Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.9, (2020), Issue 1, pp.43-56

MOTION OF PARALLEL CURVES AND SURFACES IN EUCLIDEAN 3-SPACE $\mathbb{R}^3$

MARYAM T. ALDOSSARY* AND MASHNIAH A. GAZWANI**
DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE,
IMAM ABDULRAHMAN BIN FAISAL UNIVERSITY,
P. O. BOX 1982, CITY (DAMMAM), SAUDI ARABIA
*E-MAIL:MAALDOSARI@IAU.EDU.SA, **E-MAIL:MAGAZWANI@IAU.EDU.SA

ABSTRACT. The main goal of this paper is to investigate motion of parallel curves and surfaces in Euclidean 3-space $\mathbb{R}^3$. The characteristic properties for such objects are given. The geometric quantities are described. Finally, the evolution equations of the curvatures and the intrinsic geometric formulas are derived.

keywords: Curvature, Evolution, Motion, Parallel curves, Parallel surfaces.

1. INTRODUCTION AND MOTIVATIONS

Image processing and the evolution of curves and surfaces has significant applications in computer vision [20]. As a scale space by linear and nonlinear diffusion's are defined in [19, 21], image enhancement through an isotropic diffusion's were studied [17, 5, 23], and image segmentation by active contours are classified in [10, 8, 16]. The evolution of curves has been studied extensively in various homogeneous spaces. The relationship between integrable equations and the geometric motion of curves in spaces has been known for a long time. In fact, many integrable equations have been shown to describe the evolution invariant associated with certain movements of curves in particular geometric settings. The dynamics of shapes in physics, chemistry and biology are modelled in terms of motion of surfaces and interfaces, and some dynamics of shapes are reduced to motion of plane curves. Evolution of surfaces accompanies many physical phenomena: propagation of wave fronts are descried extensively in [18], and motion of interfaces, growth of crystals [9], geometric integral equations expressed in wide range (see [13, 22, 1]). Geometrically, curves and surfaces evolution means deforming a curve or a surface into another curve or a surface in a continuous way, respectively. For more recent treatment of curves and surfaces evolution, see [16-19].

2010 *Mathematics Subject Classification.* Primary 14Q05; Secondary 14Q10.
Key words and phrases. Curvature, Evolution, Motion, Parallel curves, Parallel surfaces.

2. Preliminary

In this section we present the main results related to the motion of curves and surfaces in Euclidean 3-space $\mathbb{R}^3$ [16, 23].

2.1. Motion of curves in $\mathbb{R}^3$ [15]. Let $\boldsymbol{\alpha}$ be a regular curve in Euclidean 3-space, where

$$\boldsymbol{\alpha} : [a,\ b] \subset \mathbb{R} \to \mathbb{R}^3.$$

Let $\boldsymbol{\alpha}(u,\ t)$ denote the position vector of a point on the curve at time t by

$$\widetilde{\boldsymbol{\alpha}}\,(\widetilde{u},\ t) = \boldsymbol{\alpha}(u) + W\,(u,t)\,\mathbf{T}\,(u,t) + U\,(u,t)\,\mathbf{n}\,(u,t) + V\,(u,t)\,\mathbf{B}\,(u,t)\,, \tag{2.1}$$

such that $\boldsymbol{\alpha}(u,0) = \boldsymbol{\alpha}(u)$, where a metric on the curve is given by

$$\mathbf{g}(u,\ t) = \left\langle \frac{\partial \boldsymbol{\alpha}}{\partial u}, \frac{\partial \boldsymbol{\alpha}}{\partial u} \right\rangle, \tag{2.2}$$

and the arc length along the curve is equal to

$$s(u,\ t) = \int_o^u \sqrt{\mathbf{g}(u,\ t)}\,\mathrm{d}u. \tag{2.3}$$

We may use either $\{u,\ t\}$ or $\{s,\ t\}$ as coordinates of a point on the curve. The Frenet-Serret frame $\{\mathbf{T},\ \mathbf{n},\ \mathbf{B}\}$ is defined in the usual way (i.e., $\mathbf{T} = \frac{\partial \boldsymbol{\alpha}}{\partial s} = \mathbf{g}^{-\frac{1}{2}} \frac{\partial \boldsymbol{\alpha}}{\partial u}$) which satisfies Frenet-Serret equations (2.13). Motion of a point on the curve can be specified by the form

$$\frac{\partial \boldsymbol{\alpha}}{\partial t} = W\,\mathbf{T} + U\,\mathbf{n} + V\,\mathbf{B}, \tag{2.4}$$

where $\{W,\ V,\ U\}$ are arbitrary functions represent the component of the velocity in direction of $\{\mathbf{T},\ \mathbf{n},\ \mathbf{B}\}$ respectively. The motion is local which means that $\{W,\ V,\ U\}$ depend only on local values of $\{\kappa,\ \tau\}$ and their derivatives according to arc length s. Now we will use the compatibility conditions along above, our proofs in this section is

$$\frac{\partial}{\partial t}\frac{\partial}{\partial u}\boldsymbol{\alpha}(u,\ t) = \frac{\partial}{\partial u}\frac{\partial}{\partial t}\boldsymbol{\alpha}(u,\ t). \tag{2.5}$$

If $\boldsymbol{\alpha}(u,t)$ evolves in $\mathbb{R}^3$ locally according to the equation (2.4) such that $\boldsymbol{\alpha}(u,0) = \boldsymbol{\alpha}(u)$, then the evolution equations of the Frenet-Serret frame $\{\mathbf{T},\ \mathbf{n},\ \mathbf{B}\}$ associated to the curve, are given by

$$\frac{\partial \mathbf{T}}{\partial t} = \left(\frac{\partial U}{\partial s} - \tau V + \kappa W\right)\mathbf{n} + \left(\frac{\partial V}{\partial s} + \tau U\right)\mathbf{B}. \tag{2.6}$$

$$\frac{\partial \mathbf{n}}{\partial t} = -\left(\frac{\partial U}{\partial s} - \tau V + \kappa W\right)\mathbf{T} + \left(\frac{1}{\kappa}\frac{\partial}{\partial s}(\frac{\partial V}{\partial s} + \tau U) + \frac{\tau}{\kappa}(\frac{\partial U}{\partial s} - \tau V + \kappa W)\right)\mathbf{B}. \tag{2.7}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\left(\frac{\partial V}{\partial s} + \tau U\right)\mathbf{T} - \left(\frac{1}{\kappa}\frac{\partial}{\partial s}(\frac{\partial V}{\partial s} + \tau U) + \frac{\tau}{\kappa}(\frac{\partial U}{\partial s} - \tau V + \kappa W)\right)\mathbf{n}. \tag{2.8}$$

The evolution equation for curvature κ of $\boldsymbol{\alpha}(u,\ t)$ is given by

$$\frac{\partial \kappa}{\partial t} = \frac{\partial}{\partial s}\left(\frac{\partial U}{\partial s} - \tau V + \kappa W\right) - \tau\left(\frac{\partial V}{\partial s} + \tau U\right). \tag{2.9}$$

The evolution equation for torsion τ of $\boldsymbol{\alpha}(u, t)$ is given by

$$\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial s}\left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial V}{\partial s} + \tau U\right) + \frac{\tau}{\kappa}\left(\frac{\partial U}{\partial s} - \tau V + \kappa W\right)\right] + \kappa\left(\frac{\partial V}{\partial s} + \tau U\right). \tag{2.10}$$

The evolution equation for metric g on $\boldsymbol{\alpha}(u, t)$ is given by

$$\frac{\partial g}{\partial t} = 2g\left(\frac{\partial W}{\partial s} - \kappa U\right). \tag{2.11}$$

The evolution equation for the arc length s along the curve $\boldsymbol{\alpha}(u, t)$ is given by

$$\frac{\partial s}{\partial t} = \int_0^u \sqrt{g}\left(\frac{\partial W}{\partial s} - \kappa U\right) \mathrm{d}u. \tag{2.12}$$

Proposition 2.1. *The skew-symmetry of Frenet-Serret equations of $\boldsymbol{\alpha}(s)$ is given in matrix form by*

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{n}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{B} \end{bmatrix}. \tag{2.13}$$

2.2. **Motion of Surfaces in $\mathbb{R}^3$.** Let $\mathbf{M}$ be a surface; evolving is space according to

$$\widetilde{\mathbf{X}}\left(u^i, t\right) = \mathbf{X}\left(u^i\right) + \varphi^k\left(u^i, t\right)\mathbf{X}_k\left(u^i, t\right) + \psi\left(u^i, t\right)\mathbf{N}\left(u^i, t\right), \; i = 1, 2 \tag{2.14}$$

such that $\widetilde{\mathbf{X}}\left(u^i, 0\right) = \widetilde{\mathbf{X}}\left(u^i\right)$ and its motion of is described by

$$\frac{\partial \mathbf{X}}{\partial t} = V^i \mathbf{X}_i + U\mathbf{N},$$

where $\mathbf{X}$ is a patch of $\mathbf{M}$, V^i and U are the velocity components in the tangents vectors $\mathbf{X}_i$ and normal $\mathbf{N}$ direction, respectively. Then the evolution equations of the frame $\{\mathbf{X}_i, \mathbf{N}\}$ associated to the surface are given by [14]

$$\frac{\partial}{\partial t}\begin{bmatrix} \mathbf{X}_i \\ \mathbf{N} \end{bmatrix} = \begin{bmatrix} \bigtriangledown_i V^k - L_i^k U & (\frac{\partial U}{\partial u^i} + V^j L_{ji}) \\ -g^{ki}(\frac{\partial U}{\partial u^i} + V^j L_{ji}) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}_k \\ \mathbf{N} \end{bmatrix}, \tag{2.15}$$

where u^i are its local coordinates. Moreover, g_{ij} and L_{ij} are the metric and curvature tensors, respectively. Also, the evolution equations of the metric tensor and its inverse with determinant, from [14] and [7], respectively are given by

$$\frac{\partial g_{ij}}{\partial t} = \bigtriangledown_i V_j + \bigtriangledown_j V_i - 2UL_{ij}, \tag{2.16}$$

$$\frac{\partial g^{lk}}{\partial t} = 2L^{lk}U - g^{il} \bigtriangledown_i V^k - g^{lk} \bigtriangledown_l V^l, \tag{2.17}$$

$$\frac{\partial g}{\partial t} = 2g\left(\bigtriangledown_k V^k - 2HU\right). \tag{2.18}$$

It can also express the evolution equations of the curvature tensor, its inverse and determinant which follows from [14].

$$\frac{\partial L_{ij}}{\partial t} = \bigtriangledown_j \bigtriangledown_i U - UL_j^k L_{ik} + L_{ik} \bigtriangledown_j V^k + L_{jk} \bigtriangledown_i V^k + V^k \bigtriangledown_j L_{ik}. \tag{2.19}$$

$$\frac{\partial L^{jk}}{\partial t} = -(\bigtriangledown_j \bigtriangledown_i U - UL^k_j L_{ik} + L_{ik} \bigtriangledown_j V^k + L_{jk} \bigtriangledown_i V^k \quad (2.20)$$

$$+ V^k \bigtriangledown_j L_{ik}) L^{jk} L^{ij}.[7] \quad (2.21)$$

$$\frac{\partial L}{\partial t} = L \left[L^{ij} \left(V^k \bigtriangledown_j L_{ik} + \bigtriangledown_j \bigtriangledown_i U \right) + 2 \bigtriangledown_k V^k - 2HU \right]. \quad (2.22)$$

Finally, the evolution equations for the Gaussian Curvature K and mean Curvature H are given by

$$\frac{\partial K}{\partial t} = K \left[L^{ij} \left(\bigtriangledown_j \bigtriangledown_i U + V^k \bigtriangledown_j L_{ik} \right) + 2HU \right]. \quad (2.23)$$

$$\frac{\partial H}{\partial t} = \frac{1}{2} \left[2L_{ij} L^{ij} U + g^{ij} \left(\bigtriangledown_j \bigtriangledown_i U + V^k \bigtriangledown_j L_{ik} \right) - L^j_k L^k_j U \right]. \quad (2.24)$$

3. Parallel curves and surfaces

In this section, we explore the geometry of parallel curves, surfaces and curves lying on parallel surfaces. We investigate the properties of the geometry of such objects in $\mathbb{R}^3$. We work to derive the associated moving frame to the parallel object and present the frame found in terms of the original frame, associated to the original curve or surface. All local invariants of the parallel object will be derived.

3.1. **Parallel curves.**

Definition 3.1. *Let $\boldsymbol{\alpha}(s)$ be a unit speed curve, we construct a parallel curve $\overline{\boldsymbol{\alpha}}(\bar{s})$ that parametrized by arc length $\bar{s}$ in Euclidean 3-space $\mathbb{R}^3$ as follow*

$$\overline{\boldsymbol{\alpha}}(\bar{s}) = \boldsymbol{\alpha}(s) + r\mathbf{B}(s), \quad (3.1)$$

where $r \neq 0$ is a real constant, $s = s(\bar{s})$ is the arc length of $\boldsymbol{\alpha}$ and $\mathbf{B}$ is the binormal vector to the curve $\boldsymbol{\alpha}(s)$. We will study the associated geometry of parallel curves.

Lemma 3.1. *Let $\overline{\boldsymbol{\alpha}}(\bar{s})$ be a parallel curve to a unit speed curve $\boldsymbol{\alpha}(s)$. Then the associated Frenet-Serret frame $\{\overline{\mathbf{T}}, \overline{\mathbf{n}}, \overline{\mathbf{B}}\}$ to $\overline{\boldsymbol{\alpha}}$ in terms of the frame $\{\mathbf{T}, \mathbf{n}, \mathbf{B}\}$ of the original curve $\boldsymbol{\alpha}$ as follow*

$$\overline{\mathbf{T}} = \omega\mathbf{T} - r\omega\tau\mathbf{n} \quad (3.2)$$

$$\overline{\mathbf{n}} = \left(\frac{\omega\omega' + r\kappa\tau\omega^2}{\Omega} \right) \mathbf{T} + \left(\frac{\omega^2\kappa - (r\tau\omega)'\omega}{\Omega} \right) \mathbf{n} - \frac{r\tau^2\omega^2}{\Omega} \mathbf{B} \quad (3.3)$$

$$\overline{\mathbf{B}} = (\frac{r^2\tau^3\omega^3}{\Omega})\mathbf{T} + (\frac{r\tau^2\omega^3}{\Omega})\mathbf{n} + \left(\frac{\omega^3\kappa - (r\tau\omega)'\omega^2 + r\tau\omega^2\omega' + r^2\kappa\tau^2\omega^3}{\Omega} \right) \mathbf{B}, \quad (3.4)$$

where

$$\frac{\mathrm{d}}{\mathrm{d}s} = {}' \qquad ,\omega = \frac{1}{\sqrt{1 + r^2\tau^2}},$$

and

$$\Omega = \sqrt{(\omega\omega' + r\kappa\tau\omega^2)^2 + (\omega^2\kappa - (r\tau\omega)'\omega)^2 + (r\tau^2\omega^2)^2}.$$

Proof. Differentiating (3.1) by arc length $\bar{s}$ we get

$$\overline{\mathbf{T}} = \frac{\mathrm{d}\bar{\boldsymbol{\alpha}}}{\mathrm{d}\bar{s}} = \left(\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}s} + r\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\right)\frac{\mathrm{d}s}{\mathrm{d}\bar{s}}, \tag{3.5}$$

using (2.13) into (3.5) yields to

$$\overline{\mathbf{T}} = (\mathbf{T} - r\tau\mathbf{n})\frac{\mathrm{d}s}{\mathrm{d}\bar{s}}. \tag{3.6}$$

Since

$$\left|\overline{\mathbf{T}}\right| = |\mathbf{T} - r\tau\mathbf{n}|\frac{\mathrm{d}s}{\mathrm{d}\bar{s}},$$

assuming

$$\frac{\mathrm{d}s}{\mathrm{d}\bar{s}} = \frac{1}{\sqrt{1 + r^2\tau^2}} = \omega.$$

Therefore,

$$\overline{\mathbf{T}} = \omega\mathbf{T} - r\tau\omega\mathbf{n}.$$

Also, since

$$\bar{\mathbf{n}} = \frac{\mathrm{d}\overline{\mathbf{T}}/\mathrm{d}\bar{s}}{\left|\mathrm{d}\overline{\mathbf{T}}/\mathrm{d}\bar{s}\right|},$$

then

$$\begin{aligned}\frac{\mathrm{d}\overline{\mathbf{T}}}{\mathrm{d}\bar{s}} &= \frac{\mathrm{d}\overline{\mathbf{T}}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\bar{s}} \\ &= \left(\omega\omega' + r\kappa\tau\omega^2\right)\mathbf{T} + \left(\omega^2\kappa - (r\tau\omega)'\omega\right)\mathbf{n} - r\tau^2\omega^2\mathbf{B}.\end{aligned}$$

Taking

$$\left|\frac{\mathrm{d}\overline{\mathbf{T}}}{\mathrm{d}\bar{s}}\right| = \sqrt{\left(\omega\omega' + r\kappa\tau\omega^2\right)^2 + \left(\omega^2\kappa - (r\tau\omega)'\omega\right)^2 + (r\tau^2\omega^2)^2}.$$

Hence

$$\bar{\mathbf{n}} = \left(\frac{\omega\omega' + r\kappa\tau\omega^2}{\Omega}\right)\mathbf{T} + \left(\frac{\omega^2\kappa - (r\tau\omega)'\omega}{\Omega}\right)\mathbf{n} - \frac{r\tau^2\omega^2}{\Omega}\mathbf{B},$$

where

$$\Omega = \sqrt{\left(\omega\omega' + r\kappa\tau\omega^2\right)^2 + \left(\omega^2\kappa - (r\tau\omega)'\omega\right)^2 + (r\tau^2\omega^2)^2}.$$

Finaly, since

$$\overline{\mathbf{B}} = \overline{\mathbf{T}} \times \bar{\mathbf{n}},$$

then

$$\overline{\mathbf{B}} = (\frac{r^2\tau^3\omega^3}{\Omega})\mathbf{T} + (\frac{r\tau^2\omega^3}{\Omega})\mathbf{n} + \left(\frac{\omega^3\kappa - (r\tau\omega)'\omega^2 + r\tau\omega^2\omega' + r^2\kappa\tau^2\omega^3}{\Omega}\right)\mathbf{B},$$

which completes the proof. ∎

Theorem 3.1. *Let $\bar{\boldsymbol{\alpha}}(\bar{s})$ be a parallel curve to a unit speed curve $\boldsymbol{\alpha}(s)$. Then the metric $\bar{\mathbf{g}}$ of curve $\bar{\boldsymbol{\alpha}}(\bar{s})$ is given by*

$$\bar{\mathbf{g}} = \mathbf{g}\left(1 + r^2\tau^2\right). \tag{3.7}$$

Proof. As we know that

$$\frac{ds}{d\bar{s}} = \frac{1}{\sqrt{1+r^2\tau^2}},$$

Then, we have

$$d\bar{s} = ds\sqrt{1+r^2\tau^2}.$$

Therefore,

$$\bar{\mathbf{g}} = \mathbf{g}\left(1+r^2\tau^2\right),$$

which ends the proof. ∎

3.2. **Parallel Surfaces.**

Definition 3.2. *Let* $\mathbf{M}$ *be an orientable surface and let* $\mathbf{N}$ *be a unit normal vector field of* $\mathbf{M}$*. A surface* $\overline{\mathbf{M}}$ *is said to be parallel to* $\mathbf{M}$ *if there is a normal geodesic congruence between* $\mathbf{M}$ *and* $\overline{\mathbf{M}}$ *such that the distance between corresponding points is constant, i.e. for each* $\mathbf{X} \in \mathbf{M}$ *we have*

$$\overline{\mathbf{M}} : \overline{\mathbf{X}}(u,v) = \mathbf{X}(u,v) + r\mathbf{N}(u,v), \tag{3.8}$$

where, $r \neq 0$ is a real constant. We can say that $\mathbf{M}$ and $\overline{\mathbf{M}}$ are parallel surfaces at distance r. The relation between the Gaussian and mean curvatures $K,\ H$ and $\overline{K},\ \overline{H}$ of $\mathbf{M}$ and $\overline{\mathbf{M}}$, respectively, are given by [9]

$$\overline{K} = \frac{K}{\mu}, \tag{3.9}$$

$$\overline{H} = \frac{H - rK}{\mu}, \tag{3.10}$$

where

$$\mu = 1 - 2rH + r^2K \neq 0.$$

Also if $\kappa_1,\ \kappa_2$ and $\overline{\kappa_1},\ \overline{\kappa_2}$ denote the principal curvatures of $\mathbf{M}$ and $\overline{\mathbf{M}}$, respectively, then we have [9]

$$\overline{\kappa_1} = \frac{\kappa_1}{1+r\kappa_1},\ \ \overline{\kappa_2} = \frac{\kappa_2}{1+r\kappa_2}.$$

By direct calculations we find the frame $\left\{\overline{\mathbf{X}}_i,\ \overline{\mathbf{N}}\right\}$ associated to the surface $\overline{\mathbf{M}}$ expressed in terms of the original frame $\{\mathbf{X}_i,\ \mathbf{N}\}$ to $\mathbf{M}$

$$\begin{bmatrix} \overline{\mathbf{X}}_i \\ \overline{\mathbf{N}} \end{bmatrix} = \begin{bmatrix} a^{ij} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_j \\ \mathbf{N} \end{bmatrix}, \tag{3.11}$$

where,

$$a^{ij} = \delta^i_j - rg^{jk}L_{ki}.$$

Lemma 3.2. *Let* $\overline{\mathbf{M}}$ *be a parallel surface to* $\mathbf{M}$ *in* $\mathbb{R}^3$*. Then*

i) The metric $\overline{g}_{lk}$ *on* $\overline{\mathbf{M}}$ *is related to metric* g_{mj} *on* $\mathbf{M}$ *by*

$$\overline{g}_{lk} = a^{lj}a^{km}g_{mj}, \tag{3.12}$$

and the determinant $\overline{g}$ *of metric* $\overline{g}_{ij}$ *on* $\overline{\mathbf{M}}$ *is given by*

$$\overline{g} = det(\overline{g}_{ij}) = \overline{g}_{ij}\,\overline{G}(i,j), \tag{3.13}$$

where $\overline{g}\,\overline{g}^{ij} = \overline{G}(i,j)$.

ii) The curvature tensor $\overline{L}_{ij}$ on $\overline{\mathbf{M}}$ is related to the curvature tensor L_{kj} on $\mathbf{M}$ by

$$\overline{L}_{ij} = a^{ik} L_{kj}, \tag{3.14}$$

and the determinant $\overline{L}$ of The curvature tensor $\overline{L_{ij}}$ on $\overline{\mathbf{M}}$ are given by

$$\overline{L} = det(\overline{L}_{ij}) = \overline{L}_{ij}\, \overline{L}(i,j), \tag{3.15}$$

where $\overline{L}\, \overline{L}^{ij} = \overline{L}(i,j)$.

Proof. Since

$$\overline{g}_{lk} = \left\langle \overline{X}_l, \overline{X}_k \right\rangle,$$

using (3.11) we get

$$\overline{g}_{lk} = a^{lj} a^{km} g_{mj}.$$

By definition of parallel surfaces we conclude that $\overline{N} = N$,

since

$$\overline{X}_{ij} = a^{ik}_{,j}\, X_k + a^{ik} X_{kj},$$

$$\overline{L}_{ij} = \left\langle \overline{X}_{ij}, \overline{N} \right\rangle,$$

by using $\langle X_i, N \rangle = 0$, then we conclude that

$$\overline{L}_{ij} = a^{ik} L_{kj},$$

which completes the proof. ■

4. Motion of parallel curves and surfaces in $\mathbb{R}^3$

In this section, we study the motion of parallel curves and surfaces in $\mathbb{R}^3$, such as the evolution equations of the frame associated, curvatures, metric, components of the first and second fundamental forms for the general parallel curves and surface. We depend on integrability conditions in all our calculations.

4.1. **Motion of parallel curves.**

Theorem 4.1. *The evolution equations for Frenet-Serret frame $\left\{\overline{\mathbf{T}},\ \overline{\mathbf{n}},\ \overline{\mathbf{B}}\right\}$ associated to the parallel curve $\overline{\boldsymbol{\alpha}}(\overline{s})$ are given by*

$$\begin{aligned}
\frac{\partial \overline{\mathbf{T}}}{\partial t} &= \Big[\frac{\partial \omega}{\partial t} + (r\omega\tau)\big(\frac{\partial U}{\partial s} - \tau V + \kappa W\big)\Big]\mathbf{T} \\
&\quad + \Big[\omega\big(\frac{\partial U}{\partial s} - \tau V + \kappa W\big) - \frac{\partial (r\omega\tau)}{\partial t}\Big]\mathbf{n} \\
&\quad + \Big[\omega\big(\frac{\partial V}{\partial s} + \tau U\big) - (r\omega\tau)\Big(\frac{1}{\kappa}\frac{\partial}{\partial s}\big(\frac{\partial V}{\partial s} + \tau U\big) + \frac{\tau}{\kappa}\big(\frac{\partial U}{\partial s} - \tau V + \kappa W\big)\Big)\Big]\mathbf{B}.
\end{aligned}$$

$$\begin{aligned}\frac{\partial \overline{\mathbf{n}}}{\partial t} &= \Big[\frac{\partial}{\partial t}(\frac{\omega\omega' + r\kappa\tau\omega^2}{\Omega}) - (\frac{\omega^2\kappa - (r\tau\omega)'\omega}{\Omega})(\frac{\partial U}{\partial s} - \tau V + \kappa W) \\ &\quad + \frac{r\tau^2\omega^2}{\Omega}(\frac{\partial V}{\partial s} + \tau U)\Big]\mathbf{T} \\ &\quad + \Big[\frac{(\omega\omega' + r\kappa\tau\omega^2)}{\Omega}(\frac{\partial U}{\partial s} - \tau V + \kappa W) + \frac{\partial}{\partial t}(\frac{\omega^2\kappa - (r\tau\omega)'\omega}{\Omega}) \\ &\quad + \frac{r\tau^2\omega^2}{\Omega}(\frac{1}{\kappa}\frac{\partial}{\partial s}(\frac{\partial V}{\partial s} + \tau U) + \frac{\tau}{\kappa}(\frac{\partial U}{\partial s} - \tau V + \kappa W))\Big]\mathbf{n} \\ &\quad + \Big[-\frac{\partial}{\partial t}(\frac{r\tau^2\omega^2}{\Omega}) \\ &\quad + \frac{(\omega\omega' + r\kappa\tau\omega^2)}{\Omega}(\frac{\partial V}{\partial s} + \tau U) + \frac{(\omega^2\kappa - (r\tau\omega)'\omega)}{\Omega}(\frac{1}{\kappa}\frac{\partial}{\partial s}(\frac{\partial V}{\partial s} + \tau U) \\ &\quad + \frac{\tau}{\kappa}(\frac{\partial U}{\partial s} - \tau V + \kappa W))\Big]\mathbf{B}.\end{aligned}$$

$$\begin{aligned}\frac{\partial \overline{\mathbf{B}}}{\partial t} &= \Big[\frac{\partial}{\partial t}(\frac{r^2\tau^3\omega^3}{\Omega}) - (\frac{r\tau^2\omega^3}{\Omega})(\frac{\partial U}{\partial s} - \tau V + \kappa W) \\ &\quad - (\frac{\omega^3\kappa - (r\tau\omega)'\omega^2 + r\tau\omega^2\omega' + r^2\kappa\tau^2\omega^3}{\Omega})(\frac{\partial V}{\partial s} + \tau U)\Big]\mathbf{T} \\ &\quad + \Big[(\frac{r^2\tau^3\omega^3}{\Omega})(\frac{\partial U}{\partial s} - \tau V + \kappa W) + \frac{\partial}{\partial t}(\frac{r\tau^2\omega^3}{\Omega}) \\ &\quad - (\frac{\omega^3\kappa - (r\tau\omega)'\omega^2 + r\tau\omega^2\omega' + r^2\kappa\tau^2\omega^3}{\Omega})(\frac{1}{\kappa}\frac{\partial}{\partial s}(\frac{\partial V}{\partial s} + \tau U) \\ &\quad + \frac{\tau}{\kappa}(\frac{\partial U}{\partial s} - \tau V + \kappa W))\Big]\mathbf{n} \\ &\quad + \Big[(\frac{r^2\tau^3\omega^3}{\Omega})(\frac{\partial V}{\partial s} + \tau U) + (\frac{r\tau^2\omega^3}{\Omega})(\frac{1}{\kappa}\frac{\partial}{\partial s}(\frac{\partial V}{\partial s} + \tau U) \\ &\quad + \frac{\tau}{\kappa}(\frac{\partial U}{\partial s} - \tau V + \kappa W)) + \frac{\partial}{\partial t}(\frac{\omega^3\kappa - (r\tau\omega)'\omega^2 + r^2\tau\omega^2\omega' + r\kappa\tau^2\omega^3}{\Omega})\Big]\mathbf{B}.\end{aligned}$$

Proof. Differentiating (3.2)-(3.4) w.r.t parameter t, respectively, and using (2.6)-(2.8) completes the proof. ■

Corollary 4.1. *The evolution equation of curvature to the parallel curve $\overline{\boldsymbol{\alpha}}(\overline{s})$ is given by*

$$\frac{\partial \overline{\kappa}}{\partial t} = \frac{\frac{\partial}{\partial t}\left(\sqrt{(\omega\omega' + r\kappa\tau\omega^2)^2 + (\omega^2\kappa - (r\tau\omega)'\omega)^2 + (r\tau^2\omega^2)^2}\right)}{2\sqrt{(\omega\omega' + r\kappa\tau\omega^2)^2 + (\omega^2\kappa - (r\tau\omega)'\omega)^2 + (r\tau^2\omega^2)^2}}.$$

Corollary 4.2. *The evolution equation of torsion to the parallel curve $\overline{\boldsymbol{\alpha}}(\overline{s})$ is given by*

$$
\begin{aligned}
\frac{\partial\overline{\tau}}{\partial t} = \frac{\partial}{\partial t}\Big(&- \frac{(\omega\omega' + r\kappa\tau\omega^2)}{\Omega}\big[(\frac{-r\kappa\tau^2\omega^3}{\Omega}) + (\frac{r^2\tau^3\omega^3}{\Omega})'\big] \\
&- \frac{(\omega^2\kappa - (r\tau\omega)'\omega)}{\Omega}\big[(\frac{r\tau^2\omega^3}{\Omega})' + (\frac{r^2\kappa\tau^3\omega^3}{\Omega}) \\
&- \frac{\tau}{\Omega}(\omega^3\kappa - (r\tau\omega)'\omega^2 + r\tau\omega^2\omega' + r^2\kappa\tau^2\omega^3)\big] \\
&+ \frac{r\tau^2\omega^2}{\Omega}\big[(\frac{\omega^3\kappa - (r\tau\omega)'\omega^2 + r\tau\omega^2\omega' + r^2\kappa\tau^2\omega^3}{\Omega})' + \frac{r\tau^3\omega^3}{\Omega}\big]\Big).
\end{aligned}
$$

Theorem 4.2. *The evolution equation of metric $\overline{\mathbf{g}}$ of the parallel curve $\overline{\boldsymbol{\alpha}}(\overline{s})$ is given by*

$$\frac{\partial\overline{\mathbf{g}}}{\partial t} = \mathbf{g}\left(2(\frac{\partial W}{\partial s} - \kappa U)(1 + r^2\tau^2) + \frac{\partial}{\partial t}(1 + r^2\tau^2)\right).$$

Proof. Differentiating (3.7) w.r.t parameter t, and using (2.11) completes the proof. ■

4.2. **Motion of parallel surfaces.**

Theorem 4.3. *The evolution equations for the associated frame $\{\overline{\mathbf{X}}_i,\ \overline{\mathbf{N}}\}$ on $\overline{\mathbf{M}}$ are given by*

$$\frac{\partial}{\partial t}\begin{bmatrix} \overline{\mathbf{X}}_i \\ \overline{\mathbf{N}} \end{bmatrix} = \begin{bmatrix} \frac{\partial a^{ik}}{\partial t} + a^{ij}(\bigtriangledown_j V^k - L_j^k U) & a^{ij}(\frac{\partial U}{\partial u^i} + V^j L_{ji}) \\ -g^{ki}(\frac{\partial U}{\partial u^i} + V^j L_{ji}) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}_k \\ \mathbf{N} \end{bmatrix}. \tag{4.1}$$

Proof. From (3.11) we have

$$\overline{\mathbf{X}}_i = a^{ij}\mathbf{X}_j. \tag{4.2}$$

Differentiating (4.2) w.r.t parameter t, we get

$$\frac{\partial\overline{\mathbf{X}}_i}{\partial t} = \frac{\partial a^{ij}}{\partial t}\mathbf{X}_j + a^{ij}\mathbf{X}_j,$$

then substituting (2.15) we derive the evolution equations of tangent of $\overline{\mathbf{X}}_i$ we complete prove $\frac{\partial\overline{\mathbf{X}}_i}{\partial t}$.
Since $\overline{\mathbf{N}} = \mathbf{N}$ by definition this yields to

$$\frac{\partial\overline{\mathbf{N}}}{\partial t} = \frac{\partial\mathbf{N}}{\partial t},$$

which completes the proof. ■

Theorem 4.4. *The evolution equations for the metric tensor $\overline{g_{lk}}$ on $\overline{\mathbf{M}}$ are given by*

$$\frac{\partial\overline{g}_{lk}}{\partial t} = \frac{\partial}{\partial t}\left(a^{lj}a^{km}\right) g_{mj} + a^{lj}a^{km}\left(\bigtriangledown_j V_m + \bigtriangledown_m V_j - 2L_{mj}U\right). \tag{4.3}$$

Proof. Differentiating (3.12) w.r.t parameter t, and using (2.16) completes the proof. ■

Theorem 4.5. *The evolution equations for the curvature tensor $\overline{L}_{ij}$ on $\overline{\mathbf{M}}$ are given by*

$$\frac{\partial \overline{L}_{ij}}{\partial t} = \frac{\partial a^{ik}}{\partial t} L_{kj} + a^{ik} \left[\bigtriangledown_j \bigtriangledown_k U - L_{km} L^m_j U + L_{km} \bigtriangledown_j V^m + L_{jm} \bigtriangledown_k V^m + V^m \bigtriangledown_j L_{km} \right]. \quad (4.4)$$

Proof. Differentiating (3.14) w.r.t parameter t, and substituting (2.19) we complete the proof. ■

Lemma 4.1. *The evolution equations for the Gaussian curvature $\overline{K}$ on $\overline{\mathbf{M}}$ are given by*

$$\frac{\partial \overline{K}}{\partial t} = \frac{K}{\mu^2} [\, (L^{ij}(\bigtriangledown_j \bigtriangledown_i U + V^k \bigtriangledown_j L_{ik}) + 2HU)(1 - 2rH) + r(2L^{ij}U + g^{ki} \left(\bigtriangledown_i \bigtriangledown_j U + V^k \bigtriangledown_i L_{kj} \right) - L^j_k L^k_j U) \,].$$

Proof. Differentiating (3.9) w.r.t parameter t, and substituting (2.23) and (2.24) we complete the above equation, we get the results. ■

Theorem 4.6. *The evolution equations for the mean curvature $\overline{H}$ on $\overline{\mathbf{M}}$ are given by*

$$\frac{\partial \overline{H}}{\partial t} = \frac{1}{\mu^2} [\frac{(1 - r^2 K)}{2} (2L^{ij} L_{ij} U + g^{ij} (\bigtriangledown_j \bigtriangledown_i U + V^k \bigtriangledown_j L_{ik}) - L^j_k L^k_j U) - r \, K(1 - rH) \big(L^{ij} (\bigtriangledown_j \bigtriangledown_i U + V^k \bigtriangledown_j L_{ik}) + 2HU \big)].$$

Proof. Differentiating (3.10) w.r.t parameter t, we get

$$\begin{aligned} \frac{\partial \overline{H}}{\partial t} &= \frac{1}{\mu^2} \left[\mu \left(\frac{\partial H}{\partial t} - r \frac{\partial K}{\partial t} \right) - (H - rK) \left(-2r \frac{\partial H}{\partial t} + r^2 \frac{\partial K}{\partial t} \right) \right], \\ &= \frac{1}{\mu^2} \left[\left(\mu + 2Hr - 2r^2 K \right) \frac{\partial H}{\partial t} - r \left(\mu + rH - r^2 K \right) \frac{\partial K}{\partial t} \right], \end{aligned}$$

substituting (2.23) and (2.24) we complete the proof. ■

Corollary 4.3. *The evolution equations for the determinant $\overline{g}$ of metric tensor on $\overline{\mathbf{M}}$ are given by*

$$\frac{\partial \overline{g}}{\partial t} = \overline{G}(i,j) \left[\frac{\partial a}{\partial t} g_{mj} + a^{lj} a^{km} \left(\bigtriangledown_j V_m + \bigtriangledown_m V_j - 2L_{mj} U \right) \right] \quad (4.5)$$

Proof. Differentiating (3.13) w.r.t parameter t, we get

$$\frac{\partial \overline{g}}{\partial t} = \frac{\partial \, \overline{g}}{\partial \, \overline{g_{ij}}} \frac{\partial \, \overline{g_{ij}}}{\partial t} = \overline{G}(i,j) \frac{\partial \, \overline{g_{ij}}}{\partial t},$$

substituting (4.3) we complete the proof. ■

Corollary 4.4. *The evolution equations for the determinant $\overline{L}$ of curvature tensor on $\overline{\mathbf{M}}$ are given by*

$$\frac{\partial \overline{L}}{\partial t} = \overline{L}(i,j) [\frac{\partial a^{ik}}{\partial t} L_{kj} + a^{ik} \left(\bigtriangledown_j \bigtriangledown_k U - L_{km} L^m_j U + L_{km} \bigtriangledown_j V^m + L_{km} \bigtriangledown_k V^m + V^m \bigtriangledown_j L_{im} \right)].$$

Proof. Differentiating (3.15) w.r.t parameter t, we get

$$\frac{\partial \overline{L}}{\partial t} = \frac{\partial \overline{L}}{\partial \overline{L}_{ij}} \frac{\partial \overline{L_{ij}}}{\partial t} = \overline{L}(i,j) \frac{\partial \overline{L}_{ij}}{\partial t},$$

substituting (4.4) we complete the proof. ∎

5. Applications

Here we show some examples to show the geometric meaning of the evolution equations for the geometric quantities.

5.1. **Application.** Let $\boldsymbol{\alpha}(s)$ be a helix curve evolving in $\mathbb{R}^3$ by (2.1), where

$$\kappa = \frac{r}{r^2 + c^2}, \ \tau = \frac{c}{r^2 + c^2}.$$

We now graph the evolving a curve by time in different situations to study the evolving model. For graphing requirements we take $r = 1, \ c = 1$ and $t = \{0, \ 1, \ 2, \ 3, 4\}$.

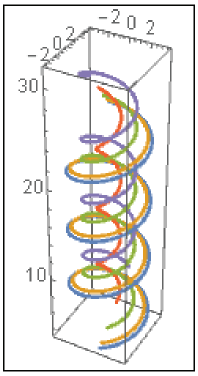

Figure 1. $(W = \tau t^2, \ U = 0, \ V = 0)$.

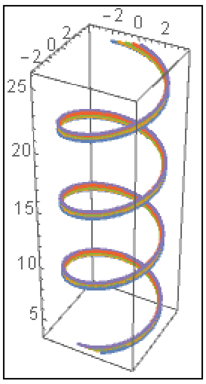

Figure 2. $(W = 0, \ U = \kappa t, \ V = 0)$.

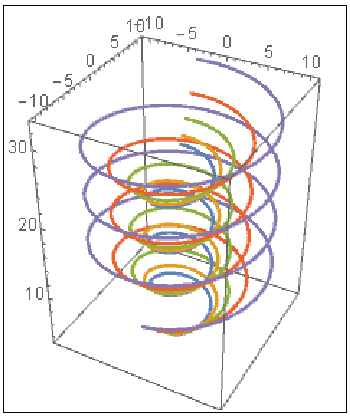

Figure 3. $(W = 0,\ U = 0,\ V = \kappa(t+1)t)$.

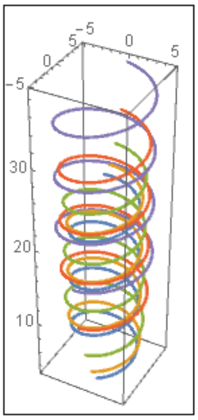

Figure 4. $(W = \tau t^2,\ U = \kappa t,\ V = \kappa(t+1)t)$.

From figure 1, we see that the evolving curve in the tangent direction of the original curve compress the parallel helix and extending its radius at the same time, while the evolution in the normal direction causes rapid increase of the radius of the helix in figure 2. If the helix evolves in the binormal direction we see a faster increment of the radius of the helix in parallel with an obvious shrinking of its length in figure 3. Finally, we sum all affects as we can see in figure 4.

5.2. **Application.** Let $\overline{\mathbf{X}}\left(u^1, u^2\right)$ be a unit sphere evolving in $\mathbb{R}^3$ by (2.14). We now graph the evolving sphere by time in different situations to study the evolving model with $r = 3, t = \{0,\ 1,\ 2,\ 3,\ 4\}$.

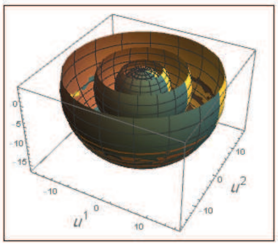

Figure 5. $(\phi^1 = t^2K,\ \phi^2 = 0,\ \psi = 0)$.

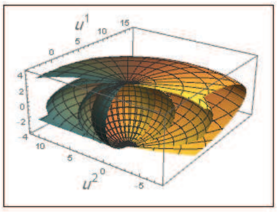

Figure 6. $(\phi^1 = 0,\ \phi^2 = t^2H,\ \psi = 0)$.

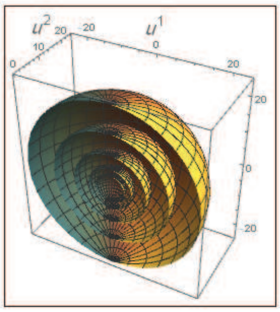

Figure 7. $(\phi^1 = \phi^2 = 0, \psi = t(t+1)K)$.

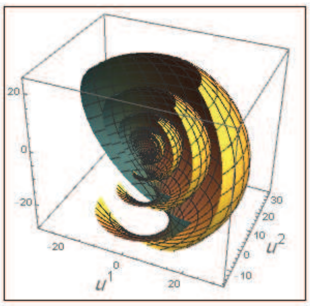

Figure 8. $(\phi^1 = t^2K, \phi^2 = t^2H,$
$\psi = t(t+1)K)$.

Figure 5 illustrates that as the sphere evolves in its $\mathbf{X}_1$ direction we see that the sphere gradual turns $z = -\sqrt{x^2+y^2}$ where $|z| \to \infty$ as $|t| \to \infty$, while in figure 6 we see that the sphere shrinks in u^2 direction and expand in the u^1 direction at the same time. The sphere blows up as it evolves in the **N** direction figure 7. Finally, figure 8 shows the combination of the three previous figures as it evolves in the directions of $\mathbf{X}_1$, $\mathbf{X}_2$ and **N**.

Acknowledgments. The authors wish to thank the referees for their many valuable and helpful suggestions in order to improve this manuscript. The authors would like to express his gratitude to prof. Mohammad Hasan Shahid (New Delhi - Jamia Millia Islamia University) and Prof. H. N. Abd-Ellah (Egpt - Assiut University) for editing and constructive criticism of the manuscript.

REFERENCES

[1] Abd-Ellah, H. N. *Motion of tubular surfaces*. Italian J. Pure and Appl. Math., N. 22 (2007): 161-176.

[2] Abdel-All, Nassar H., M. A. Abdel-Razek, H. S. Abdel-Aziz, and A. A. Khalil., *Evolution of a helix curve by observing its acceleration*. Wulfenia J. Vol 21, No. 11 (2014): 382-395.

[3] Abd-Ellah, H. N., *Evolution of translation surfaces in Euclidean 3-space E^3*. Appl. Math 9, no. 2 (2015): 661-668.

[4] Abd-Ellah, H. N., *Motion of Bishop Frenet offsets of ruled surfaces in E^3*. J. Appl. Math. ID 218956 (2015): 1-11.

[5] Alvarez, Luis; Lions, Pierre-Louis; Morel, Jean-Michel. *Image selective smoothing and edge detection by nonlinear diffusion*. II. SIAM J. Numer. Anal. 29 (1992), no. 3, 845–866

[6] Abdel-All, Nassar H., and Mariam T. Al-Dossary. *Motion of curves specified by acceleration field in $\mathbb{R}^n$*. Appl. Math. Sci. Vol 7, no. 69 (2013): 3403-418.

[7] Abdel-All, Nassar H., H. S. Abdelaziz, M. A. Abdel-Razek, S. G. Mohamed, and A. A. Khalil. *Evolution of Surfaces Patched by Principal Patch.* Wulfenia J. 22, no. 1 (2015): 395-406.

[8] Caselles, Vicent, Ron Kimmel, and Guillermo Sapiro. *Geodesic active contours*.Int. J. Comp. Vision. **22**, no. 1 (1997): 61-79.

[9] Gray, A., *Modern Differential Geometry of Curves and Surfaces*. Inc, U. S. A; CRC Press, (1993).

[10] Kass, M., Andrew Witkin, and Demetri Terzopoulos. *Active Contour Models.* Int. J. Comp. Vision. **1**, no. 4 (1988): 321-331.

[11] Kiziltuǵ, Sezai, Ömer Tarakci., *On parallel surfaces of Weingarten tubular surface İn Euclidean 3-Space.* Global Journal of Advanced Research on Classical and Modern Geometries. Vol.2, (2013); 30-35.

[12] Kosinski, Witold. *Field singularities and wave analysis in continuum mechanics*. Ellis Horwood, Polish Scientific Publishers, Warsaw (1985).

[13] Langer, Joel, and Ron Perline. *Local geometric invariants of integrable evolution equations*. J. Math. Phy. **35**, no. 4 (1994): 1732-1737.

[14] Mclachlan, R. I., and Segur, H., *A note on the motion of surfaces.* Phys. Rev. Lett., Vol.194,(1994):165-172.

[15] Nakayama, Kazuaki, Harvey Segur, and Miki Wadati., *Integrability and the motion of curves.* Physical review letters. 69, no. 18 (1992); 2603-2606.

[16] Pelce, P., *Dynamics of curved fronts.* Academic Press, New York (1988).

[17] Perona, P. and Malik, J., *Scale-space and edge detection using anisotropic diffusion*. IEEE Transactions on pattern analysis and machine intelligence, 12(7), (1990); 629-639.

[18] Romano, A., *Thermo mechanics of Phase Transitions in Classical Field Theory.* World Scientific, Singapore (1993).

[19] Romeny, B, *Geometry-driven diffusion in computer vision*." Kluwer Acad. Pub. Dor., Netherlands (1994).

[20] Sapiro, G, *Geometric Partial Differential Equations and Image Analysis*. Cambridge: Cambridge Univ. Press, (2001).

[21] Sapiro, G., Tannenbaum, A. *Affine invariant scale-space*. Int. J. Comput. Vision. 11, (1993); 25-44.

[22] Sethian, James A, J., *Differential Geometry*, **31** (1990), 13.

[23] Weickert, J. *Anisotropic Diffusion in Image Processing, Teubner-Verlag, Stuttgart*, Germany (1998).