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A NEW IMPROVISATION ON THE TOPIC "INTERSECTION OF A PYRAMID WITH A PLANE"

MARTA TEOFILOVA AND RADKA TODOROVA

ABSTRACT. An application of the dynamic geometric software GeoGebra to the topic plane–pyramid intersection is studied by using infinite points and the swap of finite and infinite points. An open set of problems about generating intersections which are special quadrangles (parallelogram, rectangle, rhomb and square) is presented. The introduced method allows presentations of solutions of a whole set of problems by solving one instance and using a pre–made applet at any stage of the solution process.

1. INTRODUCTION

Teaching geometry in a dynamic digital learning environment provides students with an opportunity to experiment, to develop deeper mathematical knowledge, to enhance their creative skills and encourages them to actively participate in the learning process. Dynamic geometry software has been shown to be useful in teaching the topic of polyhedron–plane intersection, which has proved rather challenging for a lot of students [1, 2, 3] as well as in the investigation of new properties in the plane geometry [9, 10, 11].

The present work is an application of the innovative method for studying intersections of a polyhedron with a plane by using infinite points and the function *"Swap finite & infinite points"* in the *GeoGebra* environment presented in [1]. An open set of problems and their solutions on the topic of intersections of parallelepipeds, quadrangular and triangular prisms, pyramids, and truncated pyramids with planes can be generated from one instance with the help of the applet *"Swap finite & infinite points"*. This function, which was included for the first time in the menu of the specialized dynamic software Sam [2, 3] and later in *GeoGebra* [5, 6], not only greatly optimizes the drawing work, but also provides a research style for studying geometry [4]. In the present paper we generate four types of special intersections (parallelogram, rectangle, rhomb, and square) of pyramids with planes. With this work, as well as with [1], we offer to the readers a creative and entertaining mathematical game and we expect that it will be enriched in the future.

2010 *Mathematics Subject Classification.* Primary 97Gxx; Secondary 51A05.
Key words and phrases. Dynamic geometry software, GeoGebra, Infinite point, Intersection of a plane and a polyhedron.

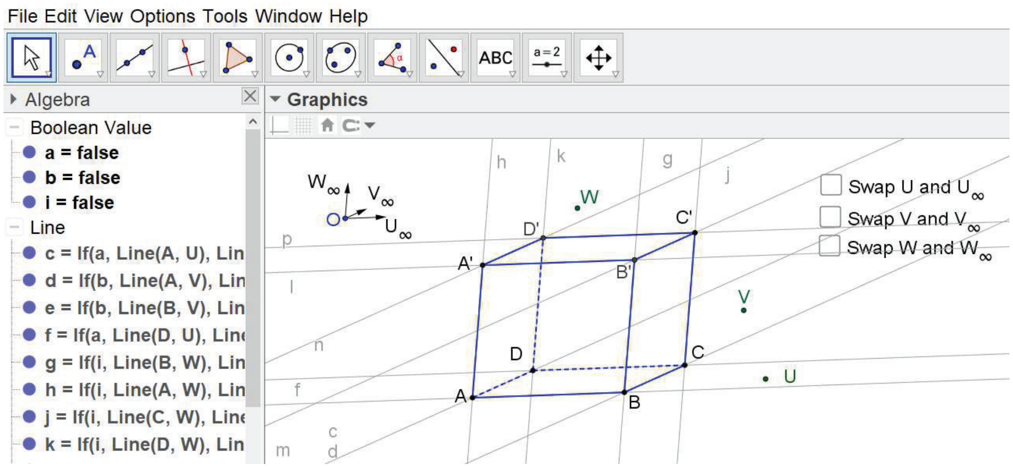

FIGURE 1.

2. PRELIMINARIES

The new approach is based on the projective property a preservation of the incidence of the basic objects (points, lines) by a central projection. The refusal to use the affine property that a plane intersects the parallel faces of a parallelepiped in parallel lines is because after the transition from parallelepipeds into prisms with bases different from parallelograms, or into pyramids (truncated or not), this property is not preserved (except for the bases).
The applet *”Swap finite & infinite points“* allows the visualization of the wide variety of quadrangular and triangular prisms, pyramids, and truncated pyramids with a single click of a button [1, 2, 3, 4]. The construction of the universal parallelepiped highlights the functionality of the function *”Swap finite & infinite points“*. This construction is completely explained in [1, 4]. For the sake of completeness, we describe it here again.

2.1. **Universal parallelepiped.** (Figure 1). The free points in the construction of the universal parallelepiped $ABCDA'B'C'D'$ are A, U, V, W, U_∞, V_∞, W_∞. After constructing these points, we define three Boolean variables, a, b and i with the names *”Swap U and U_∞“*, *”Swap V and V_∞“*, and *”Swap W and W_∞“*, respectively [1, 6]. The points B, D, A' are free points on the lines:

$$
\begin{aligned}
c &= If(a, Line(A,U), Line(A,U_\infty)),\\
d &= f(b, Line(A,V), Line(A,V_\infty)),\\
h &= If(i, Line(A,W), Line(A,W_\infty)),
\end{aligned}
$$

respectively.
The vertex C is the intersection point of the two lines $= If(b, Line(B,V), Line(B,V_\infty))$ and $f = If(a, Line(D,U), Line(D,U_\infty))$.
The choice of the point A' on the line h defines the length of the surrounding edges. The surrounding edges BB', CC', DD' lie on the lines, which are constructed similar to the line h, while the edges of the upper base $A'B'C'D'$ and the remaining vertices are constructed with the help of the tools ”parallel line“ and ”intersection of two lines“.

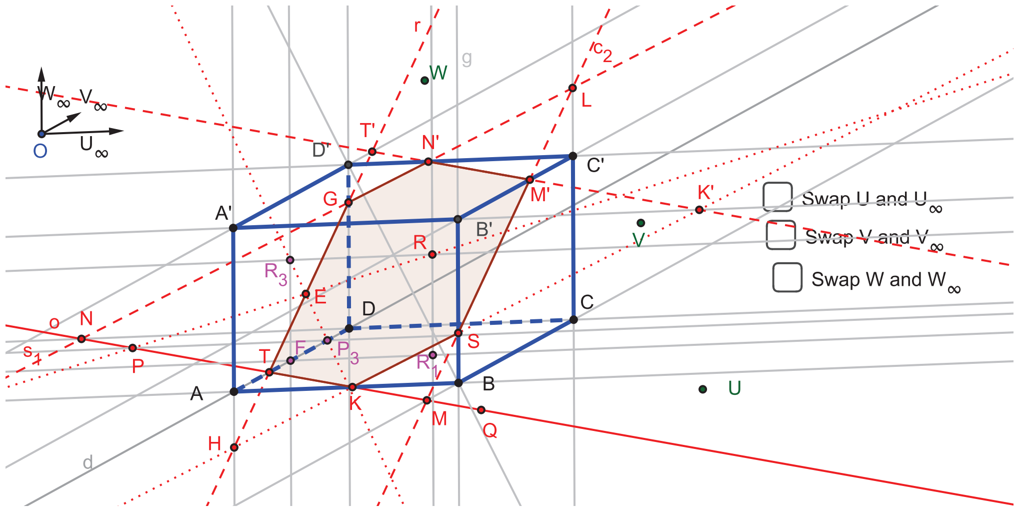

FIGURE 2.

2.2. **Basic construction for finding the intersection of a parallelepiped with a plane.** The new strategy for constructing the intersection of parallelepipeds with a plane in the environment of *GeoGebra* using the *"Swap finite & infinite points"* applet is presented in [1] through two main tasks and 17 of their so called *satellite problems* (i.e. problems derived from a basic problem by applying the swap function). The plane of the intersection $\alpha(P, Q, R)$ is determined by three non–collinear points. The first problem is classic when the three points are placed on three edges, each two of which are skew (i.e. not coplanar). We will recall the condition of the second problem that we will use as a starting point in our current research.

Basic problem. *Find the intersection of the parallelepiped $ABCDA'B'C'D'$ with the plane (P, Q, R), provided that two of the points P, Q and R lie on the plane of one and the same face of the parallelepiped $ABCDA'B'C'D'$ and the third point is an arbitrary one.*

By the first problem it is shown that without difficulty we can obtain two points from the intersection plane α in the plane of each face of the parallelepiped. That is why, without loss of generality, we choose the points P and Q to lie in the plane $\gamma(ABCD)$.

Although is an arbitrary point, the determination of its position in the space requires the assignment of one of its projections on a parallelepiped wall from that of the three infinite points incident with the three groups of parallel edges of the parallelepiped that does not lie in that wall (for example the point $R_1 = RW_\infty \cap \gamma$ [1]). Without loss of generality, the planes $\gamma(ABCD)$ and $\delta(ADD'A')$ are chosen as the first ones, whose intersections with $\alpha(P, Q, R)$ are found (Figure 2).

3. An improvisation on the basic problem when the parallelepiped is replaced by pyramids

According to [3, 4], the figures, which are obtained from each other by applying the function *"Swap finite & infinite points"*, are called connected figures. Thus the pyramid $ABCDW$ or the truncated pyramid $ABCDA'B'C'D'$ with the bases of arbitrary quadrangles are connected figures with the parallelepiped $ABCDA'B'C'D'$.

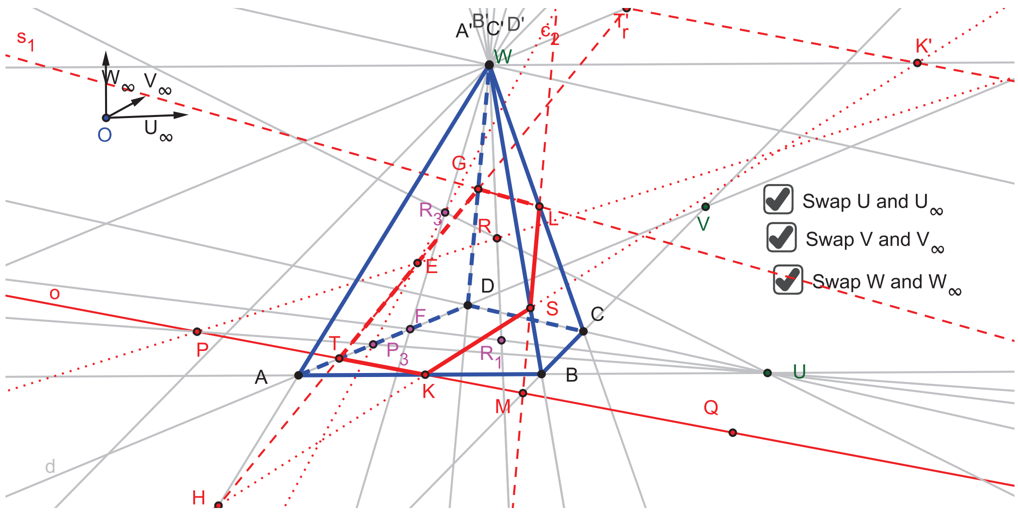

FIGURE 3.

The intersection of the pyramid $ABCDW$ with a plane α is generated from Figure 2 after activating the applets *"Swap U and U_∞"*, *"Swap V and V_∞"*, *"Swap W and W_∞"*, i.e. after exchanging the Boolean variable false with true. Thus we receive Figure 3 and the solution of one satellite of the Basic problem.
Let us notice that in order to reach the desired shape of the pyramid and its intersection with α, we have to take advantage of the freedom of movement of points A ,B, D, A', U, V, W, P, Q, R.

Problem 1. *(Main Problem). Select the plane α such that its intersection $HGLS$ with the pyramid $ABCDW$, where $ABCD$ is an arbitrary quadrangle, is a parallelogram.*

SOLUTION: (Figure 4) We construct the following lines: $UV = u$, $WU = h_2$, $WV = s$. They define the plane $\beta(U, V, W)$ and $\gamma(ABCD)$. Obviously, the intersection of the planes $\beta(U, V, W)$ and $\gamma(ABCD)$ is the line u.
Following [7], we will call $UV = u$ a disappearing line.
Let us denote by φ the central projection with a center W of the plane γ on the plane α. Then we can write $\varphi(A, B, C, D, U, V) = H, S, L, G, U', V'$. The intersection $HGLS$ will be a parallelogram if and only if the lines HG and SL, as well as the lines HS and GL, are parallel, i.e. when the points U', V' are infinite. Hence the plane α should be parallel to the plane β. But then the intersection line o of the planes α and γ will be parallel to the line u.
Using the capabilities of *GeoGebra* and in particular the window *Object property* we predefine: The line o as a line through the arbitrary point P and parallel to the line u;
The point Q as $Intersect(o, d)$, where $d = AV$, i.e. Q coincides with the point T.
The following steps are connected with the fixing of a point R:
We construct the lines g_2 through the point P and parallel to the line $h_2 = WU$; i_2 through the point Q and parallel to the line $s = WV$.
In the window *Object property* we predefine: Point R, writing $Intersect(g_2, i_2)$; Point R_1, writing $Intersect(d, i_1)$, where $i_1 = RW$.

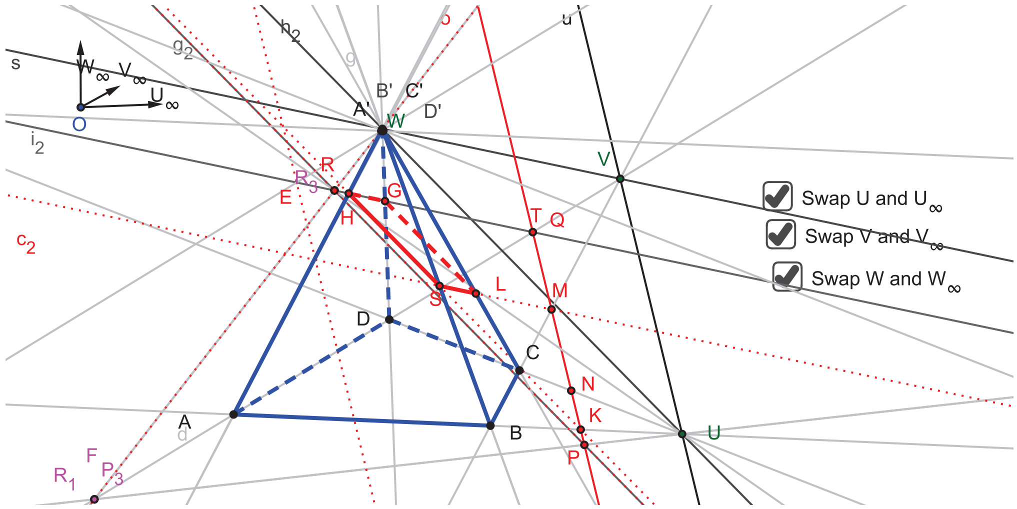

FIGURE 4.

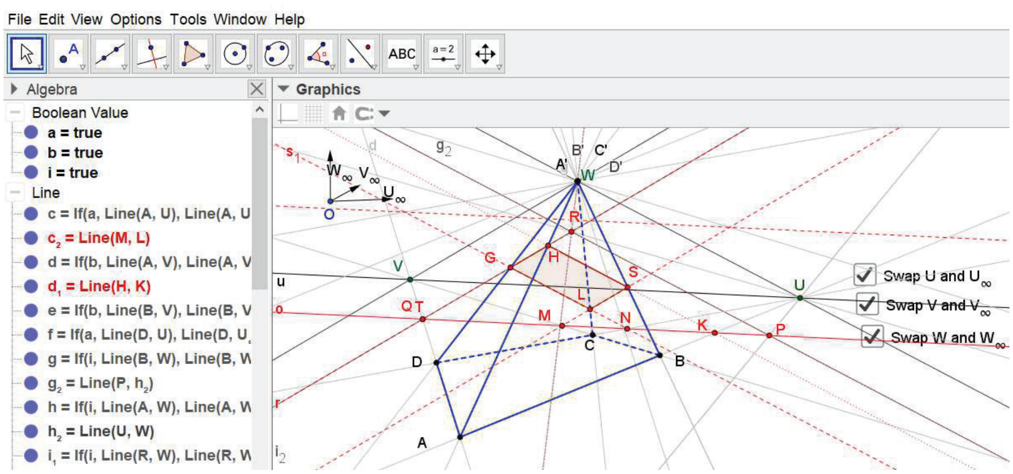

FIGURE 5.

Thus we ensure the relation $R \in \delta(ADD'A')$, which immediately leads to the matching of the points $R, R_3, E = PR \cap \delta$ and to the matching of the points R_1 and F.
The point $P_3 = PU \cap AV$ lies in the planes γ, δ and $\tau(g_2, h_2) = \tau(P, R, U, W)$. Therefore, $P_3 = \gamma \cap \delta \cap \tau = AV \cap WR \cap PU$ or P_3 coincides with R_1.
Let us note that changing the free points' positions does not change the intersection type. One frame is shown in Figure 5. Any section that is parallel to the obtained one will already be of the same type. Figure 6 presents the section $H_1G_1L_1S_1$, which is parallel to $HGLS$ and it is a parallelogram, too. The free vertex is G_1.

Problem 2. *Select the apex W of the pyramid $ABCDW$, where $ABCD$ is an arbitrary quadrangle such that its intersection $HSLG$ with the plane α is a rectangle.*

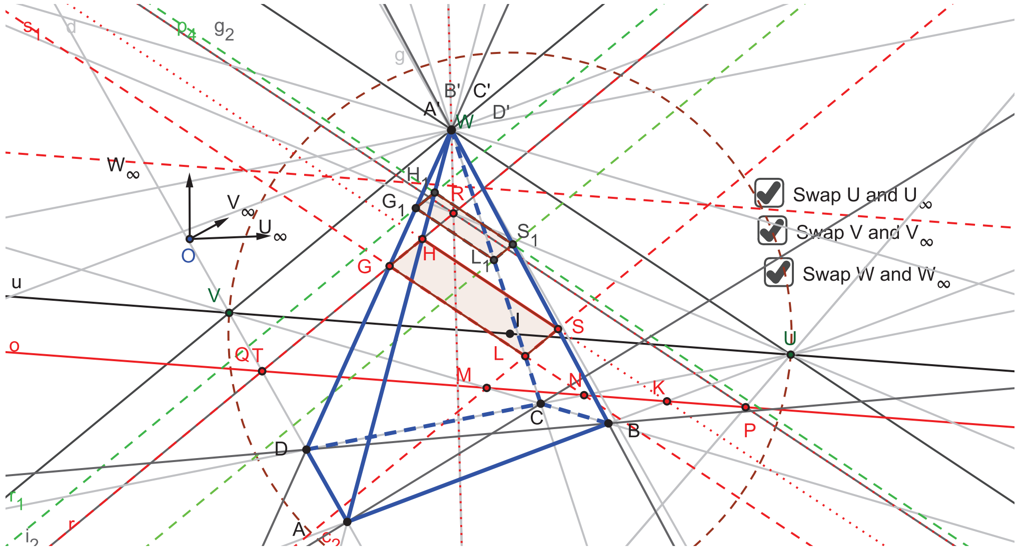

FIGURE 6.

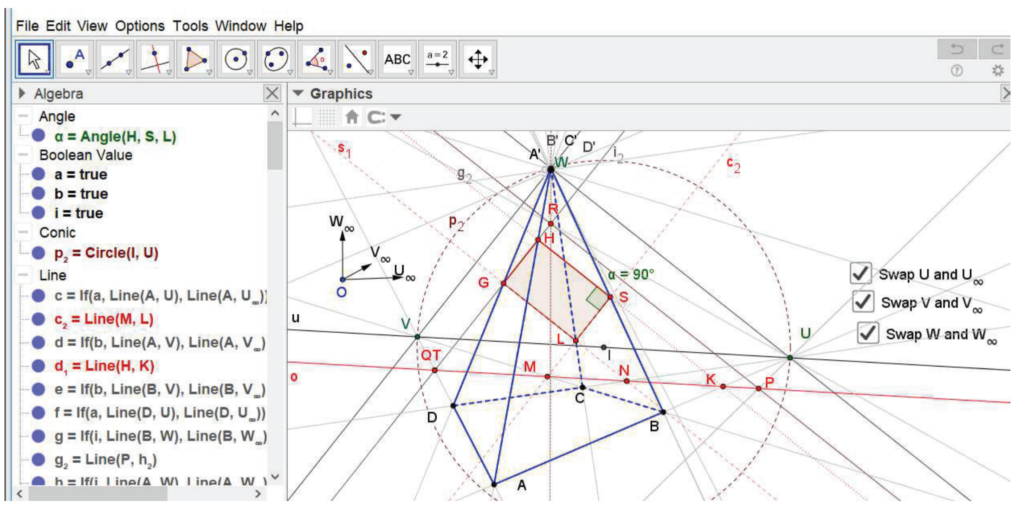

FIGURE 7.

SOLUTION (Figure 7) We find the midpoint I of the segment UV, construct the circle $p_2(I, r = IU)$ and predefine the point W by writing $Point(p_2)$. Then the directions U' and V', whose representatives are WU and WV, will be perpendicular and the intersection $HSLG$ will be a rectangle. Obviously, the problem has countless solutions for the point W – each point incident with p_2.

Problem 3. *Select the apex W of the pyramid $ABCDW$, where $ABCD$ is an arbitrary quadrangle, such that its intersection $HSLG$ with the plane α is a rhomb.*

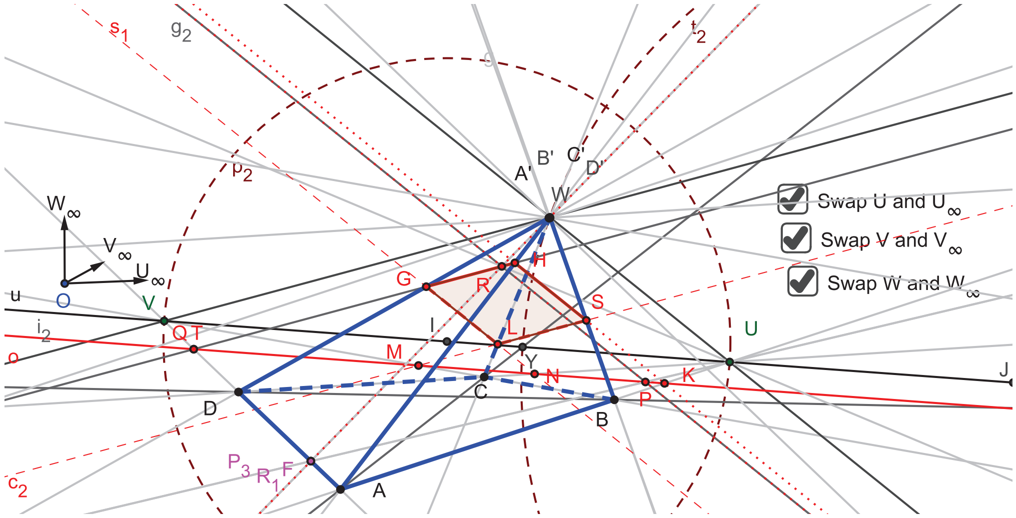

FIGURE 8.

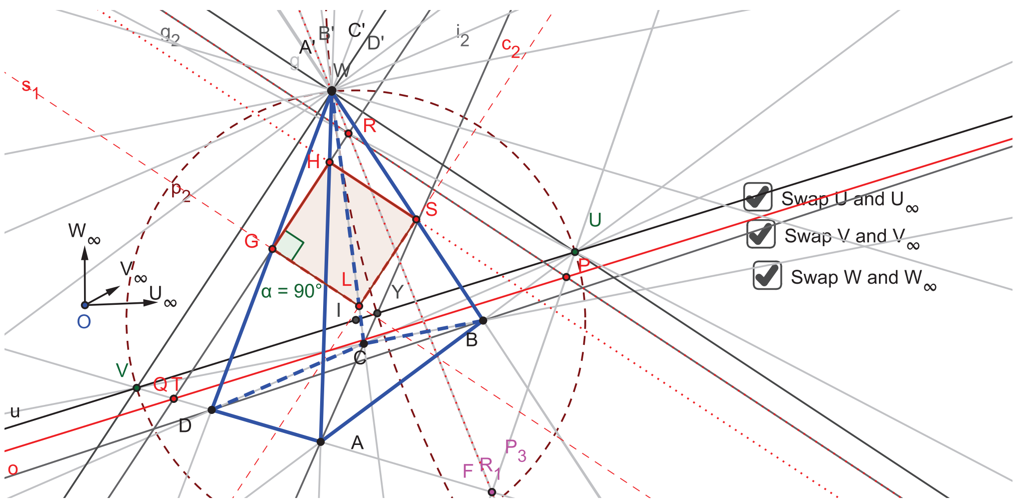

FIGURE 9.

SOLUTION: (Figure 8) Let us recall the statement: A parallelogram is a rhomb if and only if its diagonals are perpendicular. This statement defines the following solution: if $AC \cap u = Y$ and $BD \cap u = Z$, then it is enough for W to be a point on the circle t_2 with diameter YZ. Therefore, we have to predefine W as $Point(t_2)$. Again, the problem has countless solutions.

Problem 4. *Select the apex W of the pyramid $ABCDW$, where $ABCD$ is an arbitrary quadrangle, such that its intersection $HGLS$ with the plane α is a square.*

SOLUTION: (Figure 9) It is enough for the point W to be predefined as $Intersect(p_2, t_2)$.

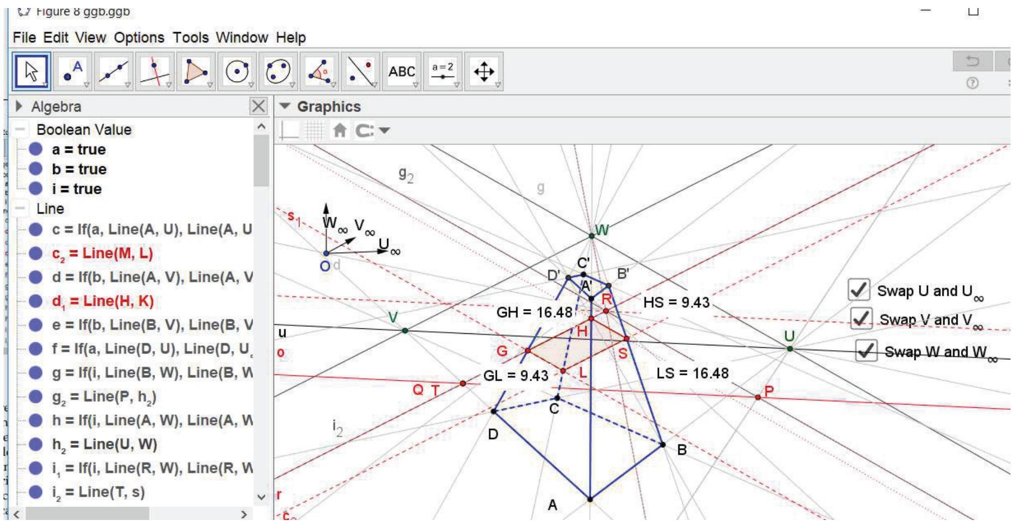

FIGURE 10.

The reader can experiment on the *GeoGebra* files available at the following web address: http://web.uni-plovdiv.bg/marta/GeoGebra/Plane-Pyramid/.
Using the Distance tool, it is possible to measure the lengths of the sides of the intersection to check that another necessary and sufficient condition for the special parallelogram is fulfilled.
The innovative approach for constructing intersections of a parallelepiped with a plane, described in [1], allows us not only to offer the present improvisation on the topic ”Intersection of a pyramid with a plane“, but also to offer solutions to similar problems for truncated pyramids and for double pyramids without wasting time for drawing. We will illustrate these cases with two examples.

Problem 5. *Select the plane α such that its intersection $HSLG$ with the truncated pyramid $ABCDW$, where $ABCD$ is an arbitrary quadrangle, is a parallelogram.*

SOLUTION (Figure 10) It is enough to use Figure 4 or 5 and to move the point A' on the edge AW.
The construction of the universal parallelepiped allows us to visualize on the screen a figure consisting of two pyramids with a common apex and surrounding edges lying on common lines. Let us call these figures *double pyramids*. Only one movement of A' along the edge AW beyond the apex W is sufficient for their appearance on the screen.

Problem 6. *Select the apex W of the double pyramid $ABCDWA'B'C'D'$, where $ABCD$ is an arbitrary quadrangle, such that its intersection $HSLG$ of the upper part with the plane α is a rhomb.*

SOLUTION (Figure 11) It is enough to use Figure 8 and to move the point A' along the edge AW until the relation W/AA' appears on the screen. Of course, it is also necessary to specify the plane of the intersection α through the free point P.

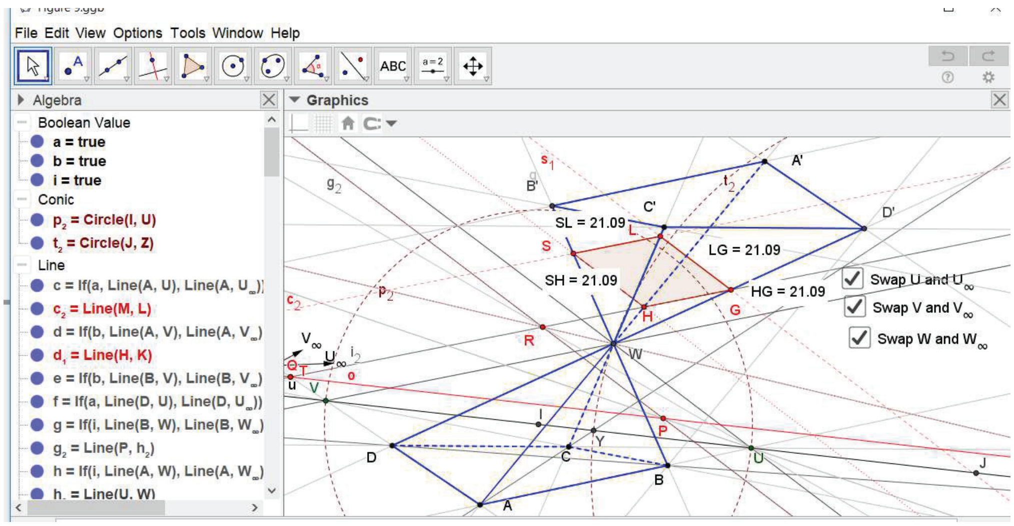

FIGURE 11.

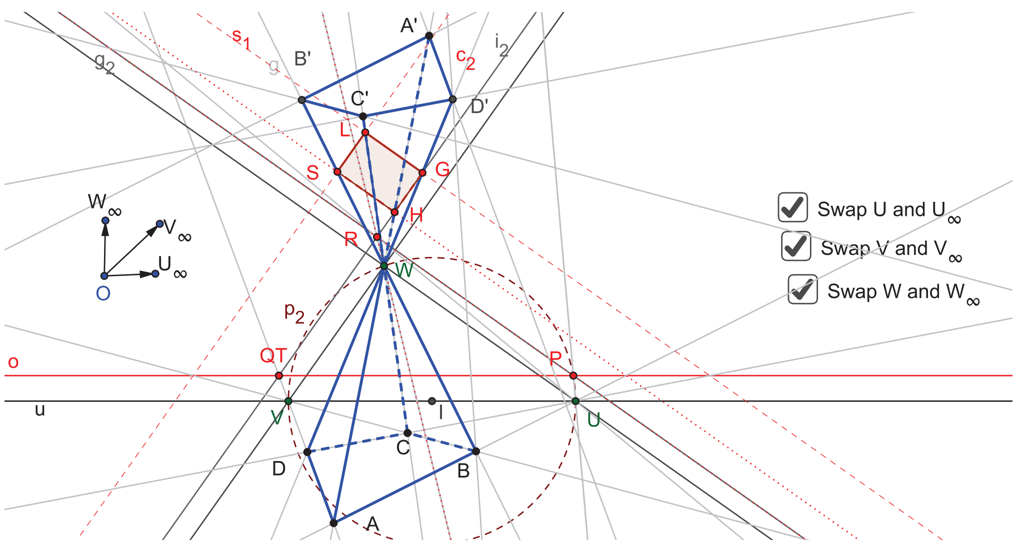

FIGURE 12.

Problem 7. *Select the apex W of the double pyramid $ABCDWA'B'C'D'$, where $ABCD$ is an arbitrary quadrangle, such that its intersection $HSLG$ of the upper part with the plane α is a rectangle.*

SOLUTION (Figure 12) It is sufficient to use Figure 7 or Figure 10 and to move the point A' on the edge AW.
By some examples we demonstrate how the specialization of the section $HGLS$ automatically leads to the same specialization of the section $H_1G_1L_1S_1$.

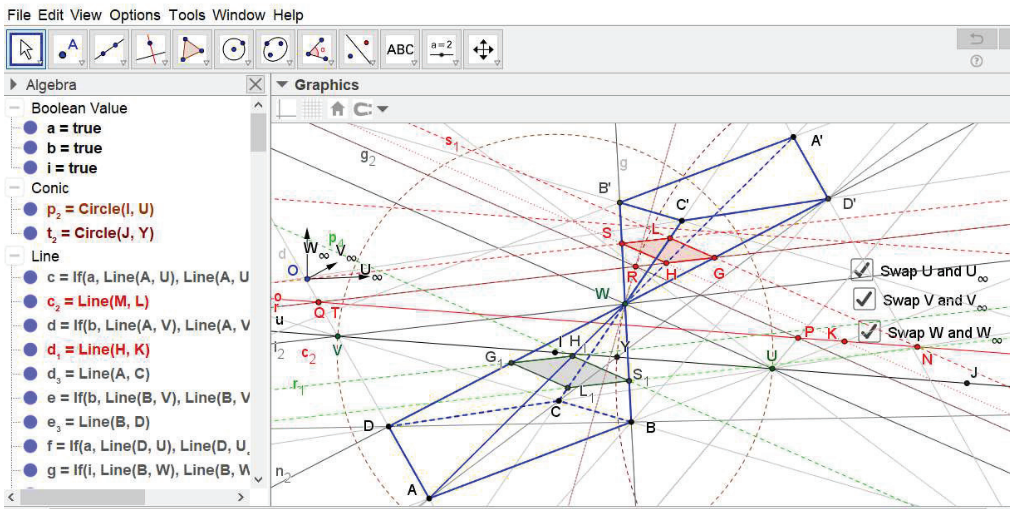

FIGURE 13.

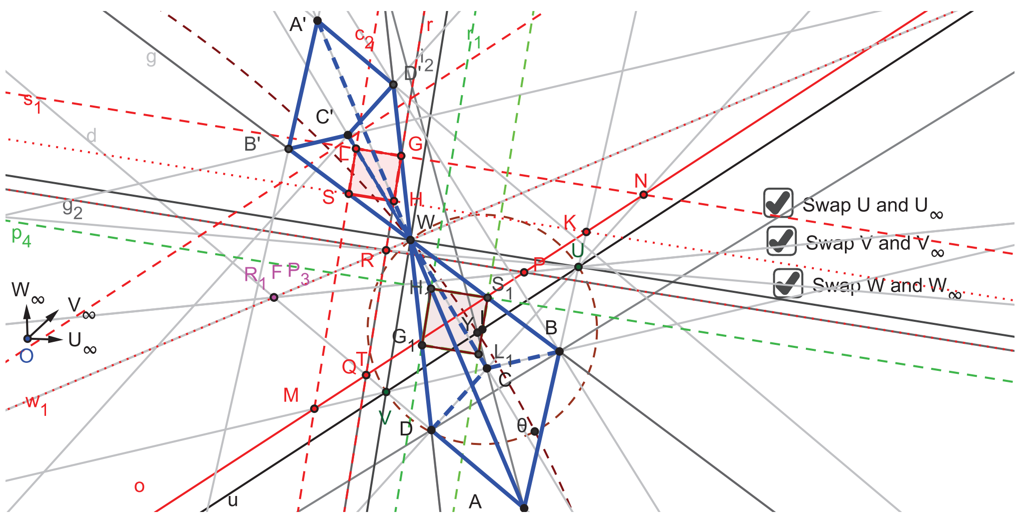

FIGURE 14.

Problem 8. *Select the apex W of the double pyramid $ABCDWA'B'C'D'$, where $ABCD$ is an arbitrary quadrangle, such that its intersection $HSLG$ of the upper part with the plane α and its parallel intersection are rhombs or squares.*

Solution Figures 13 and 14.

4. Conclusion

Dynamic geometric software allows a creative learning process and deeper understanding of geometry. It is especially helpful in studying and visualizing 3D geometric objects and their properties. The present generation of students is not only prepared to actively use computers in their training, but they are eager to apply their digital skills to this process [4]. Meetings with students from the fifth to the eleventh grade in schools of different types have shown us that the new concept for them of an infinite point introduced by the simplified definition ”We will say that two straight lines intersect at an infinite point when they are parallel“ neither disturbs nor overloads them [4]. According to their capabilities, they include it in their personal creativity [6, 8]. An undeniable fact is that mathematical knowledge obtained by self-conducted observation and research is deeper and more durable. Following this concept and [1], we have presented an open set of problems on the topic of a pyramid–plane intersection and an approach to solve and expand these problems in the dynamic environment of *GeoGebra*.

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FACULTY OF MATHEMATICS AND INFORMATICS,
UNIVERSITY OF PLOVDIV PAISII HILENDARSKI
24 TZAR ASEN, 4000 PLOVDIV, BULGARIA
Email address: marta@uni-plovdiv.bg

MATHEMATICAL SCHOOL ACADEMIC KIRIL POPOV
11 CHEMSHIR, 4000 PLOVDIV, BULGARIA
Email address: rad_tod@abv.bg