} + \frac{3}{\alpha^2}L_\alpha\right)$.
The Cartan tensor in such of space can be computed as follows(see [12]):

$$2C_{ijk} = \underset{(i,j,k)}{\sigma}\left\{\rho_{-1}a_{ij}b_k + \rho_{-2}a_{ij}\mathcal{Y}_k + \frac{1}{3}r_{-1}b_ib_jb_k + r_{-2}b_ib_j\mathcal{Y}_k \right.$$
$$\left. + r_{-3}b_i\mathcal{Y}_j\mathcal{Y}_k + \frac{1}{3}r_{-4}\mathcal{Y}_i\mathcal{Y}_j\mathcal{Y}_k\right\}, \tag{1.15}$$

where $\sigma_{(i,j,k)}$ is the cyclic sum in the indices i, j, k.
The variational problem for Finsler spaces endowed with (α, β)-metrics is an important topic in Finsler geometry. For such spaces, the Euler-Lagrange equations $E_i(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right)$=0, can be give in the following way:

$$E_i(L) = E_i(\alpha^2) + 2\frac{\rho_1}{\rho}E_i(\beta) + 2\frac{d\alpha}{dt}\frac{\partial\alpha}{\partial y^i} \tag{1.16}$$

The following result is very important:

Theorem 1.1. *([12]) In the natural parametrization, $t = s$; the Euler-Lagrange equations of the Lagrangian $L(\alpha, \beta)$, are given by:*

$$E_i(\alpha^2) + 2\frac{\rho_1}{\rho}F_{ij}(x)y^j = 0;\ y^i = \frac{dx^i}{ds}. \tag{1.17}$$

Remark 1.6. *If we use the following equations $E_i(\beta) = F_{ij}(x)\frac{dx^j}{ds}$;*

$$F_{ij} = \frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j} = b_{j|i} - b_{i|j},$$

then (5) can be rewritted in the following way:

$$E_i(\alpha^2) + 2\frac{\rho_1}{\rho}\left(b_{j|i} - b_{i|j}\right) = 0;\ y^i = \frac{dx^i}{ds}. \tag{1.18}$$

Another important result obtained by [32], is the following one:

Theorem 1.2. *([32]) Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an (α, β)-metric on an n-dimensional manifold $M^n, (n \geq 3)$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i \neq 0$ is an 1-form on M. Suppose that F is not Riemannian and $\phi'(s) \neq 0$; $\phi'(0) \neq 0$; $\beta \neq 0$. Then F is a locally dually flat on M if and only if α, β and $\phi = \phi(s)$, satisfy:*

- 1.$s_{lo} = \frac{1}{3}(\beta\theta_l - \theta b_l)$,
- 2.$r_{00} = \frac{2}{3}\theta\beta + \left[\theta + \frac{2}{3}(b^2\theta - \theta_l b^l)\right]\alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3b^2)\theta\beta^2$,
- 3.$G^l_\alpha = \frac{1}{3}\left[2\theta + (3k_1 - 2)\theta\beta\right]y^l + \frac{1}{3}(\theta^l - \tau b^l)\alpha^2 + \frac{1}{2}k_3\tau\beta^2 b^l$,
- 4.$\tau\left[s(k_2 - k_3s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')\right] = 0$,

where $\tau = \tau(x)$ is a scalar function; $\theta = \theta_i(x)y^i$ is an 1-form on M, $\theta^l = a^{lm}\theta_m$,

$$k_1 = \Pi(0); k_2 = \frac{\Pi'(0)}{Q(0)}; k_3 = \frac{1}{6Q(0)^2}\left[3Q''(0)\Pi'(0) - 6\Pi(0)^2 - Q(0)\Pi'''(0)\right], \tag{1.19}$$

and $Q = \frac{\phi'}{\phi - s\phi'}$; $\Pi = \frac{\phi'^2 + \phi\phi''}{\phi(\phi - s\phi')}$.

Finally, we will recall the following:

Theorem 1.3. *([25]) The function $F = \alpha\phi(\frac{\beta}{\alpha})$ is a Finsler metric for any $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$, with $\|\beta_x\|_\alpha < b_0$ if and only if $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$, satisfying the following conditions:*

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0,\ \ |s| \leq b < b_0$$
$$\phi(s) - s\phi'(s) > 0,\ \ |s| < b_0$$
$$\phi(s) > 0,\ \ |s| < b_0$$

2. Main Result

2.1. **Construction of a new type of (α, β)-metrics.** In this section we will construct a new type of (α, β)-metrics, using the Laplace transform. As we well know, the Laplace transform is used in electrotechnics and can be defined by $F(s) = \int_0^\infty e^{-st} f(t)dt$, if such an integral exists. Let's recall now the Bessel functions which can are defined as follows:

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{p+2n}}{2^{p+2n} n! \Gamma(p+n+1)}. \tag{2.1}$$

For the case $p = 1$, easily can be obtained the following Bessel function:

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1} n!(n+1)!}. \tag{2.2}$$

This Bessel function $J_1(x)$ is very important in physics because describe the Fraunhofer diffraction phenomena in the diffraction theory of modern physics. Diffraction phenomena can be described easily as been any deviation from geometrical optics that result from an obstruction of a wavefront of light. Fraunhofer diffraction appear when both the incident and diffracted waves are effectively plane. This occurs when the distance from the source to the aperture is large so that the aperture is assumed to be uniformly illuminated and the distance from the aperture plane to the observation plane is also large. So, the Fraunhofer diffraction pattern for a uniformly illuminated circular aperture can be described using the Bessel function $J_1(x)$. Now, for this function, it can be obtained after simple computations, the Laplace transform:

$$\mathcal{L}(J_1(t)) = 1 - \frac{s}{\sqrt{s^2+1}} = \phi(s). \tag{2.3}$$

Using this function $\phi(s)$, we will construct the attached (α, β)-metric. This new metric is:

$$F(\alpha, \beta) = \alpha \left(1 - \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right). \tag{2.4}$$

As we know from literature, recently, some progress was done for the study of Bessel and Fourrier transforms, for example, please see [15].
We will investigate in the following lines the new metric (2.4).

3. The variational problem for the (α, β)-metric which arise from Laplace transform

As we have seen in the previous section, we can construct a new (α, β)-metric using the Laplace transform for the Bessel function of the first kind $J_1(x)$. In this section we will find the Main Scalar for this new metric and also we will investigate the variational problem. The fundamental function attached to the new metric is:

$$L(\alpha, \beta) = \left(\alpha - \frac{\alpha\beta}{\sqrt{\alpha^2 + \beta^2}}\right)^2. \tag{3.1}$$

Next, we will compute the following:

$$\frac{\partial}{\partial\alpha}L(\alpha,\beta) = \frac{2\alpha\left(\sqrt{\alpha^2+\beta^2}-\beta\right)\left(\left(\alpha^2+\beta^2\right)^{3/2}-\beta^3\right)}{\left(\alpha^2+\beta^2\right)^2}$$

$$\frac{\partial}{\partial\beta}L(\alpha,\beta) = \frac{-2\alpha^4\left(\sqrt{\alpha^2+\beta^2}-\beta\right)}{\left(\alpha^2+\beta^2\right)^2}$$

$$\frac{\partial^2}{\partial^2\alpha^2}L(\alpha,\beta) = \frac{4\,\beta^6+6\,\alpha^4\beta^2-2\,\sqrt{\alpha^2+\beta^2}\left(-\alpha^2\beta^2+2\,\beta^4\right)\beta+2\,\alpha^6}{\left(\alpha^2+\beta^2\right)^3}$$

$$\frac{\partial^2}{\partial\alpha\partial\beta}L(\alpha,\beta) = \frac{-2\alpha\left(\left(-4\,\beta^3-\left(\alpha^2+\beta^2\right)^{3/2}+3\,\sqrt{\alpha^2+\beta^2}\beta^2\right)\alpha^2+2\,\alpha^2\left(\alpha^2+\beta^2\right)^{3/2}\right)}{\left(\alpha^2+\beta^2\right)^3}$$

$$\frac{\partial^2}{\partial^2\beta^2}L(\alpha,\beta) = \frac{2\alpha^4\left(\alpha^2-3\,\beta^2+3\,\beta\,\sqrt{\alpha^2+\beta^2}\right)}{\left(\alpha^2+\beta^2\right)^3}$$

$$P(\alpha,\beta) = \frac{2\alpha^2\left(\sqrt{\alpha^2+\beta^2}-\beta\right)^3\left(\left(\alpha^2+\beta^2\right)^{3/2}-\beta^3\right)}{\left(\alpha^2+\beta^2\right)^3}$$

$$P_0(\alpha,\beta) = \frac{6\alpha^6\left(\sqrt{\alpha^2+\beta^2}-\beta\right)^2\left(\alpha^2-\beta^2+\beta\,\sqrt{\alpha^2+\beta^2}\right)}{\left(\alpha^2+\beta^2\right)^4}$$

$$P_{-1}(\alpha,\beta) = \frac{-2\alpha^4\left(\sqrt{\alpha^2+\beta^2}-\beta\right)^2\left(-6\,\beta^3-\left(\alpha^2+\beta^2\right)^{3/2}+3\,\sqrt{\alpha^2+\beta^2}\beta^2\right)}{\left(\alpha^2+\beta^2\right)^4} - \frac{8\alpha^4\left(\sqrt{\alpha^2+\beta^2}-\beta\right)^2}{\left(\alpha^2+\beta^2\right)^{5/2}}$$

$$P_{-2}(\alpha,\beta) = \frac{-2\left(\sqrt{\alpha^2+\beta^2}-\beta\right)^2}{\left(\alpha^2+\beta^2\right)^4} \times\left(-4\,\beta^6-2\,\beta^4\alpha^2-6\,\alpha^4\beta^2+\sqrt{\alpha^2+\beta^2}\left(-\alpha^4+4\,\beta^4\right)\beta-2\,\alpha^6\right).$$

Using the above computations and also the the results from [13], the Main Scalar for the studied metric (2.3), with the fundamental function (3.1), can be easily obtained replacing P, P_{-1}, P_{-2} and respectively P_0 in

$$\epsilon I^2 = \left(\frac{L(\alpha,\beta)}{\alpha}\right)^4\left[\frac{\gamma^2(T_2)^2}{4T^3}\right]. \tag{3.2}$$

Here ϵ represent the signature of the space, $\gamma^2 = b^2\alpha^2-\beta^2$ and $T_2 = \frac{\partial T}{\partial\beta}$.

Theorem 3.1. *The mean Cartan torsion of the (α, β)-metric (2.4), is given by:*

$$I_i = \frac{1}{2}\frac{\partial}{\partial y^i}\Bigg[-(1+n)\left(\sqrt{s^2+1}-s\right)(s^2+1) - 3\,\frac{(n-2)\,s^2}{(s^2+1)\left((s^2+1)^{3/2}-s^3\right)} - 3\,\frac{-s^4+4\,s^2+4\,b^2s^2-b^2}{(s^2+1)\left((s^2+1)^{5/2}-s^5-4\,s^3+3\,sb^2\right)}\Bigg]. \tag{3.3}$$

Proof. The proof of this theorem is immediate from (1.12) and using some computations in Maple we get the asertion of the theorem. □

Now we will proof that this new metric (2.4), is a Finsler metric. In this respect, we will use Theorem 1.3 and we obtain the following:

Theorem 3.2. *The metric (2.4) is a Finsler metric, because the following conditions holds:*

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \ \ |s| \le b < b_0$$

$$\phi(s) - s\phi'(s) > 0, \ \ |s| < b_0$$

$$\phi(s) > 0, \ \ |s| < b_0$$

for any $\alpha = \sqrt{a_{ij}y^iy^j}$ *and* $\beta = b_iy^i$*, with* $\|\beta_x\|_\alpha < b_0$ *and* $\phi = \phi(s)$ *is a positive* C^∞ *function on* $(-b_0, b_0)$*.*

Proof. We will investigate all this conditions one by one:
The first one,

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \ \ |s| \le b < b_0$$

is equivalent after computations with

$$\frac{(s^2+1)\left((s^2+1)\sqrt{s^2+1}-s^3\right)+3s(b^2-s^2)}{(s^2+1)^{\frac{5}{2}}} > 0$$

and is easy to observe that this condition holds for any $|s| \le b < b_0$.
The second one,

$$\phi(s) - s\phi'(s) > 0, \ \ |s| < b_0$$

is equivalent with

$$\frac{(s^2+1)\sqrt{s^2+1}-s^3}{(s^2+1)^{\frac{3}{2}}} > 0$$

and is easy to observe that this condition holds for any $|s| < b_0$.

Finally, the third condition is equivalent with

$$\frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}} > 0$$

and is easy to observe that this condition holds for any $|s| < b_0$.

□

Next, we will compute the S-curvature for this metric because as we know in Finsler geometry, the S-curvature of an (α, β)-metric has an very important role. Now, we will compute for the (α, β)-metric (2.4), with $\phi(s) = 1 - \frac{s}{\sqrt{s^2+1}}$ the following:

$$Q(s) = -\left(\left(s^2+1\right)^{3/2} - s^3\right)^{-1}$$

$$\Delta(s) = \frac{1+2\,s^6 - 2\,s^5\sqrt{s^2+1} + s^4 + 3\,s^2 - 3\,b^2s^2 - s\sqrt{s^2+1} + 3\,sb^2\sqrt{s^2+1}}{\left(\sqrt{s^2+1}s^2 + \sqrt{s^2+1} - s^3\right)^2};$$

$$\Psi(s) = 3/2\,\frac{s\left(\sqrt{s^2+1} - s\right)}{1+2\,s^6 - 2\,s^5\sqrt{s^2+1} + s^4 + 3\,s^2 - 3\,b^2s^2 - s\sqrt{s^2+1} + 3\,sb^2\sqrt{s^2+1}};$$

$$\begin{aligned}\Phi(s) = \left(2\,s^2+1+s\sqrt{s^2+1}\right)^{-4} \Big(&54n\,s^7 + 54\,\sqrt{s^2+1}ns^6 + \\ &\left(20+36\,nb^2+62\,n+18\,b^2\right)s^5 + \sqrt{s^2+1}\left(16+37n+36nb^2+18b^2\right)s^4 + \\ &\left(23+23\,n+15\,b^2+27\,nb^2\right)s^3 + \sqrt{s^2+1}\left(12\,b^2+9nb^2+14+11n\right)s^2 + \\ &\left(3\,n+3+3\,nb^2-3\,b^2\right)s - \left(3\,b^2+n+1\right)\sqrt{s^2+1}\Big);\end{aligned} \quad (3.4)$$

$$\Theta(s) = -\frac{\sqrt{s^2+1}(4s^2+1) - 4\,s^3}{2(1+2\,s^6+s^4+3\,s^2-3\,b^2s^2-(2s^5+s-3sb^2)\sqrt{s^2+1})};$$

$$T(s) = \left(\sqrt{s^2+1} - s\right)\left(1 - \frac{s^3}{\left(s^2+1\right)^{3/2}}\right)^{n-2}\left(\left(s^2+1\right)^{5/2} - s^5 - 4\,s^3 + 3\,sb^2\right)\left(s^2+1\right)^{-3}.$$

Using all the above relations (3.4), and also Remark 1.4, we are ready now to formulate

Theorem 3.3. *The S-curvature for the metric (2.4), can be computed by*

$$\begin{aligned} S = &\left\{Q' - 2\Psi Qs - 2[\Psi Q]'(b^2 - s^2) - 2(n+1)Q\Theta + 2\lambda\right\} s_0 + \\ &2\left\{\Psi + \lambda\right\} r_0 + \alpha^{-1}\left\{(b^2 - s^2)\Psi' + (n+1)\Theta\right\} r_{00},\end{aligned} \quad (3.5)$$

where $\lambda = -\frac{\mu'(b)}{2b\mu(b)}$ *and*

$$\mu(b) = \frac{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left[\int_0^{\pi} \frac{\sin^{n-2}\theta}{\phi^n(b\cos\theta)}\right]^{-1}. \quad (3.6)$$

Here,

$$Q' - 2\Psi Qs - 2[\Psi Q]'(b^2 - s^2) - 2(n+1)Q\Theta + 2\lambda =$$

$$\frac{\left(-12s^7 + 3\,s^5 + \left(18b^2 - 9\right)s^3 - 3s\right)\sqrt{s^2+1} + 12s^8 + 3s^6 + \left(-18b^2+6\right)s^4 + \left(-9b^2+3\right)s^2}{\left(s^3 - (s^2+1)\sqrt{s^2+1}\right)^2\left(-2\,s^6 - s^4 - 3s^2 - 1 + \sqrt{s^2+1}(2s^5+s-3sb^2) + 3b^2s^2\right)} -$$

$$\frac{\left(\left(8s^8 + \left(10-12b^2\right)s^4 + \left(-6b^2+8\right)s^2 + 6s^6 + 2\right)\sqrt{s^2+1}\right)\lambda}{\left(s^3 - (s^2+1)\sqrt{s^2+1}\right)^2\left(-2s^6 - s^4 - 3s^2 - 1 + \sqrt{s^2+1}(2s^5+s-3sb^2) + 3b^2s^2\right)} -$$

$$\frac{\left(12s\left(-2/3s^8-5/6s^6+\left(-1+b^2\right)s^4+\left(b^2-1/2\right)s^2-1/6+1/2b^2\right)\right)\lambda}{\left(s^3-(s^2+1)\sqrt{s^2+1}\right)^2\left(-2s^6-s^4-3s^2-1+\sqrt{s^2+1}(2s^5+s-3sb^2)+3b^2s^2\right)}-$$

$$\frac{\left(-1+\left(-4\,n-4\right)s^2-n\right)\sqrt{s^2+1}+12\ \left(1/3\,n+1/3\right)s^3}{\left(s^3-(s^2+1)\sqrt{s^2+1}\right)^2\left(-2\,s^6-s^4-3s^2-1+\sqrt{s^2+1}(2s^5+s-3sb^2)+3b^2s^2\right)}$$

$$\Psi'(s)=\frac{3(1-\sqrt{s^2+1}(3s^3+2s-11s^5+32s^9+28s^7)-32\,s^{10}-44\,s^8)}{2\left(s^3-(s^2+1)\sqrt{s^2+1}\right)\left(-2s^6-s^4-3s^2-1+\sqrt{s^2+1}(2s^5+s-3sb^2)+3b^2s^2\right)^2}.$$

Next, we can reformulate Theorem 3.1 forthe computation of the mean Cartan torsion for the (α,β)-metric (2.4), but this time using the above Remark 1.5.

Theorem 3.4. *The mean Cartan torsion for the (α,β)-metric (2.4), with $\phi(s)=1-\frac{s}{\sqrt{s^2+1}}$, is given by:*

$$I_i=\frac{(-54ns^6+(-18-36n)b^2-16-37n)\sqrt{s^2+1}}{2\left(2s^2+1+\sqrt{s^2+1}s\right)\left(-\sqrt{s^2+1}+s\right)^2(s^2+1)\,M(s)}+$$

$$\frac{\left(\left(\left(-9\,n-12\right)b^2-14-11\,n\right)s^2-n+3\,b^2-1\right)\sqrt{s^2+1}}{2\left(2s^2+1+\sqrt{s^2+1}s\right)\left(-\sqrt{s^2+1}+s\right)^2(s^2+1)\,M(s)}+$$

$$\frac{54ns^7+36\left(n+\frac{1}{2}\right)b^2s^5+20s^5+62ns^5+(15+27n)b^2s^3}{2\left(2s^2+1+\sqrt{s^2+1}s\right)\left(-\sqrt{s^2+1}+s\right)^2(s^2+1)\,M(s)}+$$

$$\frac{23(n+1)s^3+s^3((15+27n)b^2+23(n+1))}{2\left(2s^2+1+\sqrt{s^2+1}s\right)\left(-\sqrt{s^2+1}+s\right)^2(s^2+1)\,M(s)}$$

where

$$M(s)=\left(-1-2s^6+(2s^5+s-3sb^2)\sqrt{s^2+1}-s^4-3s^2+3b^2s^2-3sb^2\right).$$

To obtain the proof of this theorem we have made all the computations in Maple 13.

Theorem 3.5. *Let $F=\alpha\phi(s)$, $s=\frac{\beta}{\alpha}$ be the (α,β)-metric on an n-dimensional manifold $M^n,(n\geq 3)$, given in (2.4), with $\phi(s)=1-\frac{s}{\sqrt{s^2+1}}$, where $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i\neq 0$ is an 1-form on M. Knowing that F is not Riemannian and $\phi'(s)\neq 0$; $\phi'(0)\neq 0$; $\beta\neq 0$, then F is a locally dually flat on M if and only if α,β and $\phi=\phi(s)$, satisfy:*

- 1.$s_{lo}=\frac{1}{3}(\beta\theta_l-\theta b_l)$,
- 2.$r_{00}=\frac{2}{3}\theta\beta+\left[\theta+\frac{2}{3}(b^2\theta-\theta_lb^l)\right]\alpha^2+\frac{1}{3}(-14+12b^2)\theta\beta^2$,
- 3.$G^l_\alpha=\frac{1}{3}\left[2\theta+(3k_1-2)\theta\beta\right]y^l+\frac{1}{3}(\theta^l-\tau b^l)\alpha^2+\frac{1}{2}k_3\tau\beta^2b^l$,
- 4.$\tau\left[\frac{\left(-14\,s^4+14\,s^3\sqrt{s^2+1}-12\,s^2+17\,s\sqrt{s^2+1}+6\right)s^2}{(s^2+1)^3}\right]=0$,

where $\tau = \tau(x)$ *is a scalar function;* $\theta = \theta_i(x)y^i$ *is an 1-form on M,* $\theta^l = a^{lm}\theta_m$*,*

$$k_1 = \Pi(0); k_2 = \frac{\Pi'(0)}{Q(0)}; k_3 = \frac{1}{6Q(0)^2}\left[3Q''(0)\Pi'(0) - 6\Pi(0)^2 - Q(0)\Pi'''(0)\right], \tag{3.7}$$

and $Q = \frac{\phi'}{\phi - s\phi'}$; $\Pi = \frac{\phi'^2 + \phi\phi''}{\phi(\phi - s\phi')}$.

Proof. We use Theorem 1.2, where we compute for the metric (2.4), the following:

$$k_1 = \Pi(0); k_2 = \frac{\Pi'(0)}{Q(0)}; k_3 = \frac{1}{6Q(0)^2}\left[3Q''(0)\Pi'(0) - 6\Pi(0)^2 - Q(0)\Pi'''(0)\right],$$

and also:

$$Q(s) = -\left(\left(s^2+1\right)^{3/2} - s^3\right)^{-1}$$

$$\Pi(s) = \frac{1 + 2s^6 - 2s^5\sqrt{s^2+1} + s^4 + 3s^2 - 3b^2s^2 - s\sqrt{s^2+1} + 3sb^2\sqrt{s^2+1}}{\left(s^2\sqrt{s^2+1} + \sqrt{s^2+1} - s^3\right)^2}.$$

Finally we obtain:

$$k_1 = 1; k_2 = -4; k_3 = -4,$$

Replacing all of this in Theorem 1.2, finally we proved the above theorem. □

Next, using an important result from [30], we will compute the norm of the mean Cartan torsion for the new metric (2.4). First, let's recall this classical result:

Theorem 3.6. *([30]) Let* $F = \alpha\phi(s)$ *be a non-Riemann* (α, β)*-metric on a manifold M of dimension* $n \geq 3$*. Then the norm of Cartan and mean Cartan torsion of F, satisfy the following relation:*

$$\|C\| = \sqrt{\frac{3p^2 + 6pq + (n+1)q^2}{n+1}}\,\|I\|, \tag{3.8}$$

where $p = p(x,y)$*,* $q = q(x,y)$ *are scalar function on TM, satisfying* $p + q = 1$ *and given by the following:*

$$p = \frac{n+1}{a_1 A}\left[s(\phi\phi'' + \phi'\phi') - \phi\phi'\right] \tag{3.9}$$

$$a_1 = \phi\left\{\phi - s\phi'\right\} \tag{3.10}$$

$$A = (n-2)\frac{s\phi''}{\phi - s\phi'} - (n+1)\frac{\phi'}{\phi} - \frac{-3s\phi'' + (b^2 - s^2)\phi''}{\phi - s\phi' + (b^2 - s^2)\phi''}. \tag{3.11}$$

After tedious computations in Maple 13 of all above relations, we can formulate:

Theorem 3.7. *Let* $F = \alpha\left(1 - \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)$*, be the* (α, β)*-metric defined in (2.4) on the manifold M of dimension* $n \geq 3$*.*

$$\|C\| = \sqrt{\frac{3p^2 + 6pq + (n+1)q^2}{n+1}}\,\|I\|,$$

where $p = p(x,y)$*,* $q = q(x,y)$ *are scalar function on TM, satisfying* $p + q = 1$ *and given by*

$$a_1 = -\left(\left(s^2+1\right)^{3/2} - s^3\right)^{-1}; \tag{3.12}$$

$$A = \frac{\left(-6\,s^8 + (-8-8\,n)\,s^7 + \left(6\,b^2-9\right)s^6\right)\sqrt{s^2+1}}{\left(-s^5+\sqrt{s^2+1}(s^2+1)^2-4s^3+3sb^2\right)\left(-\sqrt{s^2+1}+s\right)\left(s^3-\sqrt{s^2+1}(s^2+1)\right)(s^2+1)} +$$
$$\frac{\left((2-25\,n)\,s^5 + \left(-3+9\,b^2\right)s^4 + \left((12\,n-15)\,b^2-8-8\,n\right)s^3 + 3\,b^2s^2 + 3\,b^2\,(n+1)\,s\right)\sqrt{s^2+1}}{\left(-s^5+\sqrt{s^2+1}(s^2+1)^2-4s^3+3sb^2\right)\left(-\sqrt{s^2+1}+s\right)\left(s^3-\sqrt{s^2+1}(s^2+1)\right)(s^2+1)} +$$
$$\frac{6\,s^9 + (8+8\,n)\,s^8 + \left(-6\,b^2+12\right)s^7 + (29\,n+2)\,s^6 + \left(9-12\,b^2\right)s^5}{\left(-s^5+\sqrt{s^2+1}(s^2+1)^2-4s^3+3sb^2\right)\left(-\sqrt{s^2+1}+s\right)\left(s^3-\sqrt{s^2+1}(s^2+1)\right)(s^2+1)} +$$
$$\frac{\left((15-12\,n)\,b^2+15+15n\right)s^4 + \left(3-9\,b^2\right)s^3 + (7\,n+7)\,s^2 - 3\,sb^2 + n + 1}{\left(-s^5+\sqrt{s^2+1}(s^2+1)^2-4s^3+3sb^2\right)\left(-\sqrt{s^2+1}+s\right)\left(s^3-\sqrt{s^2+1}(s^2+1)\right)(s^2+1)}; \tag{3.13}$$

$$p = \frac{n+1}{a_1A}\left(\frac{3\sqrt{s^2+1}s^2 - 4\,s^3 + \left(s^2+1\right)^{3/2}}{\left(s^2+1\right)^3}\right); \quad q = 1-p. \tag{3.14}$$

Lemma 3.1. *Let F be the (α,β)-metric given in (2.4). Then F is of non-Randers type if $\Phi \neq 0$*

Proof. We know from (1.4), that:

$$\Phi = -(Q(s)-sQ'(s))(n\Delta+1+sQ(s)) - (b^2-s^2)(1+sQ(s))Q''(s)$$

After tedious computations, and imposing the condition $\Phi(s)=0$, one obtains:

$$\alpha^2\Bigg[\left(\left(-3\,b^2+1+n\right)\sqrt{\beta^2+\alpha^2} + 24\left(-\frac{1}{24}+\frac{1}{8}\,nb^2-\frac{1}{24}\,n+\frac{3}{8}\,b^2\right)\beta\right)\alpha^6 +$$
$$\left(15\,\beta^3\left(nb^2-\frac{7}{5}\,b^2-n-\frac{7}{5}\right) - 3\left(nb^2-\frac{7}{3}n-\frac{10}{3}\right)\beta^2\sqrt{\beta^2+\alpha^2}\right)\alpha^4 +$$
$$\left(24\beta^5\left(-\frac{1}{2}-2n+nb^2-\frac{13}{4}\,b^2\right) - 24\left(nb^2-\frac{5}{6}-\frac{9}{4}\,b^2-\frac{23}{24}\,n\right)\beta^4\sqrt{\beta^2+\alpha^2}\right)\alpha^2 +$$
$$\left(-24+54\,n+48\,b^2\right)\beta^6\sqrt{\beta^2+\alpha^2} + 24\left(-\frac{31}{12}\,n+\frac{5}{3}-2\,b^2\right)\beta^7\Bigg] = 16\,\beta^8\left(\beta-\sqrt{\beta^2+\alpha^2}\right)(-2+n)$$

Finally, we observe that β^8 is not divisible with α^2 and from this we conclude that the metric (1) is not of Randers type because $\Phi \neq 0$. □

Theorem 3.8. *Let F be the (α,β)-metric given in (2.4) with the scalar flag curvature $K = K(x,y)$ over a Finsler space. Then, F is a weak Berwald metric if and only if F is a Berwald metric and $K=0$. Then, F must be locally Minkowskian.*

Proof. In the above Lemma, we have proved that the (α,β)-metric (2.4) can't be Riemannian. We will prove now the necessity of this theorem, because the sufficiency is obvious. We will asume that the metric F given in (2.4) is weak Berwald. By Lemma 1.1, we know that $S=(n+1)c(x)F$, with $c(x)=0$ and $r_{00}=0$; $s_{ij}=0$.
From [25], we know that for a Finsler metric F of constant curvature K, the following equality holds:

$$J_{i|m}y^m + KF^2I_i = 0$$

where J_i is given in (9).
In [25] the following is computed:

$$\overline{J} = J_i b^i = -\frac{1}{2\Delta\alpha^2}\left\{\Psi_1(r_{00} - 2\alpha Q s_0) + \alpha\Psi_2(r_0 + s_0)\right\}$$

where Ψ_1 and Ψ_2 are given as follows (see [8]):

$$\Psi_1 = \sqrt{b^2 - s^2}\Delta^{\frac{1}{2}}\left[\frac{\sqrt{b^2 - s^2}}{\Delta^{\frac{3}{2}}}\right]'; \;\; \Psi_2 = 2(n+1)(Q - sQ') + 3\frac{\Phi}{\Delta}.$$

If F is of constant flag curvature K, then we know from [30], the following:

$$J_{|m}y^m - J_l\frac{\partial\left(G^l - \overline{G}^l\right)}{\partial y^i} - 2\frac{\partial J_i}{\partial y^i}\left(G^l - \overline{G}^l\right) + K\alpha^2\phi^2 I_i = 0.$$

Contracting by b^i, for:

$$J_i = -\frac{\phi s_{i0}}{2\alpha\Delta}; \;\; \overline{J} = 0; \;\; G^i - \overline{G}^i = \alpha Q s_0^i,$$

$$I_i b^i = -\frac{\Phi(\phi - s\phi')}{2\Delta F}(b^2 - s^2),$$

one obtains:

$$\frac{\Phi s_{i0}}{2\Delta\alpha}a^{ik}s_{k0} + \frac{\Phi s_{l0}}{2\Delta\alpha}\left(sQs_0^l + Q's_0^l(b^2 - s^2)\right) - KF\frac{\Phi s_{i0}}{2\Delta}(\phi - s\phi')(b^2 - s^2) = 0$$

and from this, one obtains:

$$s_{i0}s_0^i\Delta - K\alpha^2\phi(\phi - s\phi')(b^2 - s^2) = 0. \tag{3.15}$$

Replacing in (1.4),

$$\Delta = \frac{\sqrt{\beta^2 + \alpha^2}\left(\left(2\beta^4 + 6\beta^2\alpha^2 - \alpha^4\left(3b^2 - 1\right)\right)\beta\sqrt{\beta^2 + \alpha^2} - \alpha^6 + 3\left(-1 + b^2\right)\beta^2\alpha^4 - 7\beta^4\alpha^2 - 2\beta^6\right)}{\left(-\sqrt{\beta^2 + \alpha^2} + \beta\right)^3\left(2\beta^2 + \alpha^2 + \beta\sqrt{\beta^2 + \alpha^2}\right)^2}$$

Also, when we compute $K(\phi'(s))\left(1 - \phi(s)\right)\phi'(s)(b^2 - s^2)\alpha^2$, one obtains:

$$K(\phi'(s))\left(1 - \phi(s)\right)\phi'(s)(b^2 - s^2)\alpha^2 =$$
$$\frac{K\left(\sqrt{\beta^2 + \alpha^2} - \beta\right)\left(\sqrt{\beta^2 + \alpha^2}\beta^2 + \sqrt{\beta^2 + \alpha^2}\alpha^2 - \beta^3\right)\left(b^2\alpha^2 - \beta^2\right)}{(\beta^2 + \alpha^2)^2}$$

If we multiply $s_{i0}s_0^i\Delta - K\alpha^2\phi(\phi - s\phi')(b^2 - s^2) = 0$, with

$$(\beta^2 + \alpha^2)^2\left(\beta - \sqrt{\alpha^2 + \beta^2}\right)^3\left(2\beta^2 + \alpha^2 + \beta\sqrt{\alpha^2 + \beta^2}\right)^2$$

and replacing, after computations, we get:

$$s_{i0}s_0^i\beta\Bigg[(\alpha^2 + \beta^2)^3\left(3\alpha^4 b^2 - 6\beta^2\alpha^2 - 2\beta^4 - \alpha^4\right) +$$

$$\left(\alpha^2+\beta^2\right)^{\frac{5}{2}}\left(2\beta^5+7\beta^3\alpha^2+3\alpha^4\beta(1-b^2)\right)\Bigg]$$

$$= K\alpha^2(\beta^2+\alpha^2)^2\left(\beta-\sqrt{\alpha^2+\beta^2}\right)^4\left(2\beta^2+\alpha^2+\beta\sqrt{\alpha^2+\beta^2}\right)^2. \qquad (3.16)$$

The right term of the above relation is divisible with α^2. Hence, we can get the flag curvature $K = 0$ because $a \neq 0$ and β is not divisible with α^2. Replacing $K = 0$ in (3.16), we get $s_{i0}s^i_0 = a_{ij}(x)s^j_0 s^i_0 = 0$. But $\left(a_{ij}(x)\right)$ is positive definite, so $s^i_0 = 0 \Rightarrow \beta$ is closed.
By $r_{00} = 0$ and $s_0 = 0$, we know that β is parallel with respect to α. Then, we conclude that the (α, β)-metric given by (2.4) is a Berwald metric and must be locally Minkowskian. □

4. Conclusions

In this paper we succeed to construct and to investigate from many points of view a new type of Finsler metric which can be obtained using the Laplace transform. The Laplace transform is very important not just in mathematics but also in physics because converts integral and differential equations into algebraic equations and this procedure has multiple applications in physics, for instance at the study of the study of the signals. For the new Finsler metric obtained in this paper with the use of Laplace transform for the Bessel function of the first kind $J_1(x)$, we have studied the mean Cartan torsion, the local duality, the S-curvature and also the variational problem for a Finsler space endowed with this new metric. Finally, we have proved that this new metric is not of Randers type, nor Riemann type and we proved that is a Berwald metric and so this metric is locally Minkowskian. In our future works we will try to extend this procedure of construction of such Finsler metrics and also we will try to investigate some new class of such metrics which arise from Laplce transform.

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