Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.9, (2020), Issue 1, pp.15-19

CONTACT STRUCTURES IN LIE GROUP AND DEFORMATION OF FOLIATIONS

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ABSTRACT. In this paper, we study some properties of the Riemannian structures of a contact Lie group and we give some conditions for a left invariant contact structure on a Lie group to be the deformation of a foliation.

1. INTRODUCTION

A contact form on a manifold M^{2n+1} is a differential 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$ pointwise over M. The kernel $\{\alpha = 0\}$ of α defines a maximally non-integrable smooth field of tangent hyperplanes on M^{2n+1}. The existence of contact structures on a given manifold is a fondamental question in the theory of contact geometry. Every closed oriented 3-manifold admits a contact structure (see [15]). The question remains open in higher dimensions, some answers have been obtained using surgery-like tools mainly (see [16, 10]). According to M. Gromov [11], there is a contact structure on every odd dimensional connected non-compact Lie group. Still now, in general, such contact structures are not invariant under left translations (left invariant) of the Lie group. Furthermore, the methods used by Gromov in his proofs do not, a priori, involve any kind of invariance. Beyond the geometric interest, contact Lie groups appear in a natural way in all areas using contact Geometry or Topology (for these areas, see [1, 2] and excellent review-like sources by Lutz [12] and Geiges [10]). If a Lie contact group G contain an integrable left invariant 1-form, the question is if this form is deformed to a left invariant contact form.

A foliation $\mathcal{F}$ on a contact Lie group G is deformable into contact structures if there exists a one parameter family $\mathcal{F}_t$ of hyperplane fields satisfying $\mathcal{F}_0 = \mathcal{F}$ and for all $t > 0$, $\mathcal{F}_t$ is contact. It is well known from Eliashberg-Thurston's work [8] that any oriented codimension C^1-foliation on an oriented 3-manifold can be perturbed into contact structures, except the product foliation of $\mathbb{S}^2 \times \mathbb{S}^1$ by spheres $\mathbb{S}^2$. It was then unknown if this approximation can always be done through a deformation. A particular deformation is the one called *linear*. For a foliation $\mathcal{F}$ defined by a 1-form α_0, a deformation $\mathcal{F}_t$ defined by 1-forms α_t is said to be linear if $\alpha_t = \alpha_0 + t\alpha$ where α is a 1-form on G (independent

2010 *Mathematics Subject Classification.* Primary 53B05; Secondary 53D10.
Key words and phrases. Contact metric, Lie Group, Left and right invariant vector fields, Foliation, Deformation, Contact Structure...

of t). The foliation theory has been studied by several authors (see [5], [4]...)

The aim of this paper is the study some properties of contact structures of these Lie groups having a left invariant contact form and we make some conditions in the 1-form and show that every foliation in the contact Lie group can be deformed into a contact structure.

The paper is organized as follows: in the section 2, we recall somes preliminaries results about the Lie contact and in the section 3 we will give the proof of the mains theorems.

2. Preliminaries

Let G be a Lie group, e its unit, and $\mathcal{G}$ its Lie algebra identified with the tangent space T_eG to G at e. If $X \in \mathcal{G}$, let X^+ stand for the right invariant vector field on G with value $X := X_e^+$ at e. If G has dimension $2n+1$, a right invariant differential 1-form α^+ on G is a contact form if its de Rham differential $d\alpha^+$ caps up, together with α^+, to a volume form

$$\alpha^+ \wedge (d\alpha^+)^n \neq 0$$

pointwise over G. This is equivalent to $\alpha \wedge (\partial\alpha)^n$ being a volume form in $\mathcal{G}$, where $\alpha := \alpha_e^+$ and $\partial(x,y) := -\alpha([x,y])$. In this case (G, α^+) (resp. $(\mathcal{G}, \alpha)$ is called a contact Lie group (resp. algebra). The Reeb vector field is the unique vector field ξ^+ satisfying $d\alpha(\xi^+, X^+) = 0$, $\forall$ X^+ and $\alpha^+(\xi^+) = 1$. Next we will also usually write $\partial\alpha^+$ instead of $d\alpha^+$.

For example, every 3-dimensional nonabelian Lie group is a contact Lie group, except the one (unique, up to a local isomorphism) all of whose left invariant Riemannian metrics have sectional curvature of constant sign (see [6]). Every Heisenberg Lie group $\mathbb{H}_{2n+1}$ is a contact Lie group.
Let (G, α^+) a Lie contact group of dimension $2n+1$ and ξ^+ its Reeb field. The $2n$-dimensional distribution

$$D(p) = \{v \in T_pG : \alpha^+(p)(v) = 0\},$$

which is invariant by ξ^+, is called the *contact distribution*. It carries a $(1,1)$ tensor field ϕ satisfying $\phi^2 = Id_D$ and $\phi\xi^+ = 0$.

The contact Lie group (G, α^+) carries a Riemannian metric h (not necessary unique) called a *contact metric*, adapted to α^+ and ϕ with the following conditions: for all right invariant vector fields X^+ and Y^+ in G, we have

$$h(X^+, Y^+) = h(\phi X^+, \phi Y^+) + \alpha^+(X^+)\alpha^+(Y^+), \tag{2.1}$$

$$d\alpha^+(X^+, Y^+) = 2h(X^+, \phi Y^+), \tag{2.2}$$

$$\phi^2 X^+ = -X^+ + \alpha^+(X^+)\xi^+. \tag{2.3}$$

Using the formula $d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$ we have

$$\alpha^+([\xi^+, X^+]) = \xi^+\alpha^+(X^+) \tag{2.4}$$

for all X^+. A contact metric structure on which $\mathcal{L}_{\xi^+}\phi = 0$ is called K-contact structure.

Let $\mathcal{F}$ be a foliation of codimension q on a Lie group G defined by a 1-form ω. A deformation of $\mathcal{F}$ is given by a 1-form ω_t for $t \in \mathbb{R}$ which define the family of foliation $\mathcal{F}_t$ such that $\mathcal{F}_0 = \mathcal{F}$ called the deformation of $\mathcal{F}$. More generality we give the following definition:

Definition 2.1. A foliation $\mathcal{F}$ defined by a form ω is C^k-deformed into a contact structure if there exists a C^k-deformation $\mathcal{F}_t$ defined by ω_t such that $\theta_0 = \theta$ and ω_t is a contact form for $t > 0$.

3. Main Results

Let us introduce in the beginning of this section somes properties in contact Lie groups. Before we give the following lemmas:

Lemma 3.1. *Let α^- be a left invariant form on a Lie group G and X^+ a right invariant vector field on G. Then $\mathcal{L}_{X^+}\alpha^- = 0$.*

Proof. Let α^- be a left invariant form on a Lie group G and X^+ a right invariant vector field on G. For any left invariant vector field Y^- on G we have:

$$\begin{aligned} \mathcal{L}_{X^+}\alpha^-(Y^-) &= (di_{X^+}\alpha^-)(Y^-) + i_X^+ d\alpha^-(Y^-) \\ &= Y^-\alpha^-(X^+) + d\alpha^-(X^+, Y^-) \\ &= Y^-\alpha^-(X^+) + X^+\alpha^-(Y^-) - Y^-\alpha^-(X^+) - \alpha^-[X^+, Y^-]. \end{aligned}$$

Then we have:

$$\mathcal{L}_{X^+}\alpha^-(Y^-) = 0.$$

Indeed we have $\alpha^-(Y^-)$ is a constant so $X^+(cte) = 0$ and X^+ is left invariant, Y^- right invariant so $[X^+, Y^-] = 0$. □

Lemma 3.2. *Let (G, α^+) be a contact Lie group with Reeb vector field ξ^+, then we have*

$$\xi^+ d\alpha^+(X^+, Y^+) = d\alpha^+([[\xi^+, X^+], Y^+]) + d\alpha^+([X^+, [\xi^+, Y^+]])$$

for every right invariant vector fields X^+ and Y^+ on G.

Proof. Using (2.4) we have

$$\begin{aligned} d\alpha^+([[\xi^+, X^+], Y^+]) + d\alpha^+([X^+, [\xi^+, Y^+]]) &= [\xi^+, X^+]\alpha^+(Y^+) - Y^+\alpha^+([\xi^+, X^+]) \\ &\quad -\alpha^+([[\xi^+, X^+], Y^+]) - [\xi^+, Y^+]\alpha^+(X^+) + X^+\alpha^+([\xi^+, Y^+]) \\ &\quad -\alpha^+([[Y^+, \xi^+], X^+]) \end{aligned}$$

$$\begin{aligned} d\alpha^+([[\xi^+, X^+], Y^+]) + d\alpha^+([X^+, [\xi^+, Y^+]]) &= \xi^+X^+\alpha^+(Y^+) - \xi^+Y^+\alpha^+(X^+) \\ &\quad -\xi^+\alpha^+([X^+, Y^+]) \end{aligned}$$

$$d\alpha^+([[\xi^+, X^+], Y^+]) + d\alpha^+([X^+, [\xi^+, Y^+]]) = \xi^+ d\alpha^+(X^+, Y^+)$$

□

Now we give the following theorem

Theorem 3.1. *Let (G, α^+, h, ϕ^+) be a contact Lie group where h is the contact metric adapted to α^+ and ϕ^+. Then G is K-contact Lie group if only if $\mathcal{L}_{\xi^+}h = 0$.*

Proof. Suppose that G is K-contact then $\mathcal{L}_{\xi^+}\phi = 0$. By definition we have $(\mathcal{L}_{\xi^+}h)(X^+, \phi Y^+) = 0$ for any Y^+. Now we must show $(\mathcal{L}_{\xi^+}h)(X^+, \xi^+) = 0$ for any X^+.

$$\begin{aligned}
(\mathcal{L}_{\xi^+}h)(X^+, \xi^+) &= \xi^+ h(X^+, \xi) - h(\mathcal{L}_{\xi^+}X^+, \xi^+) - h(X^+, \mathcal{L}_{\xi^+}\xi^+) \\
&= \xi^+ h(X^+, \xi) - h([\xi^+, X^+], \xi^+) \\
&= \xi^+ \alpha^+(X^+) - \alpha^+([\xi^+, X^+]) \\
&= \alpha^+([\xi^+, X^+]) - \alpha^+([\xi^+, X^+])
\end{aligned}$$

so $(L_{\xi^+}h)(X^+, \xi^+) = 0$.
Conversely for every right invariant vector fields X^+ and Y^+ on G, we have

$$\begin{aligned}
(\mathcal{L}_{\xi^+}h)(X^+, \phi Y^+) &= \xi^+ h(X^+, \phi Y^+) - h(\mathcal{L}_{\xi^+}, \phi Y^+) - h(X^+, \mathcal{L}_{\xi^+}(\phi Y^+)) \\
&= \xi^+ h(X^+, \phi Y^+) - h([\xi^+, X^+], \phi Y^+) - h(X^+, [\xi^+, \phi Y^+].
\end{aligned}$$

Using the formula $\mathcal{L}_X(fY) = (\mathcal{L}_X f)(Y) + f\mathcal{L}_X Y$, we have:

$$\begin{aligned}
(\mathcal{L}_{\xi^+}h)(X^+, \phi Y^+) &= \xi^+ h(X^+, \phi Y^+) - h([\xi^+, X^+], \phi Y^+) \\
&\quad - h(X^+, (\mathcal{L}_{\xi^+}\phi)(Y^+)) - h(X^+, \phi[\xi^+, Y^+]).
\end{aligned}$$

From the Lemma 3.2 and the relation (2.4), we have:

$$\begin{aligned}
(\mathcal{L}_{\xi^+}h)(X^+, \phi Y^+) &= \frac{1}{2}\big(d\alpha^+([\xi^+, X^+], Y^+) + d\alpha^+(X^+, [\xi^+, Y^+])\big) \\
&\quad - \frac{1}{2}d\alpha^+([\xi^+, X^+], Y^+) - h\big(X^+, (\mathcal{L}_{\xi^+}\phi)(Y^+)\big) \\
&\quad - \frac{1}{2}d\alpha^+(X^+, [\xi^+, Y^+]). \\
(\mathcal{L}_{\xi^+}h)(X^+, \phi Y^+) &= -h(X^+, (L_{\xi^+}\phi)(Y^+)).
\end{aligned}$$

Therefore, if $\mathcal{L}_{\xi^+}h = 0$ then $\mathcal{L}_{\xi^+}\phi = 0$ for all Y^+.

□

Let α be a integrable 1-form in a Lie group G i.e $\alpha \wedge d\alpha = 0$. We want to know in what condition α is deformed into a contact form. Since the kernel $\{\alpha = 0\}$ of α is a foliation in G, the question is if every foliation in G is deformed in contact structure. Moreover if ζ is a contact structure and ξ a codimension 1 foliation on M, then following J.B. Etnyre [9], ζ is the deformation of ξ if there exist a one parameter family ξ_t such that $\xi_0 = \xi$, $\xi_1 = \zeta$ and for all $t > 0$, ξ_t is a contact structure. He obtains the following theorem on 3-dimension.

Theorem 3.2. [9] *On a closed oriented 3-manifold, every positive or negative contact structure is the C^∞-deformation of a codimension 1 foliation.*

We generalize the Etnyre result on $2n + 1$ dimensional Lie group.

Theorem 3.3. *Let $\mathcal{F}$ be a foliation defined by a non-singular right invariant 1-form α_0^+ in a K-contact Lie group G of dimension $2n + 1$. If α_0^+ is harmonic relatively to the contact metric h, then the foliation $\mathcal{F}$ is deformed to a contact structure.*

Proof. Let α^+ the contact form on a K-contact Lie group G and ξ^+ its Reeb vector field. Let α_0^+ the harmonic non-singular 1-form on G which define $\mathcal{F}$. We have α_0^+ is basic relatively to ξ^+ (see [13]). The Reeb vector field associate to $\alpha_t^+ = \alpha_0^+ + t\alpha^+$ is $\frac{\xi^+}{t}$. Indeed we have

$$\alpha_t^+(\frac{\xi^+}{t}) = 1$$

and

$$d\alpha_t^+(\frac{\xi^+}{t}, X^+) = 0.$$

for any left invariant vector X^+ in G. If α^+ is K-contact with the contact metric h then, according to the theorem 3.1, $\frac{\xi^+}{t}$ is Killing relatively to the contact metric h.
Using $h(\frac{\xi^+}{t}, Y^+) = \alpha_t^+(Y^+)$ for any Y^+, we have $\mathcal{L}_{\frac{\xi^+}{t}}\alpha_t^+ = 0$. According to [14] for all t, α_t^+ is K-contact. □

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