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SYNTHETIC PROOF OF DAO'S THEOREM

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ABSTRACT. I will present a synthetic proof for Theorem Dao [1].

1. INTRODUCTION

In 2014, O.T.Dao has introduced the following theorem.

Theorem 1.1. Given a triangle ABC inscribed the conic (S) and two points P, Q conjugated with respect to (S). A_1, B_1, C_1 are the second intersecting points of PA, PB, PC and (S) respectively. A_2, B_2, C_2 are the intersecting points of QA_1, QB_1, QC_1 and BC, CA, AB respectively. Then A_2, B_2, C_2 are collinear.

In 2015, Giang Ngoc Nguyen proved Dao's Theorem using the coordinate method [2]. In this article, I will present a synthetic proof for Dao's Theorem [1].

2. A PROOF OF DAO'S THEOREM

We need two lemmas.

Lemma 2.1. Given a triangle ABC inscribed the circle (O) and two points P, Q conjugated with respect to (O). A_1, B_1, C_1 are the second intersecting points of PA, PB, PC and (O) respectively. A_2, B_2, C_2 are the intersecting points of lines (which pass through A_1, B_1, C_1 and are perpendicular to OP) and BC, CA, AB respectively. Then A_2, B_2, C_2 are collinear.

Proof. Let X, Y, Z, X_1, Y_1, Z_1 be the orthogonal projections of A, B, C, A_1, B_1, C_1 onto PO respectively (fig.1).
Noticing that the triangles PA_1B, PB_1C, PC_1A are similar to the triangles PB_1A, PC_1B, PA_1C respectively, we have

$$\prod \frac{\overline{A_2B}}{\overline{A_2C}} = \prod \frac{\overline{X_1Y}}{\overline{X_1Z}} = \prod \frac{\overline{X_1Y}.\overline{OP}}{\overline{X_1Z}.\overline{OP}} = \prod \frac{\overrightarrow{A_1B}.\overrightarrow{OP}}{\overrightarrow{A_1C}.\overrightarrow{OP}} = \prod \frac{(\overrightarrow{OB}-\overrightarrow{OA_1}).\overrightarrow{OP}}{(\overrightarrow{OC}-\overrightarrow{OA_1}).\overrightarrow{OP}} = \prod \frac{\overrightarrow{OB}.\overrightarrow{OP}-\overrightarrow{OA_1}.\overrightarrow{OP}}{\overrightarrow{OC}.\overrightarrow{OP}-\overrightarrow{OA_1}.\overrightarrow{OP}}$$
$$= \prod \frac{2\overrightarrow{OB}.\overrightarrow{OP}-2\overrightarrow{OA_1}.\overrightarrow{OP}}{2\overrightarrow{OC}.\overrightarrow{OP}-2\overrightarrow{OA_1}.\overrightarrow{OP}} = \prod \frac{(OB^2+OP^2-BP^2)-(OA_1^2+PO^2-A_1P^2)}{(OC^2+OP^2-CP^2)-(OA_1^2+PO^2-A_1P^2)}$$

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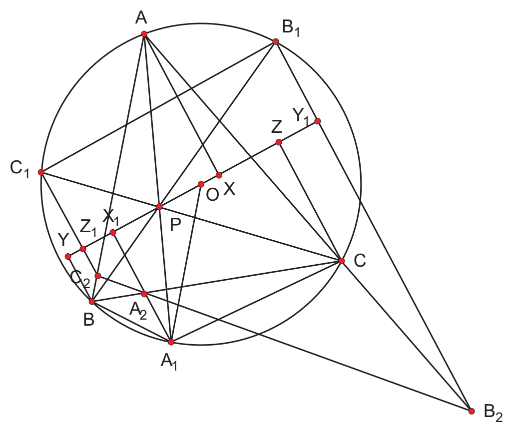

FIGURE 1

$$= \prod \frac{A_1P^2 - BP^2}{A_1P^2 - CP^2} = \prod \frac{A_1P^2 - BP^2}{B_1P^2 - AP^2} = \prod \frac{BP^2}{AP^2} = 1.$$

Thus, by Menelaus' theorem, A_2, B_2, C_2 are collinear. □

Lemma 2.2. Given a triangle ABC inscribed the circle (O) and two points P, Q conjugated with respect to (O). A_1, B_1, C_1 are the second intersecting points of PA, PB, PC and (O) respectively. A_2, B_2, C_2 are the intersecting points of QA_1, QB_1, QC_1 and BC, CA, AB respectively. Then, A_2, B_2, C_2 are collinear.

Proof. Let A_0, B_0, C_0 be the intersecting points of BC_1, CA_1, AB_1 and CB_1, AC_1, BA_1 respectively. Let A_3, B_3, C_3 be the intersecting points of lines (which pass through A_1, B_1, C_1 and are perpendicular with OP) and BC, CA, AB respectively (fig.2).
By Lemma 1, A_3, B_3, C_3 are collinear.
Hence, by Menelaus' theorem, $\prod \frac{\overline{A_3B}}{\overline{A_3C}} = 1$.
It is easy to see that A_0, B_0, C_0 are also conjugated with P with respect to (O).
Hence Q, A_0, B_0, C_0 belong to the same line which is polar of P with respect to (O), denoted by Δ, obviously $OP \perp \Delta$.
From this, noticing that A_1A_3, B_1B_3, C_1C_3 are also perpendicular to OP, It follows that A_1A_3, B_1B_3, C_1C_3 are both parallel with Δ.
Thus

$$\prod \frac{\overline{A_2B}}{\overline{A_2C}} = \prod \frac{\overline{A_2B}}{\overline{A_2C}} : \prod \frac{\overline{A_3B}}{\overline{A_3C}} = \prod \frac{\overline{A_2B}}{\overline{A_2C}} : \frac{\overline{A_3B}}{\overline{A_3C}} = \prod (BCA_2A_3) = \prod Q(BCA_2A_3)$$

$$= \prod Q(BCA_1A_3) = \prod A_1(BCQA_3) = \prod A_1(C_0B_0QA_3) = \prod \frac{\overline{QC_0}}{\overline{QB_0}} = 1.$$

From that, by Menelaus' theorem, A_2, B_2, C_2 are collinear. □

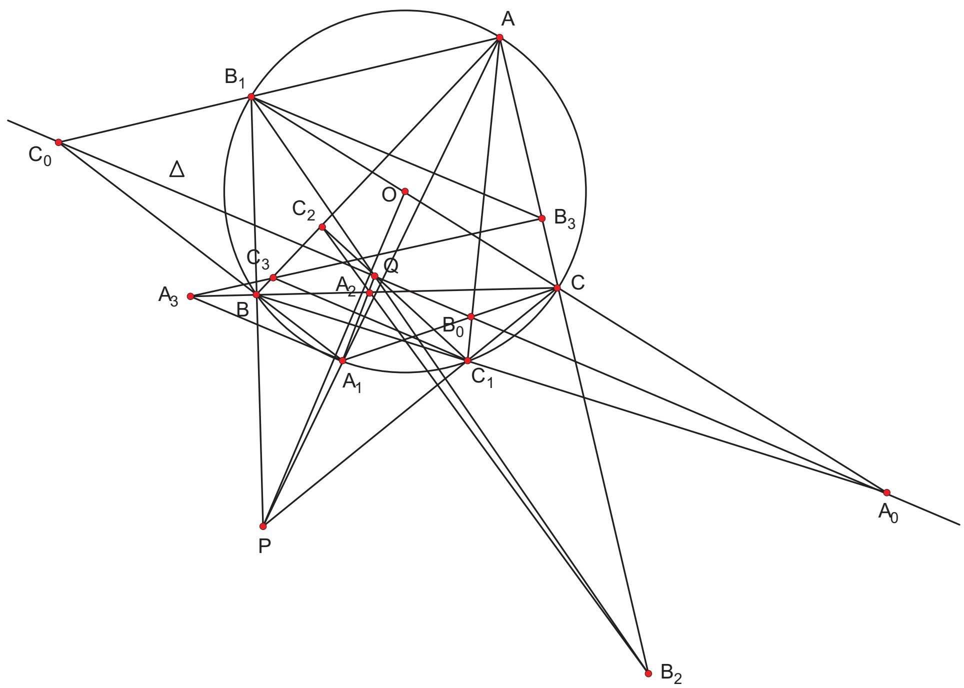

FIGURE 2

Now we return to proof of theorem 1.1.
Let (α) be the plane containing (S); let (C) be one cone of revolutionsuch that $(S) = (\alpha) \cap (C)$; let T be the vertex (C); let d be the axis (C); let (β) be one plane which is perpendicular to d and does not pass through T; let $(O) = (\beta) \cap (C)$.
It is easy to see that (O) is a circles which has the center $O = d \cap (\alpha)$.
Denote F the central projection with center T, from (α) to (β).
It is easy to see that F turns (S) to (O) ((O) without a point, (O) without two points) when (S) is an ellipse ((S) is a parabola, (S) is a hyperbola).
Since P, Q are conjugated with respect to (S), the images of P, Q through F are conjugated with respect to (O).
Hence, by Lemma 2.1, the image of A_2, B_2, C_2 through F are collinear.
Thus A_2, B_2, C_2 are collinear.

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