SYNTHETIC PROOF OF DAO’S THEOREM

NGUYEN MINH HA

ABSTRACT. I will present a synthetic proof for Theorem Dao [1].

1. INTRODUCTION

In 2014, O.T. Dao has introduced the following theorem.

**Theorem 1.1.** Given a triangle $ABC$ inscribed the conic $(S)$ and two points $P, Q$ conjugated with respect to $(S). A_1, B_1, C_1$ are the second intersecting points of $PA, PB, PC$ and $(S)$ respectively. $A_2, B_2, C_2$ are the intersecting points of $QA_1, QB_1, QC_1$ and $BC, CA, AB$ respectively. Then $A_2, B_2, C_2$ are collinear.

In 2015, Giang Ngoc Nguyen proved Dao’s Theorem using the coordinate method [2]. In this article, I will present a synthetic proof for Dao’s Theorem [1].

2. A PROOF OF DAO’S THEOREM

We need two lemmas.

**Lemma 2.1.** Given a triangle $ABC$ inscribed the circle $(O)$ and two points $P, Q$ conjugated with respect to $(O). A_1, B_1, C_1$ are the second intersecting points of $PA, PB, PC$ and $(O)$ respectively. $A_2, B_2, C_2$ are the intersecting points of lines (which pass through $A_1, B_1, C_1$ and are perpendicular to $OP$) and $BC, CA, AB$ respectively. Then $A_2, B_2, C_2$ are collinear.

**Proof.** Let $X, Y, Z, X_1, Y_1, Z_1$ be the orthogonal projections of $A, B, C, A_1, B_1, C_1$ onto $PO$ respectively (fig.1).

Noticing that the triangles $PA_1B, PB_1C, PC_1A$ are similar to the triangles $PB_1A, PC_1B, PA_1C$ respectively, we have

$$
\prod \frac{A_2B}{A_2C} = \prod \frac{X_1Y}{X_1Z} = \prod \frac{X_1Y.\overrightarrow{OP}}{X_1Z.\overrightarrow{OP}} = \prod \frac{A_1B.\overrightarrow{OP}}{A_1C.\overrightarrow{OP}} = \prod \frac{(\overrightarrow{OB} - \overrightarrow{OA_1}).\overrightarrow{OP}}{(\overrightarrow{OC} - \overrightarrow{OA_1}).\overrightarrow{OP}} = \prod \frac{\overrightarrow{OB}.\overrightarrow{OP} - \overrightarrow{OA_1}.\overrightarrow{OP}}{\overrightarrow{OC}.\overrightarrow{OP} - \overrightarrow{OA_1}.\overrightarrow{OP}}
$$

$$= \prod \frac{2\overrightarrow{OB}.\overrightarrow{OP} - 2\overrightarrow{OA_1}.\overrightarrow{OP}}{2\overrightarrow{OC}.\overrightarrow{OP} - 2\overrightarrow{OA_1}.\overrightarrow{OP}} = \prod \frac{(OB^2 + OP^2 - BP^2) - (OA_1^2 + PO^2 - A_1P^2)}{(OC^2 + OP^2 - CP^2) - (OA_1^2 + PO^2 - A_1P^2)}
$$

2010 Mathematics Subject Classification. Primary 51M04; Secondary 51M15.

Key words and phrases. Dao’s Theorem, orthogonal projections.
Thus, by Menelaus’ theorem, $A_2, B_2, C_2$ are collinear. □

**Lemma 2.2.** Given a triangle $ABC$ inscribed the circle $(O)$ and two points $P, Q$ conjugated with respect to $(O)$. $A_1, B_1, C_1$ are the second intersecting points of $PA, PB, PC$ and $(O)$ respectively. $A_2, B_2, C_2$ are the intersecting points of $QA_1, QB_1, QC_1$ and $BC, CA, AB$ respectively. Then, $A_2, B_2, C_2$ are collinear.

**Proof.** Let $A_0, B_0, C_0$ be the intersecting points of $BC_1, CA_1, AB_1$ and $CB_1, AC_1, BA_1$ respectively. Let $A_3, B_3, C_3$ be the intersecting points of lines (which pass through $A_1, B_1, C_1$ and are perpendicular with $OP$) and $BC, CA, AB$ respectively (fig.2).

By Lemma 1, $A_3, B_3, C_3$ are collinear.

Hence, by Menelaus’ theorem, $\prod \frac{A_3B}{A_3C} = 1$.

It is easy to see that $A_0, B_0, C_0$ are also conjugated with $P$ with respect to $(O)$. Hence $Q, A_0, B_0, C_0$ belong to the same line which is polar of $P$ with respect to $(O)$, denoted by $\Delta,$ obviously $OP \bot \Delta$. From this, noticing that $A_1A_3, B_1B_3, C_1C_3$ are also perpendicular to $OP$, It follows that $A_1A_3, B_1B_3, C_1C_3$ are both parallel with $\Delta$.

Thus

$$\prod \frac{A_2B}{A_2C} = \prod \frac{A_2B}{A_2C} : \prod \frac{A_3B}{A_3C} = \prod \frac{A_2B}{A_2C} : \frac{A_3B}{A_3C} = \prod (BCA_2A_3) = \prod \frac{QC_0}{QB_0} = 1.$$  

From that, by Menelaus’ theorem, $A_2, B_2, C_2$ are collinear. □
Now we return to proof of theorem 1.1.
Let \( \alpha \) be the plane containing \((S)\); let \((C)\) be one cone of revolutionsuch that \((S) = (\alpha) \cap (C)\); let \(T\) be the vertex \((C)\); let \(d\) be the axis \((C)\); let \((\beta)\) be one plane which is perpendicular to \(d\) and does not pass through \(T\); let \((O) = (\beta) \cap (C)\).
It is easy to see that \((O)\) is a circles which has the center \(O = d \cap (\alpha)\).
Denote \(F\) the central projection with center \(T\), from \((\alpha)\) to \((\beta)\).
It is easy to see that \(F\) turns \((S)\) to \((O)\) ((\(O)\) without a point, \((O)\) without two points) when \((S)\) is an ellipse ((\(S)\) is a parabola, \((S)\) is a hyperbola).
Since \(P, Q\) are conjugated with respect to \((S)\), the images of \(P, Q\) through \(F\) are conjugated with respect to \((O)\).
Hence, by Lemma 2.1, the image of \(A_2, B_2, C_2\) through \(F\) are collinear.
Thus \(A_2, B_2, C_2\) are collinear.
3. ACKNOWLEDGMENT

The author thanks Prof. Doan Quynh and Master. Nguyen Van Van for their help in the improvement of this paper.

REFERENCES


HA NOI UNIVERSITY OF EDUCATION
HA NOI, VIET NAM.
Email address: minhha27255@yahoo.com