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AN INTRODUCTION TO FINSLER ALMOST OSSERMAN MANIFOLDS

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ABSTRACT. In this paper, we introduce the notion of Finsler almost Osserman manifold and give example of Chern Osserman connection on a 2-dimensional manifold.

1. INTRODUCTION

Let M be a smooth manifold with a Finsler metric F, $TM_0 = TM \setminus \{0\}$ and $TTM_0 = T(TM \setminus \{0\})$ be the double tangent bundle. There are several methodologies of studying the Finsler geomety. The approach through the bundle TTM_0 will be used in this paper. This approach facilitates the analogy with the Riemannian geometry. The aim of this paper is to construct, as in Riemannian case, the Osserman structure in Finsler geometry. Let $\pi : TM \to M$ be the canonical projection. The pull-back bundle π^*TM, which is nothing but a collection of fibers of TM on TM_0, which that offers an adequate framework of this study. The paper is organized as follows: In section 2, we recall some basic definitions and necessary geometric concepts that are used throughout this paper. In section 3, we introduce the notion of Osserman Finsler structures on pull-back bundle π^*TM, and obtain some characterizations. Finally, in section 4, we study a particular Chern connection on a 2-dimensional Finsler manifold.

2. PRELIMINARIES

2.1. **Finsler geometry.** Let M be a Finsler manifold of dimension n. We introduce a coordinate system on TM as follows: let $V \subset M$ be an open set with local coordinate $(x^1, x^2, \dots, x^n)$. By setting $u = y^i \frac{\partial}{\partial x^i}$ for every $u \in \pi^{-1}(U)$, we introduce a local coordinate $(x, y) = (x^1, x^2, \dots, x^n, y^1, y^2, \dots, y^n)$ on $\pi^{-1}(V)$.

Definition 2.1. *A Finsler manifold of dimension n is a manifold M and a function* $F : TM \to [0, +\infty[$ *called a Finsler structure which has the following properties:*

(1) *F is smooth on* $\mathcal{C}^\infty(TM_0)$,
(2) $F(x, \lambda y) = \lambda F(x, y)$ *for all* $\lambda > 0$,
(3) *in local coordinate system* (x_i) *on M; the* $n \times n$ *Hessian matrix* (g_{ij}) *where* $g_{ij} := \frac{1}{2}(F^2)_{y^i y^j}$ *is positive definite at every point of* TM_0.

The pair (M, F) *is called Finsler manifold.*

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In Finsler geometry, we have several connections that intervene. For example Cartan connection, Hachiguchi connection, Berwald connection, Rund connection that are either associated with the Chern connection or coinside with it[1]. Chern connection is a linear connection that acts on a distinguished vector bundle π^*TM, not on TM over M. However, it serves Finsler geometry in a parallel way as the Levi-Civita connection does for Riemannian geometry [2].

The pull-back bundle π^*TM is a vector bundle over the tangent bundle TM_0 defined by

$$\pi^*TM := \{(x,y,u) \in TM_0 \times TM : u \in T_{\pi(x,y)}M\}. \tag{2.1}$$

Note that a curve $t \mapsto (x^1(t), \dots, x^n(t))$ is called Finslerian geodesic, if and only if

$$\ddot{x} + 2G^k(x, \dot{x}) = 0; \quad k = 1, \dots, n \tag{2.2}$$

where G^k are definied by:

$$G^k(x,y) := \frac{1}{4} g^{kj}\Big[2\frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{li}}{\partial x^j}(x,y)\Big] y^l y^i \tag{2.3}$$

and are called spray functions. From G^k, we can obtain natural objects:

$$N^i_j(x,y) := \frac{\partial G^i}{\partial y^j}. \tag{2.4}$$

N^i_j is known as the non linear connection on TM_0 which we use to define a vector 1-form on π^*TM denoted:

$$\theta = \frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + N^i_j dx^j). \tag{2.5}$$

Now, using the transformation formula of N^i_j [3], we can show that θ is globlaly well defined on TM_0 ant it is called Finsler-Ehresman form[11]. From θ and $\pi_* = d\pi$, the differential of the submersion $\pi : TM_0 \longrightarrow M$; we can obtain the horizontal distribution $\mathcal{H}$ and the vertical distribution $\mathcal{V}$ associated with a Finsler manifold (M, F) defined by:

$$\mathcal{H} = \ker\theta \tag{2.6}$$

$$\mathcal{V} = \ker\pi_*. \tag{2.7}$$

Locally, using the objets N^i_j we have :

$$\mathcal{H} = \operatorname{span}\Big\{\frac{\delta}{\delta x^i}\Big\} \tag{2.8}$$

$$\mathcal{V} = \operatorname{span}\Big\{F\frac{\partial}{\partial y^i}\Big\}. \tag{2.9}$$

The above subbundles give way to the following decomposition of TTM_0:

$$TTM_0 = \mathcal{H} \oplus \mathcal{V}. \tag{2.10}$$

Note that π^*TM can be naturally identified with the horizontal subbundle $\mathcal{H}$ and the vertical subbundle $\mathcal{V}$. Thus, any section $\bar{X}$ of π^*TM is considered as a section of $\mathcal{H}$ or a section of $\mathcal{V}$. We denote by $\bar{X}^H$ and $\bar{X}^V$, respectively the section of $\mathcal{H}$ and the section of $\mathcal{V}$ corresponding to $\bar{X} \in \Gamma(\pi^*TM)$:

$$\bar{X} = \frac{\partial}{\partial x^i} \otimes \bar{X}^i \in \pi^*TM \Leftrightarrow \bar{X}^H = \frac{\delta}{\delta x^i} \otimes \bar{X}^i \in \Gamma(\mathcal{H}) \tag{2.11}$$

and

$$\bar{X} = \frac{\partial}{\partial x^i} \otimes \bar{X}^i \in \pi^*TM \Leftrightarrow \bar{X}^V = F\frac{\partial}{\partial y^i} \otimes \bar{X}^i \in \Gamma(\mathcal{V}) \tag{2.12}$$

where $\left\{\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^i} = \left(\frac{\partial}{\partial x^i}\right)^H\right\}_{i=1,\dots,n}$ and $\left\{F\frac{\partial}{\partial y^i} := \left(\frac{\partial}{\partial x^i}\right)^V\right\}_{i=1,\dots,n}$ are the horizontal and vertical lifts on natural local frame field $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ with respect to the Finsler-Ehresmann connection. (See [12] for more details).

Proposition 2.2. [2] *The bundle morphisms π_* and θ satisfy*

$$\pi_*(\bar{X}^H) = \bar{X}, \quad \pi_*(\bar{X}^V) = 0 \tag{2.13}$$

and

$$\theta(\bar{X}^H) = 0, \quad \theta(\bar{X}^V) = \bar{X} \tag{2.14}$$

for every $\bar{X} \in \Gamma(\pi^*TM)$.

The existence of the Chern connection on the pull back bundle is given by the following result.

Theorem 2.3. [12] *Let (M, F) be a Finsler manifold, g a fundamental tensor of F and θ the Finsler-Ehresmann form. The pull-back bundle π^*TM admits a unique linear connection, called Chern connection such that, for all $X, Y \in \Gamma(TTM_0)$ and $\xi, \eta \in \Gamma(\pi^*TM)$ we have the following:*

(1) *Torsion freeness*

$$\nabla_X \pi_* Y - \nabla_Y \pi_* X = \pi_*[X, Y] \tag{2.15}$$

(2) *Almost g-compatibility*

$$(\nabla_X g)(\xi, \eta) = 2A(\theta(X), \xi, \eta) \tag{2.16}$$

where A is the Cartan tensor.

Locally, the Chern connection is given by

$$\nabla : \Gamma(T(TM)) \times \Gamma(\pi^*TM) \to \Gamma(\pi^*TM) \tag{2.17}$$

$$\nabla_X W = (dW^i(X) + W^j \omega^i_j(X)) \otimes \frac{\partial}{\partial x^i} \tag{2.18}$$

where $X \in \Gamma(T(TM))$, $W = W^i \frac{\partial}{\partial x^i}$ is a section of π^*TM and the coefficients ω^i_j are given by: $\omega^i_j = \Gamma^i_{jk} dx^k$.

Let us now give an intrinsic formulation of the Chern curvature that is necessary for this work.

Definition 2.4. *The curvature tensor Ω of the Chern connection is given by*

$$\begin{array}{rcl} \Omega : \Gamma(T(TM)) \times \Gamma(T(TM)) \times \Gamma(\pi^*TM) & \to & \Gamma(\pi^*TM) \\ (X, Y)W & \mapsto & \nabla_X \nabla_Y W - \nabla_Y \nabla_X W \\ & & -\nabla_{[X,Y]} W \end{array}$$

where $X, Y \in \Gamma(T(TM))$ and $W \in \Gamma(\pi^* TM)$. Locally we have:

$$\Omega(X,Y)W = W^j \Omega(X,Y)\frac{\partial}{\partial x^j}. \tag{2.19}$$

Using the decomposition: $\nabla_X = \nabla_{X^H} + \nabla_{X^V}$, where X^H et X^V are respectively, the horizontal and the vertical components of X, we have

$$\begin{aligned} \Omega(X,Y)W &= \Omega(X^H,Y^H)W + \Omega(X^H,Y^V)W \\ &\quad + \Omega(X^V,Y^H)W + \Omega(X^V,Y^V)W. \end{aligned} \tag{2.20}$$

We denote by

$$R(X,Y)W = \Omega(X^H,Y^H)W, \quad \text{horizontal part} \tag{2.21}$$

$$P(X,Y)W = \Omega(X^H,Y^V)W + \Omega(X^V,Y^H)W, \quad \text{mixed part} \tag{2.22}$$

$$Q(X,Y)W = \Omega(X^V,Y^V)W, \quad \text{vertical part.} \tag{2.23}$$

In the literature, R, P, Q are respectively called the $hh-, hv-, vv-$ curvature tensors of the Chern connection. By integrability of $\mathcal{V}$, we have for all $X, Y \in \Gamma(T(TM))$ and $W \in \Gamma(\pi^* TM)$

$$Q(X,Y)W = 0.$$

Hence the surviving part of Ω are the horizontal part R and the mixed part P. Thus, we have

$$\Omega(X,Y)W = R(X,Y)W + P(X,Y)W. \tag{2.24}$$

In local coordinates we have:

$$\begin{aligned} R\Big(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^l}\Big)\frac{\partial}{\partial x^i} &= \nabla_{\frac{\delta}{\delta x^k}}\nabla_{\frac{\delta}{\delta x^l}}\frac{\partial}{\partial x^i} - \nabla_{\frac{\delta}{\delta x^l}}\nabla_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial x^i} - \nabla_{[\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^l}]}\frac{\partial}{\partial x^i} \\ P\Big(\frac{\delta}{\delta x^k}, F\frac{\partial}{\partial y^l}\Big)\frac{\partial}{\partial x^i} &= \nabla_{\frac{\delta}{\delta x^k}}\nabla_{F\frac{\partial}{\partial y^l}}\frac{\partial}{\partial x^i} - \nabla_{F\frac{\partial}{\partial y^l}}\nabla_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial x^i} - \nabla_{[\frac{\delta}{\delta x^k}, F\frac{\partial}{\partial y^l}]}\frac{\partial}{\partial x^i} \end{aligned}$$

Next, we will denote by $\delta_i = \frac{\delta}{\delta x^i}, \partial_i = \frac{\partial}{\partial x^i}$ and $\dot{\partial}_i = F\frac{\partial}{\partial y_i}$. We refer to [3] for more information and details on Finsler manifolds.

2.2. **Osserman manifolds.** Let R be the curvature operator of a Riemannian manifold (M, g) of dimension m. The Jacobi operator $\mathcal{J}(X) : y \mapsto R(Y,X)X$ is the self-adjoint endomorphism of the tangent bundle. Following the seminal work of Osserman [14], one says that (M, g) is *Osserman* if the eigenvalues of $\mathcal{J}$ are constant on the unit sphere bundle

$$S(M,g) := \{X \in TM : g(X,X) = 1\}.$$

The works of Chi [4], Gilkey et al. [10] and Nikolayevsky [13] show that any complete and simply connected Osserman manifold of dimension $m \neq 16$ is a rank-one symmetric space; the 16-dimensional setting is exceptional and the situation is still not clear in that setting, although there are some partial result due, again, to Nikolayevsky [13].

Suppose (M, g) is a pseudo-Riemannian manifold of signature (p, q) for $p > 0$ and $q > 0$. The pseudo-sphere bundles are defined by setting

$$S^{\pm}(M,g) := \{X \in TM : g(X,X) = \pm 1\}.$$

One says that (M, g) is spacelike (resp. timelike) Osserman if the eigenvalues of $\mathcal{J}$ are constant on $S^+(M, g)$ (resp. $S^-(M, g)$). The situation is rather different here as the Jacobi operator is no longer diagonalizable and can have nontrivial Jordan normal form as shown by García-Río et al. [9]. Of course, inspired by the recent works in [5] and [8], giving the characterization of Chern connection that are Osserman would be an interesting problem.
The investigation of Osserman manifolds has been an extremely attractive and fruitful over the recent years. (See [5], [6], [7], [9] and references therein).

3. Results

Definition 3.1. *Let (M, F) be a Finsler manifold with a Chern connection and let Ω be the curvature tensor of ∇. The Finsler Jacobi operator $J_\Omega(X) : \Gamma(TTM_0) \to \Gamma(\pi^*TM)$ associated to a vector $X \in \Gamma(\pi^*TM)$ is defined by*

$$J_\Omega(X)Y = \Omega(Y, X)X \tag{3.1}$$

where $Y \in \Gamma(TTM_0)$.

From (2.24), we have

$$J_\Omega(X)Y = J_R(X)Y + J_P(X)Y \tag{3.2}$$

where $J_R(X)Y = R(Y, X)X$, $J_P(X)Y = P(Y, X)X$ and $Y \in \Gamma(TTM_0)$. The operator $J_R(X)$ (respectively $J_P(X)$) is called R-Finsler Jacobi operator (P-Finsler Jacobi operator). From the decomposition $Y = Y^H + Y^V$, we have:

$$J_\Omega(X)Y = R(Y^H, X)X + P(Y^V, X)X \tag{3.3}$$

Definition 3.2. *A Finsler manifold (M, F) is called a R-Finsler almost Osserman at $p \in M$ if the characteristic polynomial of the R-Finsler Jacobi operator $J_R(X)$ is independent of $X \in \Gamma(\pi^*TM)$. Also (M, F) is said to be R-Finsler almost Osserman if (M, F) is R-Finsler almost Osserman at each point $p \in M$.*

Definition 3.3. *A Finsler manifold (M, F) is called a P-Finsler almost Osserman at $p \in M$ if the characteristic polynomial of the P-Finsler Jacobi operator $J_P(X)$ is independent of $X \in \Gamma(\pi^*TM)$. Also (M, F) is said to be P-Finsler almost Osserman if (M, F) is P-Finsler almost Osserman at each point $p \in M$.*

Definition 3.4. *A Finsler manifold (M, F) is said to be almost Osserman at $p \in M$ if the characteristic polynomial of $J_\Omega(X)$ is independent of $X \in \Gamma(\pi^*TM)$. Also, (M, F) is said to be almost Osserman if (M, F) is almost Osserman at each point $p \in M$.*

Definition 3.5. *A Finsler manifold (M, F) is said to be almost Osserman at $p \in M$ if and only if it is R-Finsler almost Osserman and P-Finsler almost Osserman at $p \in M$.*

Let us consider the following Chern connection ∇ on a 2-dimensional Finsler manifold givin by:

$$\begin{cases} \nabla_{\delta_1}\partial_1 = f_1\partial_1 \\ \nabla_{\delta_2}\partial_2 = f_2\partial_2 \\ \nabla_{\partial_1}\partial_1 = h_1\partial_1 \\ \nabla_{\partial_2}\partial_2 = h_2\partial_2 \\ \nabla_{\dot\partial_1}\partial_1 = 0 \\ \nabla_{\dot\partial_2}\partial_2 = 0. \end{cases} \tag{3.4}$$

Let $X = \alpha_1\partial_1 + \alpha_2\partial_2$ be a non-null vector, where $\{\partial_i\}$ denotes the coordinates basis and $\alpha_i \in \mathbb{R}^*$.

(1) The associated R-Finsler Jacobi operator can be expressed, with respect to the coordinate basis, as:

$$J_R(X) = \begin{pmatrix} a_1 & c_1 \\ b_1 & d_1 \end{pmatrix} \tag{3.5}$$

with

$$\begin{aligned} a_1 &= \alpha_1^2\delta_1 h_1 - \alpha_1^2\partial_1 f_1 - \alpha_1\alpha_2\partial_2 f_1 \\ b_1 &= \alpha_2^2\delta_1 h_2 \\ c_1 &= \alpha_1^2\delta_2 h_1 \\ d_1 &= -\alpha_1\alpha_2\partial_1 f_2 + \alpha_2^2\delta_2 h_2 - \alpha_2^2\partial_2 f_2. \end{aligned}$$

The characteristic polynomial of $J_R(X)$ is:

$$P(J_R(X)) = \lambda^2 - (a_1 + d_1)\lambda + a_1 d_1 - b_1 c_1. \tag{3.6}$$

It follows that the connection given by (3.4) is R-Osserman if and only if:

$$\begin{cases} a_1 + d_1 = 0 \\ a_1 d_1 - b_1 c_1 = 0. \end{cases} \tag{3.7}$$

Straightforward calculations give

$$\begin{cases} \delta_1 h_1 - \partial_1 f_1 = 0 \\ \partial_2 f_1 - \partial_1 f_2 = 0 \\ \delta_2 h_2 - \partial_2 f_2 = 0 \\ \delta_1 h_1 \delta_2 h_2 - \delta_1 h_1 \partial_2 f_2 - \partial_1 f_1 \delta_2 h_2 + \partial_1 h_1 \partial_2 f_2 + \partial_2 f_1 \partial_1 f_2 - \delta_2 h_1 \delta_1 h_2 = 0 \\ (\delta_1 h_1 - \partial_1 f_1)\partial_1 f_2 = 0 \\ (\delta_2 h_2 - \partial_2 f_2)\partial_2 f_1 = 0. \end{cases} \tag{3.8}$$

(2) The associated P-Finsler Jacobi operator can be expressed, with respect to the coordinate basis, as:

$$J_P(X) = \begin{pmatrix} a_2 & c_2 \\ b_2 & d_2 \end{pmatrix} \tag{3.9}$$

with

$$a_2 = \alpha_1^2\dot{\partial}_1 h_1 \tag{3.10}$$

$$b_2 = \alpha_2^2\dot{\partial}_1 h_2 \tag{3.11}$$

$$c_2 = \alpha_1^2\dot{\partial}_2 h_1 \tag{3.12}$$

$$d_2 = \alpha_2^2\dot{\partial}_2 h_2. \tag{3.13}$$

The characteristic polynomial of $J_P(X)$ is:

$$P(J_P(X)) = \lambda^2 - (a_2 + d_2)\lambda + a_2 d_2 - b_2 c_2. \tag{3.14}$$

It follows that the connection given by (3.4) is P-Osserman if and only if:

$$\begin{cases} a_2 + d_2 = 0 \\ a_2 d_2 - b_2 c_2 = 0. \end{cases} \tag{3.15}$$

Straightforward calculation give

$$\left\{\begin{array}{l}\dot{\partial}_1 h_1 = 0\\ \dot{\partial}_2 h_2 = 0\\ \dot{\partial}_1 h_2 = 0\\ \dot{\partial}_2 h_1 = 0.\end{array}\right. \tag{3.16}$$

We have the following result:

Proposition 3.6. *The Chern connection (3.4).*

(1) *is R-Finsler almost Osserman if the funtions f_1, f_2, h_1, h_1 satisfy:*

$$\left\{\begin{array}{l}\partial_2 f_1 = 0\\ \partial_1 f_2 = 0\\ \delta_1 h_1 - \partial_1 f_1 = 0\\ \delta_2 h_2 - \partial_2 f_2 = 0\\ -\partial_1 f_1 \delta_2 h_2 + \partial_1 h_1 \partial_2 h_2 - \delta_2 h_1 \delta_1 h_2 = 0.\end{array}\right. \tag{3.17}$$

(2) *is P-Finsler almost Osserman if the following condition holds:*

$$\left\{\begin{array}{l}\dot{\partial}_1 h_1 = 0\\ \dot{\partial}_2 h_2 = 0\\ \dot{\partial}_1 h_2 = 0\\ \dot{\partial}_2 h_1 = 0.\end{array}\right. \tag{3.18}$$

Proof. From (3.8) and (3.16) we get the results. □

Proposition 3.7. *Define the Chern connection ∇ on the 2-dimensional Finsler manifold given by:*

$$\left\{\begin{array}{l}\nabla_{\delta_1}\partial_1 = f_1\partial_1\\ \nabla_{\delta_2}\partial_2 = f_2\partial_2\\ \nabla_{\partial_1}\partial_1 = h_1\partial_1\\ \nabla_{\partial_2}\partial_2 = h_2\partial_2.\end{array}\right. \tag{3.19}$$

Then ∇ is Finsler almost Osserman if and only if the following conditions hold:

$$\begin{array}{rcl} h_1 &=& \phi_1(x_1) + \phi_2(x_2), \quad h_2 = \psi_1(x_1) + \psi_2(x_2);\\ f_1 &=& \phi_1(x_1) + p(y_1, y_2), \quad f_2 = \psi_2(x_2) + l(y_1, y_2).\end{array}$$

and the functions ϕ_1, ϕ_2, ψ_1 and ψ_2 are solutions of the following:

$$-\frac{\partial\phi_1}{\partial x_1}\frac{\delta\psi_2}{\delta x_2} + \frac{\partial\phi_1}{\partial x_1}\frac{\partial\psi_2}{\partial x_2} - \frac{\delta\phi_2}{\delta x_2}\frac{\delta\psi_1}{\delta x_1} = 0.$$

Proof. From (3.18), we obtain:

$$h_1 = \phi(x_1, x_2) \quad \text{and} \quad h_2 = \psi(x_1, x_2). \tag{3.20}$$

From (3.17), we have:

$$\partial_2 f_1 = 0 \quad \text{and} \quad \partial_1 f_1 = \delta_1 h_1 \Rightarrow h_1 = \phi_1(x_1) + \phi_2(x_2). \tag{3.21}$$

Hence,

$$f_1 = \phi_1(x_1) + p(y_1, y_2). \tag{3.22}$$

Also, from

$$\partial_1 f_2 = 0 \quad \text{and} \quad \partial_2 f_2 = \delta_2 h_2 \Rightarrow h_2 = \psi_1(x_1) + \psi_2(x_2). \tag{3.23}$$

So

$$f_2 = \psi_2(x_2) + l(y_1, y_2). \tag{3.24}$$

□

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REFERENCES

[1] M. Anastasiei, *A historical remark on the connection of Chern and Rund*, Contemp. Math., 196 (1996), 171-176. MR1403589

[2] T. Aikou and L. Kozma, Global aspects of Finsler geometry. Handbook of global analysis. 1-39, 1211, Elsevier Sci. B. V., Amsterdam, 2008. MR2389632

[3] D. Bao, S. -S. Chern and Z. Shen, An introduction to Riemann-Finsler Geometry. Graduate Texts in Mathematics, 200. Springer-Verlag, New York, 2000. MR1747675

[4] Q. -S. Chi, *A curvature characterization of certain locally rank-one symmetric spaces*, J. Differential Geom., 28 (1988), (2), 187-202. MR0961513

[5] A. S. Diallo, *Affine Osserman connections on* 2*-dimensional manifolds*, Afr. Diaspora J. Math., 11 (2011), (1), 103-109. MR2792213

[6] A. S. Diallo and M. Hassirou, *Two families of affine Osserman connections on* 3*-dimensional manifolds*, Afr. Diaspora J. Math., 14 (2012), (2), 178-186. MR3093242

[7] A. S. Diallo, P. K. Gtougaimbou and M. Hassirou, *Affine Osserman connections which are locally symmetrics*, Glob. J. Adv. Res. Class. Mod. Geom., 3, (2014), (1), 1-6. MR3216245

[8] A. S. Diallo, M. Hassirou and I. Ousmane Toudou, *Example of Walker Osserman metric of signature* $(3,3)$, Bull. Math. Anal. Appl., 9, (2017), (4), 21-30. MR3840089

[9] E. García-Rio, D. N. Kupeli and R. Vázquez-Lorenzo, *Osserman Manifolds in Semi-Riemannian Geometry*, Lectures Notes in Mathematics 1777, Springer-Verlag, Berlin, 2002. MR1891030

[10] P. Gilkey, A. Swann and L. Vanhecke, *Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator*, Quart. J. Math. Oxford Ser. (2) 46 (1995), (183), 299-320. MR1348819

[11] J. S. Mbatakou and L. Todjihoundé *Conformal change of Finsler-Ehresmann connections*, Appl. Sciences, 16 (2014), 32-47. MR3224497

[12] J. S. Mbatakou, *Intrinsic proofs of the existence of generalized Finsler connections*, Int. Election. J. Geom., 8 (2015), (1), 1-13. MR3340277

[13] Y. Nikolayevsky, *On Osserman manifolds of dimension* 16, Contemporary Geometry and Related Topics (Belgrade: Faculty of Mathematics, University of Belgrade), (2006), 379-398. MR2963642

[14] R. Osserman, *Curvature in the eighties*, Amer. Math. Monthly, 97 (1990), (8) 731-756. MR1072814

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