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A SIMILARITY PROPERTY OF TRAPEZIA

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ABSTRACT. In this article we study a property of trapezia connected with constructions using isometries. The constructions create for each quadrangle another one, which when the starting quadrangle is a trapezium, then the constructed quadrangle is a similar to it trapezium. This property is proved to be characteristic of non-isosceli trapezia.

1. INTRODUCTION

Our starting point here is the configuration of figure 1, which is created by a simple recipe from the quadrangle $EFGH$. Next lemma formulates this construction using for the side-lines the notation $\{e = HE\ ,\ f = EF\ ,\ g = FG\ ,\ h = GH\ \}$ and $\{a = EG\ ,\ b = FH\}$ for the diagonal-lines of $EFGH$. I use also the same symbol to denote a line and the reflection on that line. The composition of reflections is meant and applied to points of the plane, as usual, from right to left.

Lemma 1. *Let $r_O = hgfe$ be the composition of successive reflections on the corresponding sides of a generic convex non-cyclic quadrangle $EFGH$. Then r_O is a rotation whose center and rotation angle ϕ are determined by the quadrangle's data.*

Proof: It is known and trivial to prove ([1, I, p.50]), that the product (=composition) of two reflections in intersecting lines $\{a, b\}$ is a rotation by double the measure of the oriented angle $\widehat{ab}$ and this rotation can be represented by any other pair of lines $\{a', b'\}$ intersecting at the same point with $\{a, b\}$ and forming there the same angle.

Thus, the partial compositions (fe) and (hg) are rotations, hence they can be represented by some other reflections $\{(ae')\ ,\ (h'a)\}$ such that the angles $\{\widehat{ae'}\ ,\ \widehat{h'a}\}$ are equal to the corresponding $\{\widehat{fe}\ ,\ \widehat{hg}\}$. From this remark follows the construction of the center O as the intersection point of $e' \cap h'$, the rotation angle being the double in measure of the angle $\widehat{e'h'}$. Assuming the quadrangle positively oriented and taking into account that $\{\widehat{e'a} = \widehat{E}\ ,\ \widehat{h'a} = \widehat{G}\}$, we see that the angle of rotation is

$$\phi \ = \ 2\widehat{e'h'} \ = \ 2(\pi - \widehat{E} - \widehat{G}) \ = \ \widehat{F} + \widehat{H} - \widehat{E} - \widehat{G}\,. \tag{1.1}$$

□

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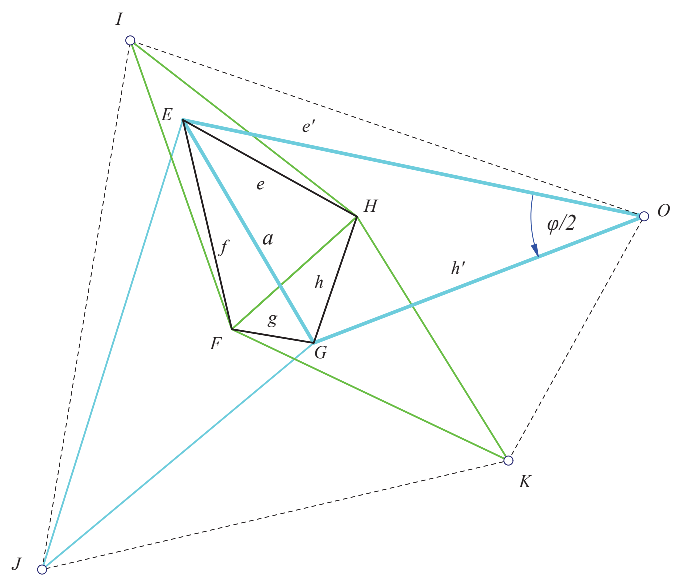

FIGURE 1. Reflecting on the sides of the quadrangle

Repeating the previous construction with the products of reflections, resulting from $(hgfe)$ by cyclically permuting the letters, we come to three other rotations $\{r_K, r_J, r_I\}$ by the same angle up to sign. Next table and subsequent lemma summarizes the result defining the new *"generated"* quadrangle $OKJI$.

center	composition	angle	rotation
O	$(hgfe)$	$+\phi$	r_O
K	$(ehgf)$	$-\phi$	r_K
J	$(fehg)$	$+\phi$	r_J
I	$(gfeh)$	$-\phi$	r_I

Lemma 2. *The rotation $r_O = (hgfe)$ together with the rotations $\{r_K, r_J, r_I\}$ of the preceding table define by their centers the "generated" quadrangle $OKJI$ of $EFGH$. The pairs of opposite vertices $\{(O,J), (K,I)\}$ lie symmetrically w.r. to the diagonals $\{EG, FH\}$, thus the angle of the diagonals is the same for the two quadrangles.*

2. RELATIONS BETWEEN THE TWO QUADRANGLES

Next theorem expresses a basic property relating the two quadrangles considered in the preceding section.

Theorem 3. *The side-lines of a generic convex non-cyclic quadrangle $EFGH$ are orthogonal bisectors of corresponding sides of the generated quadrangle $KJIO$ of $EFGH$. More precisely*

$$e = HE \perp OK, \quad f = EF \perp KJ, \quad g = FG \perp JI, \quad h = GH \perp IO.$$

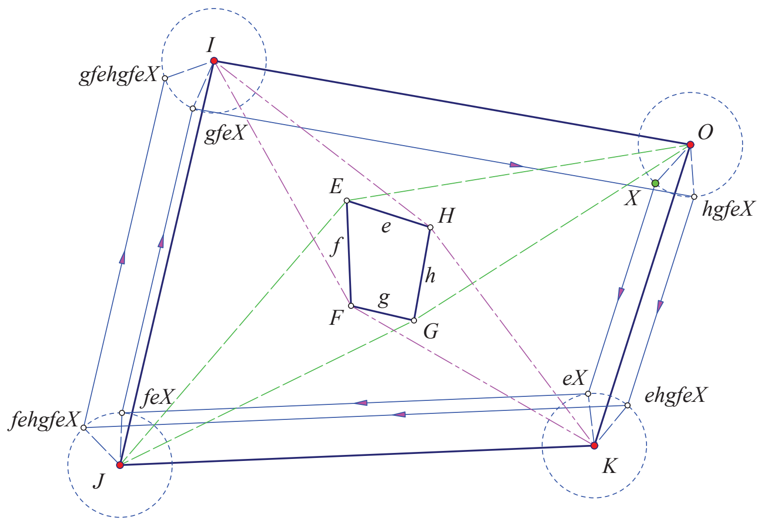

FIGURE 2. Reflecting on the sides of the quadrangle

Proof: Consider the orbit of a point X, i.e. the transformed points under the reflections on the side-lines of $EFGH$:

$$X\,,\quad eX\,,\quad feX\,,\quad gfeX\,,\quad hgfeX\,,\quad ehgfeX\,,\quad fehgfeX\,,\,\ldots$$

To see the orthogonality $e = HE \perp OK$ we look at the segments with endpoints related by the reflection e :

$$(X\,,\,eX) \quad \text{and} \quad (hgfeX\,,\,ehgfeX)\,.$$

Both are orthogonal at their middle to e, hence they are parallel. Their endpoints are also related by the rotations $\{r_O\,,\,r_K\}$ which have opposite oriented rotation angles $\pm\phi$:

$$r_O(X) \;=\; hgfeX \quad \text{and} \quad r_K(eX) \;=\; ehgfeX\,.$$

It follows that the quadrangle with these endpoints is an isosceles trapezium and the isosceli based at its lateral sides are equal. This implies that KO is also orthogonal to e at its middle. Analogous arguments show the remaining orthogonalities. □

Corollary 4. *The angles of the two quadrangles, which are inversely oriented, are correspondingly supplementary, the positive measures of their angles satsifying:*

$$\widehat{K}+\widehat{E} \;=\; \widehat{J}+\widehat{F} \;=\; \widehat{I}+\widehat{G} \;=\; \widehat{O}+\widehat{H} \;=\; \pi\,. \tag{2.1}$$

3. THE CASE OF TRAPEZIA

Here we continue with the notation and conventions of the preceding section. In the case of trapezia, their fundamental property, to have two pairs of consecutive angles supplementary, combined with corollary 4, implies that the two quadrangles have equal angles (See Figure 3). More is true:

Theorem 5. *In the case $EFGH$ is a trapezium, the generated quadrangle is a similar to it trapezium.*

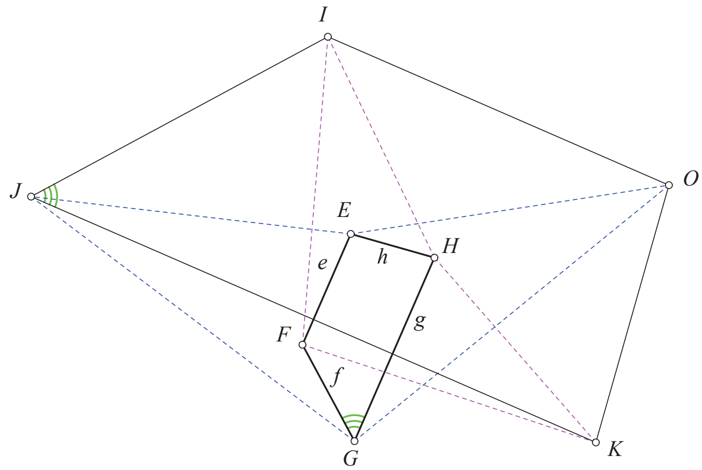

FIGURE 3. The trapezium case

Proof: The proof results from the equality of angles and the equality of angles of the diagonals of the two quadrangles. Latter results from that fact, that the diagonals of $OKJI$ are orthogonally bisected by corresponding diagonals of $EFGH$. The proof results then from the following lemma. □

Lemma 6. *Two trapezia with equal angles and equal angles of diagonals, in the proper orientation, are similar.*

Proof: Assume that one of the trapezia is $ABCD$ with AB one of its parallel sides, the non-parallel sides intersecting at the point E. Then, place a similar to the second trapezium so, that one of its parallel sides coincides with AB and its non-parallel sides intersect also at E (see figure 4). Then the other parallel side of this trapezium, FG say, must coincide with CD. This

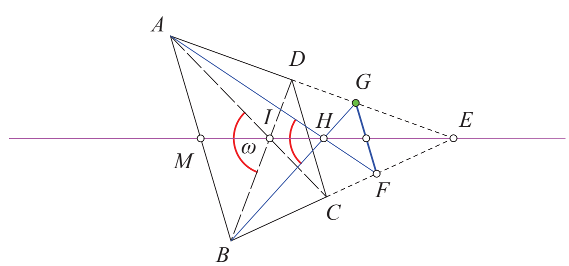

FIGURE 4. Similarity of trapezia

because in the contrary case, the intersection points $\{I,H\}$ of their diagonals would be different, would be viewing the segment AB under the same angle $\omega = \widehat{AIB} = \widehat{AHB}$ and also lie on the median EM of $\triangle ABE$, which is impossible. □

4. Similarity center and ratio

Here we continue with the notation and conventions of the preceding sections. The center S of the similarity s mapping the trapezium $EFGH$ to $KOJI$ is found by the standard procedure, which locates the similarity center ([1, II, p.43]) of the oriented segments $\{HG\ ,\ KJ\}$. Accord-

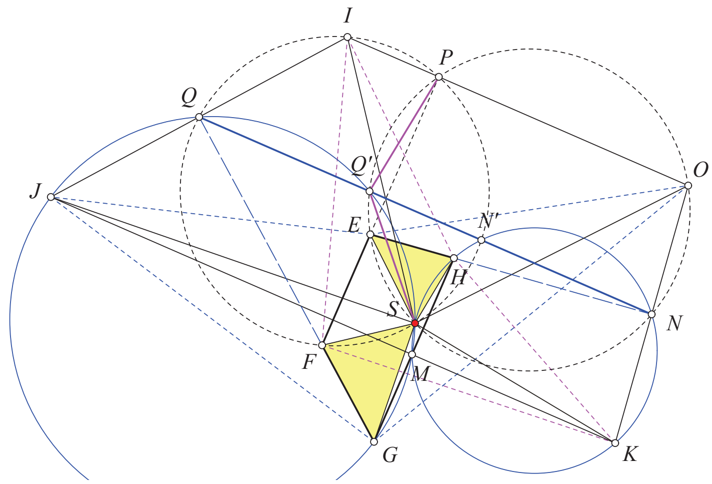

Figure 5. The similarity center

ing to this, point S is the second intersection of the circles $\{(MGJ)\ ,\ (MHK)\}$, where M is the intersection point of the two, corresponding under the similarity, oriented sides (See Figure 5). Next theorem identifies this point with the center of a similarity more directly associated to the trapezium.

Theorem 7. *The following are valid properties of the configuration of figure 5:*

1. *Points* $\{Q = FG \cap IJ\ ,\ N = EH \cap OK\}$ *are respectively on the circles* $\{(SGJ)\ ,\ (SHK)\}$.
2. *Line* QN *is parallel to the parallel sides of the trapezium.*
3. *The second intersection points* $\{Q'\ ,\ N'\}$ *of line* QN *with the circles* $\{(SEO)\ ,\ (SFI)\}$ *are on line* QN.
4. *Triangles* $\{SFG\ ,\ SHE\}$ *are similar.*
5. *The similarity center* S *coincides with the center of the similarity mapping the oriented side* EH *to* FG.

Proof: *Nr-1*: since FG is the medial line of JI, point Q is viewing JG under a right angle, as does S. Analogous is the proof for N.

Nr-2: is trivial, since $\{Q, N\}$ are respectively the middles of $\{JI, OK\}$.

Nr-3: $\widehat{PQ'N} = 180° - \widehat{PON} = \widehat{QNO}$. This shows that Q' is on QN. Analogously is seen that N' is on QN.

Nr-4: $\widehat{SEH} = \widehat{SQ'N} = \widehat{QGS}$. Analogously is seen that $\widehat{SHE} = \widehat{SFG}$.

Nr-5: follows directly from the previous *nr*. □

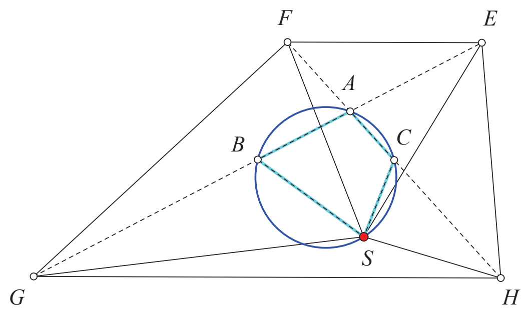

FIGURE 6. The harmonic quadrangle

Theorem 8. *The similarity center S is the fourth vertex of the harmonic quadrangle whose three other vertices are the intersection point A of the diagonals of $EFGH$ and the middles $\{B, C\}$ of the diagonals (See Figure 6).*

Proof: This is an earlier result of mine concerning the similarity mapping the directed non-parallel sides EH to FG ([2]). □

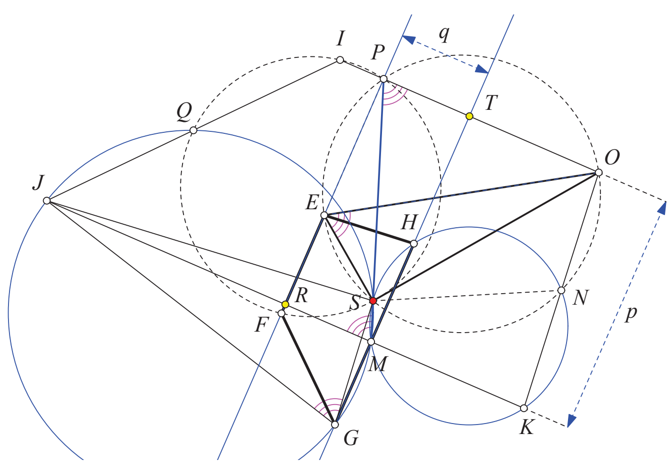

FIGURE 7. The similarity ratio p/q

Theorem 9. *The similarity ratio of s is $r = p/q$, where p is the distance of the parallel sides and q is the distance of the medial lines of the parallel sides (See Figure 7).*

Proof: The proof results by observing first that point S is on line MP. This follows from the similarity of the right-angled triangles $\{SGJ\ ,\ SEO\}$, since $\{s(E) = O\ ,\ s(G) = J\}$. From the cyclic quadrangle $SMGJ$ we have $\widehat{SMJ} = \widehat{SGJ}$ and from the cyclic quadrangle $SPON$ we have $\widehat{SEO} = \widehat{SPO}$ hence $\widehat{SPO} = \widehat{SMJ}$ showing the collinearity. Thus, the similarity ratio is realized by the ratio of the orthogonal sides of the right-angled triangle PMR, where R is the middle of JK. This proves the theorem. □

5. The converse

In the converse case we start with a generated $OKIJ$, which is inversely oriented to $EFGH$ and similar to it. There are four cases of pairings of the angles of the two quadrangles:

(1) $(\widehat{E}, \widehat{F}, \widehat{G}, \widehat{H}) = (\widehat{O}, \widehat{K}, \widehat{J}, \widehat{I})$. This is the *normal* case, for which the condition $\widehat{K}+\widehat{E} = 180^\circ$ implies that $\widehat{K}+\widehat{O} = 180^\circ$, hence the claim.

(2) $(\widehat{E}, \widehat{F}, \widehat{G}, \widehat{H}) = (\widehat{K}, \widehat{J}, \widehat{I}, \widehat{O})$. In this case equations (2.1) imply that the quadrangles are rectangles, which is excluded by the assumption of non-cyclicity.

(3) $(\widehat{E}, \widehat{F}, \widehat{G}, \widehat{H}) = (\widehat{J}, \widehat{I}, \widehat{O}, \widehat{K})$. Again the condition $\widehat{K}+\widehat{E} = 180^\circ$ implies that $\widehat{K}+\widehat{J} = 180^\circ$ and similarly $\widehat{J}+\widehat{I} = 180^\circ$, implying that the quadrangles are parallelograms.

(4) $(\widehat{E}, \widehat{F}, \widehat{G}, \widehat{H}) = (\widehat{I}, \widehat{O}, \widehat{K}, \widehat{J})$. Now the condition $\widehat{K}+\widehat{E} = 180^\circ$ is equivalent to $\widehat{K}+\widehat{I} = 180^\circ$ i.e. the quadrangles are cyclic, which is excluded by assumption.

This, together with the preceding discussion, completes the proof of next theorem.

Theorem 10. *The convex non-cyclic quadrangle $EFGH$ is similar to its generated $OKJI$ if and only if it is a trapezium.*

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