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THE ARBELOS IN WASAN GEOMETRY: CHIBA'S PROBLEM

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Abstract. We consider a sangaku problem involving an arbelos with two congruent circles, and give several conditions where the two congruent circles appear. Also we show several pairs of congruent circles and Archimedean circles.

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1. Introduction

We consider the arbelos appeared in Wasan geometry, and consider an arbelos formed by three semicircles α, β and γ with diameters AO, BO and AB, respectively for a point O on the segment AB. The radii of α and β are denoted by a and b, respectively. We consider the following sangaku problem proposed by Chiba (千葉栄治安規) in 1880 [5] (see Figure 1).

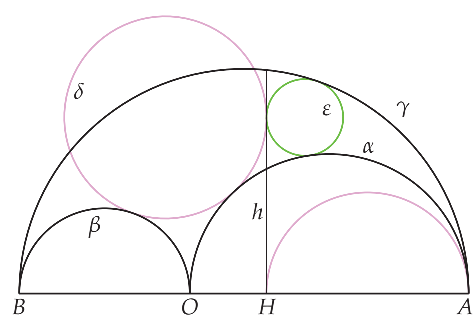

Figure 1.

Problem 1. *For a point H on the segment AO, let h be the perpendicular to AB at H. Let δ be the circle touching α and β externally and h, and let ε be the incircle of the curvilinear triangle made by α, γ and h. If δ and ε touch and δ is congruent to the circle of diameter AH, find the radius of ε in terms of a and b.*

Circles of radius $r_{\rm A} = ab/(a+b)$ are said to be Archimedean. In this paper we generalize the problem and consider two circles touching at a point on a perpendicular to the line AB. We give several conditions in which the circle δ is congruent to the circle of diameter AH together with Archimedean circles.

2. Circles touching a perpendicular to AB at the same point

In this section we consider the circles δ and ε in Problem 1 in a general way. For this purpose, we redefine two circles δ and ε as follows: Let α' be the reflection of α in AB. Similarly β' and γ' are defined. For a point H on the line AB, let h be the perpendicular to AB at H. If B lies between A and H, δ is a circle touching the semicircle β' internally and h, and ε is a circle touching the semicircle γ' internally and h (see Figure 2). If H lies between A and B, then δ is a circle touching the semicircle β externally (resp. internally) and h from the side opposite to A (resp. B), and ε is a circle touching the semicircle γ internally (resp. externally) and h from the side opposite to B (resp. A) (see Figures 3, 4 and 5). If A lies between B and H or $H = A$, δ is a circle touching β externally and h, and ε is a circle touching γ externally and h (see Figure 6). Notice that no definition is given for the case $H = B$.

The circle δ has signed radius d, where the sign is plus if δ touches β externally, otherwise minus, also the circle ε has signed radius e, where the sign is plus if ε touches γ or γ' internally otherwise minus. We use a rectangular coordinate system with origin O such that the farthest point on α from AB has coordinates (a, a). Therefore the centers of δ and ε lie on the region $y \geq 0$ by the definition of the circles.

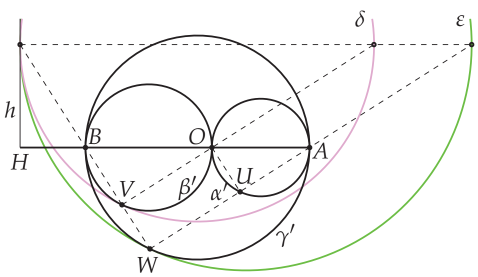

Figure 2: $d < 0, e > 0$.

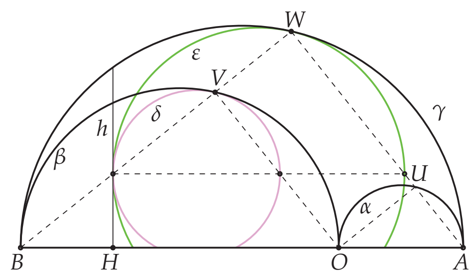

Figure 3: $d < 0, e > 0$.

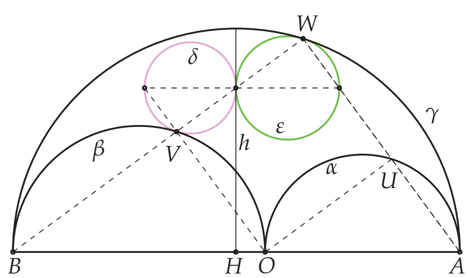

Figure 4: $d > 0, e > 0$.

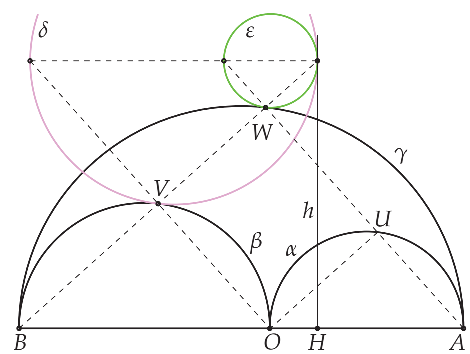

Figure 5: $d > 0, e < 0$.

Theorem 2.1. *If $H \neq B$, and δ touches β or β' at a point V and ε touches γ or γ' at a point W, the following statements are equivalent.*
(i) *The circles δ and ε touch.*
(ii) $a = d + e$.
(iii) *The points B, V and W are collinear.*
(iv) *The lines AW and OV are parallel.*
(v) *If AW meets α or α' again in a point U, then $OUWV$ is a rectangle.*

Proof. Let $x = i$ be an equation of h. The circles δ and ε touch if and only if their centers have the same y-coordinate, which is equivalent to

$$(b+d)^2 - ((i-d)+b)^2 = (a+b-e)^2 - (i+e-(a-b))^2, \tag{1}$$

since the centers of γ, δ and ε have x-coordinates $a-b$, $i-d$ and $i+e$, respectively. While (1) is equivalent to $(2b+i)(d+e-a) = 0$. Therefore (1) is equivalent to $a = d+e$ by $i \neq -2b$. Hence (i) and (ii) are equivalent. Since V is one of the centers of similitude of δ and the circle made by β and β', the line BV passes through the point of tangency of δ and h, for the tangent of β at B is parallel to h. Similarly BW passes through the point of tangency of ε and h. Therefore B, V and W are collinear if and only if the two points of tangency coincide. Hence (i) and (iii) are equivalent. Since AW and OV are perpendiculars to BW and BV, respectively, (iii) and (iv) are equivalent, also (iii) and (v) are equivalent. □

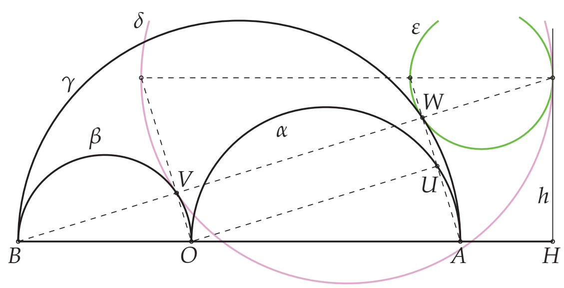

Figure 6: $d > 0, e < 0$.

Let us assume that the circles δ and ε touch in the theorem. If the circle δ touches h at the point of intersection of γ and h, then ε degenerates to the point W, i.e., $e = 0$ (see Figures 7). Also if ε touches h at the point of intersection of β and h, then δ degenerates to the point V, i.e., $d = 0$ (see Figures 8). Therefore we get the next corollary. A part of this corollary can be found in [4].

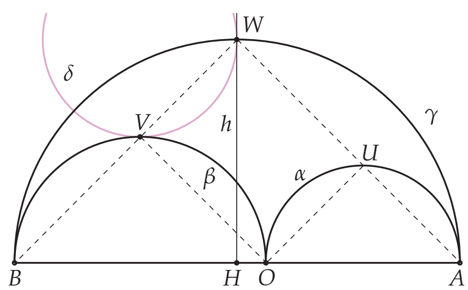

Figure 7.

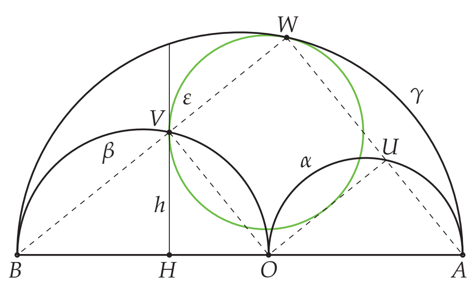

Figure 8.

Corollary 2.1. *For a point $H \neq B$ on the segment AB, let h be the perpendicular to AB at H. If a circle ζ touches β externally (resp. γ internally) and h at the point of intersection of h and γ (resp. β) from the side opposite to A (resp. B), then ζ has radius a.*

3. Chiba's problem

We now consider the figure of Problem 1 and give conditions that the circle δ and the circle of diameter AH are congruent (see Figure 9). Let T be the point of tangency of α and the tangent of α from B, which has coordinates $(2ab/t, 2av/t)$, where $t = a+2b$ and $v = \sqrt{(a+b)b}$. It is known that the Archimedean circle touching γ internally and the axis from the side opposite to B touches α externally at the point T. We will show that there is an unexpected Archimedean circle related to the point T. For two points P and Q, we denote the circle of center P passing through Q by $P(Q)$, and the circle of diameter PQ by (PQ).

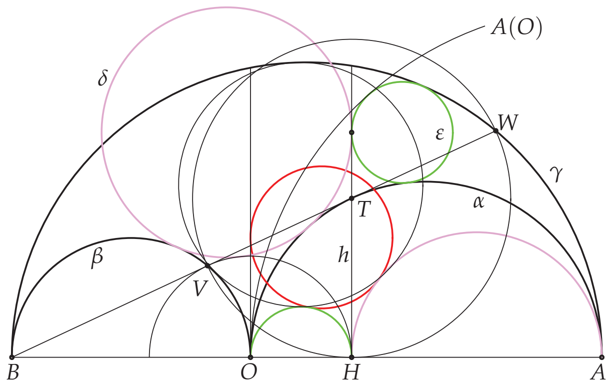

Figure 9.

Theorem 3.1. *Let h be the perpendicular to AB at H for a point H on the segment AO. If δ is the circle touching the semicircles α and β externally and h, and ε is the incircle of the curvilinear triangle made by α, γ and h, then the following statements are equivalent.*
(i) *The line h passes through the point T.*
(ii) *The circles δ and (AH) are congruent.*
(iii) *The circles ε and (HO) are congruent.*
(iv) *The circles δ and ε touch.*
(v) *The two circles touching (HO) and (AH) externally and the axis are Archimedean.*
(vi) *There is a circle touching β, (HO), (AH) externally and γ internally.*
In this event the following statements are true.
(vii) *The radii of δ and ε equal $a(a+b)/t$ and ab/t, respectively.*
(viii) *If the line BT meets β and γ in points V and W, respectively, then the two points lie on the circle $T(H)$.*
(ix) *The circle δ touches the circle $O(H)$ externally and the circle ε touches the circle $A(O)$ internally.*

Proof. We assume that $|HO| = i$, and the circles δ and ε have radii d and e, respectively. Since the center of δ has x-coordinate $i - d$, we have

$$(a+d)^2 - (a-(i-d))^2 = (b+d)^2 - (-b-(i-d))^2$$

Solving the equations for d, we have

$$d = \frac{(a+b)i}{2b}. \tag{2}$$

Therefore (ii) holds if and only if $(a+b)i/(2b) = (2a-i)/2$, which is equivalent to $i = 2ab/t$, i.e., (i) and (ii) are equivalent. Similarly (i) and (iii) are equivalent, where we solve the equation $(a+e)^2 - (a-(i+e))^2 = (a+b-e)^2 - ((a-b)-(i+e))^2$, for e and get

$$e = \frac{b(2a-i)}{2(a+b)}. \tag{3}$$

Since $d + e = a$ holds if and only if $i = 2ab/t$ by (2) and (3), (i) and (iv) are equivalent by the equivalence of (i) and (ii) of Theorem 2.1. Let r be the radius of the circles touching the circles (HO) and (AH) externally and the axis. Then we have

$$\left(r+\frac{i}{2}\right)^2 - \left(r-\frac{i}{2}\right)^2 = \left(r+\frac{2a-i}{2}\right)^2 - \left(r-\frac{2a+i}{2}\right)^2.$$

This implies $r = ai/(2a-i)$. Therefore $r = r_A$ is equivalent to $i = 2ab/t$, i.e., (i) and (v) are equivalent. If the circle touching β, (HO) and (AH) externally has center with coordinates (x,y) and radius r_1, then $(x+b)^2+y^2 = (r_1+b)^2$, $(x-i/2)^2+y^2 = (r_1+i/2)^2$ and $(x-((a+i/2))^2+y^2 = (r_1+(a-i/2))^2$ hold, and we get $r_1 = a(2b+i)i/(4ab-(2b+i)i)$. Similarly, if the circle touching β and (HO) externally γ internally has center with coordinates (x,y) and radius r_2, we have $(x+b)^2+y^2 = (r_2+b)^2$, $(x-i/2)^2+y^2 = (r_2+i/2)^2$ and $(x-(a-b))^2+y^2 = (-r_2+a+b)^2$. Hence we have $r_2 = ab(2b+i)/(2b^2+(a+b)i)$. Since $r_1 = r_2$ if and only if $i = 2ab/t$, (i) and (vi) are equivalent. The part (vii) follows from the equivalence of (i), (ii), (iii) and the x-coordinate of T. The part (viii) follows from that the points V and W have coordinates $(-2a^2b/t^2, 4abv/t^2)$ and $(2ab(3a+4b)/t^2, 4a(a+b)v/t^2)$, respectively. The part (ix) follows from that the centers of δ and ε have coordinates $(a(b-a)/t, 2\sqrt{2}av/t)$ and $(3ab/t, 2\sqrt{2}av/t)$, respectively. □

The part (vii) gives an answer of Problem 1. If h passes through T, then the circle $O(H)$ touches the line BT at V and the part (ix) shows that δ touches the circle $O(A)$ internally and ε touches the circle $A(H)$ externally. We proved the equivalence of (i) and (iii) and obtained the radius of ε in this event in [3].

4. Circles of radius $a/2$ touching a perpendicular to AB at the same point

In this section we consider the special case of section 2, in which the circles δ and ε are congruent and have radius $a/2$. The two circles touch in this case by the equivalence of (i) and (ii) in Theorem 2.1. We denote the center of α by M.

Theorem 4.1. *For a point $H \neq B$ on the segment AB, let h be the perpendicular to AB at H. Assume that δ is a circle touching β externally and h from the side opposite to A, ε is a circle touching γ internally and h from the side opposite to B. Then the two circles have radius $a/2$ if and only if they touch at a point on the circle (BM).*

Proof. We assume that δ and ε have radius $a/2$ and touch at a point with coordinates (x,y). Since the distance between the centers of ε and γ equals $a/2+b$ and the centers of γ and ε have coordinates $(a-b,0)$ and $(x+a/2,y)$, we have

$$\left(x-\left(\frac{a}{2}-b\right)\right)^2+y^2 = \left(\frac{a}{2}+b\right)^2. \tag{4}$$

This is an equation of the circle (BM). Conversely, we assume that δ and ε have radii d and e, respectively, and touch at a point with coordinates (x,y) satisfying (4) and $x \neq -2b$. Since the center of δ has coordinates $(x-d,y)$, we get $(x-d-(-b))^2+y^2 = (b+d)^2$. Solving the equation for d, we have

$$d = \frac{1}{2}\left(x+\frac{y^2}{2b+x}\right).$$

This implies

$$d-\frac{a}{2} = \frac{1}{2(2b+x)}\left(\left(x-\left(\frac{a}{2}-b\right)\right)^2+y^2-\left(\frac{a}{2}+b\right)^2\right) = 0$$

by (4). Therefore we get $d = a/2$. Then we also have $e = a/2$ by the equivalence of (i) and (ii) in Theorem 2.1. □

Remark 4.1. The point B is the external center of similitude of the circle (BM) and the circle made by β and β'. Also it is the external center of similitude of (BM) and the circle made by γ and γ'. Therefore if a line passing through B meets β, (BM) and γ again in points P_1, P_2 and P_3, respectively, then $|BP_1| : |BP_2| : |BP_3| = b : a/2+b : a+b$. This implies that P_2 is the midpoint of P_1P_3.

We now assume that the circles δ and ε satisfy the hypothesis of Theorem 4.1. We will see that the point T is closely related to the several notable special cases of Theorem 4.1.

The point T lies on the circle (BM) by Theorem 3.1(vii) and Remark 4.1. Therefore if δ and ε touch at T, they have radius $a/2$ (see Figure 10).

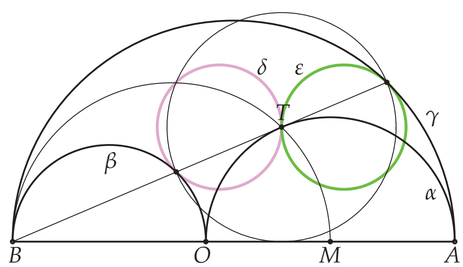

Figure 10.

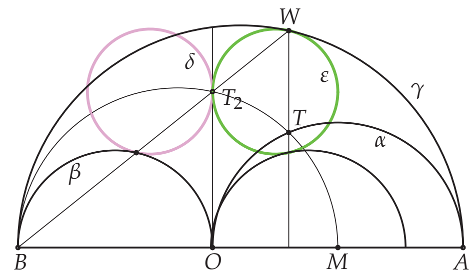

Figure 11.

We call the radical axis of α and β the axis. We assume that the perpendicular from T to AB meets γ in a point W. Then W has coordinates $(2ab/t, 2(a+b)\sqrt{2ab}/t)$ (see Figure 11). Hence if T_2 is the point of intersection of the line BW and the axis, it has coordinates $(0, \sqrt{2ab})$, which satisfy (4). Therefore T_2 lies on (BM), and if δ and ε touch at T_2 they have radius $a/2$. It is trivial that the reflection of β in the axis touches ε externally, also the reflection of γ in the axis touches δ internally. We note that the sangaku problem proposed by Satoh (佐藤幸吉定寄) in 1850 states that the circle touching the reflection of β in the axis externally, γ internally and the axis from the side opposite to B has radius $a/2$ [2, 5].

Let h_1 be the the perpendicular to AB at the center of γ. We assume that T_3 is the point of intersection of the lines AT and h_1 (see Figure 12). Then T_3 has coordinates $(a-b, v)$, which satisfy (4). We assume that δ and ε touch at T_3. Then they have radius $a/2$ and δ is the reflection of ε in h_1. Hence δ also touches γ and the line AT passes through the point of tangency of γ and δ by the equivalence of (i) and (iii) of Theorem 2.1, where its coordinates are $(-2b^2/t, 2(a+b)v/t)$. While the circles β and δ touch at the point with coordinates $(-2b^2/t, 2bv/t)$. Therefore the perpendicular from the point of tangency of γ and δ to AB also passes through the point of tangency of β and δ. Let h_2 be the perpendicular to AB at the center of (BM). The center of δ has x-coordinate $a - b - a/2 = a/2 - b$, which coincides with the x-coordinate of the center of (BM). Therefore δ is symmetric in h_2. This implies that the reflection of ε in h_2 touches β and δ externally and the perpendicular from the point of tangency of this circle and δ to AB passes through the center of β.

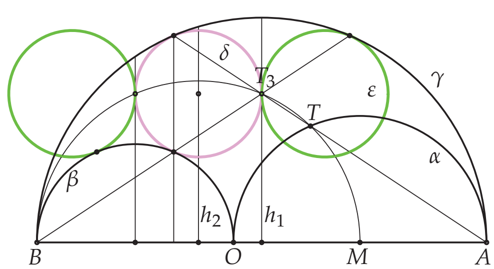

Figure 12.

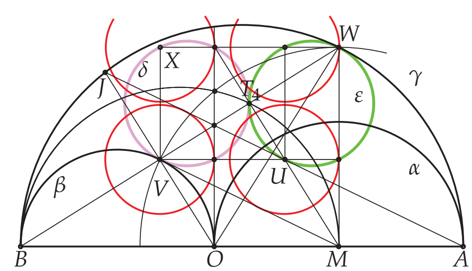

Figure 13.

We now assume that W is the point of intersection of γ and the perpendicular to AB at M, the line BW meets β in a point V, and T_4 is the midpoint of VW (see Figure 13). Then T_4 lies on (BM) by Remark 4.1, where V has coordinates $(-r_A, r_A\sqrt{t/a})$ [1]. We assume that δ and ε touch at T_4. Then they have radius $a/2$, and the point of intersection of δ and the axis closer to O has coordinates $(0, r_{\rm A}\sqrt{t/a})$. Therefore the Archimedean circle of center V touches the axis at this point. Let J be the point of intersection of γ and the reflection of the line OW in the axis. Then ε touches the segment AJ at the midpoint, and the lines AJ and BW meet in a point on the axis [1]. On the other hand, let U be the point of tangency of ε and AJ. Then U has coordinates $(r_{\rm A}a/b, r_{\rm A}\sqrt{t/a})$, since J has coordinates $(-2r_{\rm A}, 2r_{\rm A}\sqrt{t/a})$. While ε meets the line MW again in the point of coordinates $(a, r_{\rm A}\sqrt{t/a})$. Therefore the circle of center U touching MW at this point is Archimedean. The line UV is parallel to AB and the line MV touches the Archimedean circle of center U and β and δ at V. Let X be the point of intersection of the perpendicular from V to AB and the perpendicular from W to the axis. Then the circle of center X touching the axis at the point of intersection of the axis and the line MT_4 is Archimedean. The point of intersection of the perpendicular from U to AB and the line WX lies on ε and the circle of center at this point touching the line MW at W is Archimedean. The circle $M(V)$ passes through the points T_2 and W.

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