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# TWO-STEP HOMOGENEOUS GEODESICS IN HOMOGENEOUS LORENTZIAN SPACES

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ABSTRACT. A geodesic $\gamma(t)$ in a homogeneous Riemannian space $(G/H, g)$ is called a homogeneous geodesic if it is an orbit of a one-parameter subgroup of $G$, that is $\gamma(t) = exp(tX).o$, for some nonzero vector $X$ in the Lie algebra of $G$ which is called a geodesic vector. In the present paper, we discuss a generalisation of homogeneous geodesics in homogeneous Lorentzian spaces, namely geodesics of the form $\gamma(t) = exp(tX)exp(tY).o, X, Y \in \mathfrak{g} = Lie(G)$. An example of such a generalized geodesic is given.

## 1. INTRODUCTION

One of the classical problem of differential geometry has been to study geodesics of manifolds with a given metric on them. Among geodesics some of them are of paricular interest because of their special features and applications in other land of science such as physics, mechanics etc as well. For instance, homogeneous geodesics i.e geodesics which are orbits of a one parameter group of isometries of manifold are of this kind. Homogeneous geodesics of a manifold are called by V.I. Arnold “relative equilibriums”. The description of such relative equilibria is important for qualitative description of the behaviour of the corresponding mechanical system with symmetries. There is a big literature in mechanics devoted to the investigation of relative equilibria [2].

Riemannian and Finslerian Homogeneous geodesics have been studied by many authors [2], [3], [4], [16], [19], [20]. Homogeneous geodesics have been studied in homogeneous pseudo-Riemannian spaces in [9], [10], [11], [12], [13]. In pseudo-Riemannian spaces since the metric $g$ is indefinite, the reductive decomposition may not exist (see for instance [14] or [15] for examples of nonreductive pseudo-Riemannian homogeneous spaces). In pseudo-Riemannian geometry, null homogeneous geodesics are of particular interest. Plane-wave limits (Penrose limits) of homogeneous spacetimes along light-like homogeneous geodesics have been investigated in [15], [18].

One of the interest or we can say purpose of mathematicians is to generalise some facts, definitions and theorems which have been expressed for special cases, in some occasions they

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encounter with situations which show themselves as generalise situation of a known fact of mathematicians. With these descriptions one can guess we intend to discuss about generalization of concept we talked about so far that is generalization of Homogeneous geodesics.

Riemannian generalization of these geodesics namely geodesics of the form

$$\gamma(t) = exp(tX)exp(tY).o$$

which $X$ and $Y$ are in Lie algebra of group of isometries, known as *two-step homogeneous geodesics* has been studied in [5], such geodesics were initially studied by H. C. Wang in 1969 [21] as geodesics in a semisimple Lie group equipped with a metric induced by a Cartan involution of the Lie algebra of Lie group. D. Atri and Ziller in 1979 [7] , R. Dohira in 1995 [8], are authors who have had research in these forms of geodesics.

In our resarch at first we tried to examin two step homogeneous geodesics in Finslerian space but we encountered some obstacles. In Finsler geometry, the angle is not symmetric, and orthogonality is not necessarily well-defined. These are obstacles that prevent Theorem 3.1 from being expressed for the Finslerian state. Second, we focused on semi Riemannian case and specially Lorentzian space to see under what conditions it is possible to express the theorem. In this short paper we follow [5] and our main aim is to consider two step g.o spaces in Lorentzian spaces and examine the main theorem in [5] under which conditions is established for the Lorentzian space (see Theorem 3.1). We use semi Riemannian submersion of anti-de Sitte space to give example of our main theorem.

## 2. Preparation for main theorem

The definition given for two-step homogeneous geodesics in Riemannian space [5] can be used for pseudo-Riemannian spaces with a slight of change. One can see Definition1.1 in [9] for definition of homogeneous geodesics in pseudo-Riemannian case.
Let $(G/H, g)$ be a pseudo-Riemannian homogeneous manifold and consider the natural map $\pi : G \to G/H$. Let $O = \pi(e)$ be the origin of $G/H$.

We propose the following definitions.

**Definition 2.1.** A geodesic $\gamma$ through the point $p$ defined in an open interval $J$ (where $s$ is an affine parameter) is said to be two-step homogeneous if there exist (I) a diffeomorphism $s = \varphi(t)$ between the real line and the open interval $J$; and (II) There is a pair vectors $(X, Y) \in \mathfrak{g} \times \mathfrak{g} - \{(O, O)\}$ (This symbol is a formal symbol),such that $\gamma(\varphi(t)) = exp(tX)exp(tY)(p)$ for all $t \in (-\infty, +\infty)$.

The reason for the appearance of the $\varphi$ in the definition is as follows: If $t$ (the natural parameter of the 1-parameter isometry group) is not the affine parameter of the geodesic, $\varphi$ is the reparametrization of the geodesic from $t$ to the affine parameter (or conversely). This cannot happen in the Riemannian case, because covariant derivative of the tangent vector along homogeneous geodesic is always zero. However, along a light-like homogeneous geodesic this derivative can be a multiple of the tangent vector.

**Definition 2.2.** A two-step geodesic orbit space (two-step g.o space) is a pseudo-Riemannian homogeneous space so that all geodesics $\gamma$ with $\gamma(0) = 0$, are two-step homogeneous.

By contrast with the Riemannian case, not every smooth manifold can be made a Lorentz manifold.

**Proposition. 1.** [17] For a smooth manifold $M$ the following are equivalent

- There exists a Lorentz metric on $M$ .
- There exists a time-orientable Lorentz metric on $M$ .
- There is a nonvanishing vector field on $M$ .
- Either $M$ is noncompact, or $M$ is compact and has Euler number $\chi(M) = 0$.

For example, the only compact surfaces that can be made Lorentz surfaces are the torus and Klein bottle. Also, a sphere $S^n$ admits a Lorentz metric if and only if n is odd $\geqslant 3$ [17].

**Definition 2.3.** ( [1],p:500) Let $N$ be an $n$ dimensional Lorentz manifold. A linear subspace $V \leq T_pN$ is said to be

(1) spacelike if the restriction of the Lorentzian metric $\langle , \rangle$ to $V$ is positive definite; that is, $\langle , \rangle|_V$ is a Euclidean metric;
(2) timelike if the restriction of the Lorentzian metric $\langle , \rangle$ to $V$ is nondegenerate of index 1; that is, $\langle , \rangle|_V$ is a Lorentzian metric,
(3) lightlike (or null) if the restriction of the Lorentzian metric $\langle , \rangle$ to $V$ is degenerate.

## 3. Main theorem

Here we consider two-step homogeneous geodesics in homogeneous pseudo-Riemannian spaces (especially Lorentzian spaces) and prove the following theorem. It should be noted that the Riemannian version of the following theorem is proved in [5] (Theorem 2.3 )and we try to state it for a Lorentzian metric.

*Theorem* 3.1. Let $M = G/H$ be a noncompact homogeneous space admitting a naturally reductive Lorentzian metric. Let $B$ be the corresponding Lorentzian scalar product on $\mathfrak{m} = T_0(G/H)$. We assume that $\mathfrak{m}$ admits an $Ad(H)$-invariant orthogonal decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus ... \oplus \mathfrak{m}_s, \tag{3.1}$$

with respect to $B$, where each $\mathfrak{m}_i$ is not a null subspace. We can equip $G/H$ with a $G$-invariant Lorentzian metric $g$ corresponding to the $Ad(H)$-invariant Lorentzian scalar product

$$\langle , \rangle = \lambda_1 B\ |_{\mathfrak{m}_1} + ... + \lambda_s B\ |_{\mathfrak{m}_s}, \quad \lambda_1, ..., \lambda_s > 0. \tag{3.2}$$

If $(\mathfrak{m}_a, \mathfrak{m}_b)$ is pair of submodules in the decomposition (3.1) such that

$$[\mathfrak{m}_a, \mathfrak{m}_b] \subset \mathfrak{m}_a, \tag{3.3}$$

then any geodesic $\gamma$ of $(G/H, g)$ with $\gamma(0) = 0$ and $\dot{\gamma}(0) \in \mathfrak{m}_a \oplus \mathfrak{m}_b$, is a two-step homogeneous geodesic.
In particular, if $\dot{\gamma}(0) = X_a + X_b \in \mathfrak{m}_a \oplus \mathfrak{m}_b$, then for every $\varphi(t) \in J$ , $\gamma$ is given by

$$\gamma(\varphi(t)) = expt(X_a + \lambda X_b)expt(1 - \lambda)X_b.o, \tag{3.4}$$

where $\lambda = \frac{\lambda_b}{\lambda_a}$. Moreover, if one of the following relations holds:
1)$\lambda_a = \lambda_b$ or
2)$[\mathfrak{m}_a, \mathfrak{m}_b] = 0$
then $\gamma$ is a homogeneous geodesic, that is $\gamma(\varphi(t)) = expt(X_a + X_b).o$, for any $\varphi(t) \in J$.

**Proof**: We closely follow the proof of main theorem in [5], and the same notation as in [5]. Let $X = X_a + \lambda X_b, Y = (1-\lambda)X_b, \alpha(t) = exptXexptY$ and $\beta = \pi o \alpha$. Let $\gamma : J \to G/H$ be a reparametrization of $\beta$. Thus there exists a diffeomorphism $\varphi : \mathbb{R} \to J$ such that $\gamma(\varphi(t)) = \beta(t)$. We have

$$\nabla_{\dot\gamma}\dot\gamma = (\psi')^2 \nabla_{\dot\beta}\dot\beta + \psi''\dot\beta \tag{3.5}$$

where $\psi(s) = \varphi^{-1}(s) = t$ has nowhere vanishing derivative. For some $\varphi$ the curve $\gamma$ is a geodesic if and only if $\nabla_{\dot\beta}\dot\beta = -k(s)\dot\beta$ where $k = \psi''/(\psi')^2$. The affine parameters for a geodesic which is a reparametrization of $exptXexptY.o$ are $t$ and $e^{kt}$ where $k$ arises from (3.5). Let $\nabla$ be a Lorentzian connection of $(G/H, g_\lambda)$. For a space like geodesic we have

$$g(\nabla_{\dot\gamma}\dot\gamma, Z) = \dot\gamma g(\dot\gamma, Z) - \frac{1}{2} Z g(\dot\gamma, \dot\gamma) + g([Z, \dot\gamma], \dot\gamma), \tag{3.6}$$

for a timelike geodesic

$$g(\nabla_{\dot\gamma}\dot\gamma, Z) = \dot\gamma g(\dot\gamma, Z) + \frac{1}{2} Z g(\dot\gamma, \dot\gamma) + g([Z, \dot\gamma], \dot\gamma) \tag{3.7}$$

and for a null geodesic

$$g(\nabla_{\dot\gamma}\dot\gamma, Z) = \dot\gamma g(\dot\gamma, Z) + g([Z, \dot\gamma], \dot\gamma) \tag{3.8}$$

for any vector field $Z \in G/H$. The relation (13) in [5] implies that the vector field $\dot\alpha$ along the curve $\alpha$ can be extended to the vector field $X^R + Y^L$ in $G$. Then for any $t \in \mathbb{R}$ there exists a neighbourhood $U_{\pi(\alpha(t))}$ of $\pi(\alpha(t)) = \beta(t)$ in $G/H$ such that $\pi_*(X^R + Y^L)$ is a well defined vector field in $U_{\pi(\alpha(t))}$ which locally extends $\dot\beta$. Therefore, $\nabla_{\dot\beta}\dot\beta$ is well defined hence $\nabla_{\dot\gamma}\dot\gamma$ is well defined.

Next, we will show that the right-hand side of equation (3.6) vanishes for any $\varphi(t) \in J$ and for any vector field $Z$ in $G/H$, which is equivalent to show $\nabla_{\dot\beta(t)}\dot\beta(t) = -k(s = \varphi(t))\dot\beta(t)$. By replacing this in relation (3.6) we obtain

$$-kg(\dot\beta, Z) = g(\nabla_{\dot\beta}\dot\beta, Z) = \dot\beta g(\dot\beta, Z) - \frac{1}{2} Z g(\dot\beta, \dot\beta) + g([Z, \dot\beta], \dot\beta) \tag{3.9}$$

By the calculations on [5] we see that the second term for spacelike and timelike case is zero and for null case was already zero and first and third terms cancel each others. We obtain that the right hand side of equation (3.6) vanishes for all three curves, therefore $k$ should be zero and $t$ is an affine parameter of $\gamma$.

If $\lambda_a = \lambda_b$, then $\lambda = 1$, therefore

$$\beta(t) = \pi(expt(X_a + \lambda X_b)expt(1-\lambda)X_b) = \pi(expt(X_a + X_b)).$$

Eventually, if $[\mathfrak{m}_a, \mathfrak{m}_b] = \{0\}$ then the vectors $X_a + \lambda X_b$ and $X_b$ commute, therefore

$$\beta(t) = \pi(expt(X_a + \lambda X_b)expt(1-\lambda)X_b) = \pi(exp[t(X_a + \lambda X_b) + t(1-\lambda)X_b]) = \pi(expt(X_a + X_b)).$$

□

*Corollary* 3.1. Let $M = G/H$ be a noncompact homogeneous space admitting a naturally reductive Lorentzian metric. Let $B$ be the corresponding Lorentzian inner product of $\mathfrak{m} = T_o\frac{G}{H}$. We assume that $\mathfrak{m}$ admits an $Ad(H)$-invariant orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with respect to $B$ such that $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$ which $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are not null subspaces. Then $M$ admits an one parameter family of $G$-invariant Lorentzian metrics $g_\lambda, \lambda \in \mathbb{R}$, such that $(M, g_\lambda)$ is a

two step g.o space.
Each metric $g_\lambda$ corresponds to an $Ad(H)$-invariant Lorentzian inner product on $\mathfrak{m}$ of the form $\langle,\rangle = B|_{\mathfrak{m}_1} + \lambda B|_{\mathfrak{m}_2}$. This is homothetic to a metric corresponding to the inner product $\langle,\rangle = \lambda_1 B|_{\mathfrak{m}_1} + \lambda_2 B|_{\mathfrak{m}_2}, \lambda = \frac{\lambda_2}{\lambda_1}$.

According to Theorem 3.1, examples such as Flag manifolds, Generalized Wallach spaces and $k$-symmetric spaces of compact type for even k which appeared in [5] as two-step g.o. Riemannian spaces do not work in Lorentzian spaces.

Let $G$ be a Lie group and $K, H$ two compact Lie subgroups of $G$ with $K \subset H$. Let $\pi : G/K \longrightarrow G/H$ be the associated bundle with fibre $H/K$ to the $H$-principal bundle $p : G \longrightarrow G/H$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{k} \subset \mathfrak{h}$ the corresponding Lie subalgebras of $K$ and $H$. We choose an $Ad(H)$-invariant complement $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$, and an $Ad(K)$-invariant complement $\mathfrak{p}$ to $\mathfrak{k}$ in $\mathfrak{h}$. An $ad(H)$-invariant nondegenerate bilinear symmetric form on $\mathfrak{m}$ defines a $G$-invariant semi-Riemannian metric $g'$ on $G/H$ and an $ad(K)$-invariant nondegenerate bilinear symmetric form on $\mathfrak{p}$ defines a $H$-invariant semi-Riemannian metric $\hat{g}$ on $H/K$. The orthogonal direct sum for these nondegenerate bilinear symmetric forms on $\mathfrak{p} \oplus \mathfrak{m}$ defines a $G$-invariant semi-Riemannian metric $g$ on $G/K$.

*Theorem* 3.2. [6] The map $\pi : (G/K, g) \longrightarrow (G/H, g')$ is a semi-Riemannian submersion with totally geodesic fibres.

Now we are going to show that total spaces of semi Riemannian submersions is an example of Lorentzian two step g.o spaces.

**Property 1.** *Let $G$ be a noncompact Lie group admitting a left invariant Lorentzian metric and let $K, H$ be closed and connected subgroups of $G$, such that $K \subset H \subset G$. Let $B$ be the $Ad$-invariant Lorentzian inner product on the Lie algebra $\mathfrak{g}$ corresponding to the left invariant metric of $G$. We identify each of the spaces $T_0(G/K), T_0(G/H)$ and $T_0(H/K)$ with corresponding subspaces $\mathfrak{m}, \mathfrak{m}_1$ and $\mathfrak{m}_2$ of $\mathfrak{g}$, such that $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ where restriction of $B$ to $\mathfrak{m}_i$ is not null. We endow $G/K$ with the $G$-invariant Lorentzian metric $g_\lambda$ corresponding to the $Ad(K)$-invariant Lorentzian inner product $\langle,\rangle = B|_{\mathfrak{m}_1} + \lambda B|_{\mathfrak{m}_2}, \quad \lambda > 0$, on $\mathfrak{m}$. Then $(G/K, g_\lambda)$ is a Lorentzian two step g.o. space.*

**Proof:** The proof is similar to the Riemannian version in [5].

**Example 1.** We consider Hopf pseudo-Riemannian submertion. $(2n+1)$-dimensional anti-de Sitter space can be considered as the total space of the pseudo Riemannian homogeneous Hopf bundle. Let $G = SU(1,n), H = S(U(1)U(n)), K = SU(n)$, we have semi-Riemannian submersion

$$H_1^{2n+1} = SU(1,n)/SU(n) \longrightarrow \mathbb{C}H^n = SU(1,n)/S(U(1)U(n))$$

from $(2n+1)$-dimensional anti-de Sitter space with constant sectional curvature $-1$ and signature $(1,2n)$ to a complex hyperbolic space $\mathbb{C}H^n$ with fibers isometric to $H_1^1 = (S^1, -g_{S^1})$.

$$H_1^{2n+1} = \{x \in \mathbb{R}_2^{2n+2} : \langle x, x\rangle = -1\} \subset \mathbb{R}_2^{2n+1},$$

where

$$\langle,\rangle = dx_1^2 + ... + dx_{n-1}^2 - dx_n^2 - dx_{n+1}^2$$

is the pseudo-Euclidean metric of $\mathbb{R}^{2n+1}_2$ with index 2. For each $x \in H^{2n+1}_1$

$$T_x H^{2n+1}_1 = \{v \in \mathbb{R}^{2n+1}_2 : \langle v, x\rangle = 0\} = x^{\perp}$$

and the restriction of $\langle,\rangle$ to $T_x H^{2n+1}_1$ is a Lorentzian metric, because of the decomposition $\mathbb{R}^{2n+1}_2 = T_x S^{2n}_1 \oplus span\{x\}$ with $\langle x, x\rangle = -1$. Therefore, $H^{2n+1}_1$ is a Lorentzian hypersurface of $\mathbb{R}^{2n+1}_2$.

Let $g_1 = \langle,\rangle$ be the metric of $H^{2n+1}_1$. We equip $H^{2n+1}_1$ with a one parameter family of metrics $g_\lambda$.

Since $SU(1,n)$ is noncompact so it admits a left invariant Lorentzian metric corresponding to an $Ad(SU(1,n))$-invariant Lorentzian inner product $B$ on $\mathfrak{su}(1,n)$. We identify each of the spaces $T_0H^{2n+1}_1 = T_0(G/K), T_0\mathbb{C}H^n = T_0(G/H)$, and $T_0S^1 = T_0(H/K)$ with corresponding subspaces $\mathfrak{m}, \mathfrak{m}_1$, and $\mathfrak{m}_2$ of $\mathfrak{su}(1,n)$. The desired one parameter family of metrics $g_\lambda$ corresponds to the one parameter family of Lorentzian inner products

$$\langle,\rangle = B|_{\mathfrak{m}_1} + \lambda B|_{\mathfrak{m}_2}, \quad \lambda > 0 \tag{3.10}$$

on $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. By our assumptions we see that restriction of $B$ on each of $\mathfrak{m}_i$ is not degenerate and the scalar product $\langle,\rangle$ is Lorentzian. We see that the inner product (3.10) induces the standard metric $g_1$ on $H^{2n+1}_1$. Therefore Proposition 1 implies that $(H^{2n+1}_1, g_\lambda)$ is a Lorentzian two step g.o. space. Especially if we consider $X \in T_0H^{2n+1}_1$, hence the unique geodesic $\gamma$ of $(H^{2n+1}_1, g_\lambda)$ with $\gamma(0) = o$ and $\dot\gamma(0) = X$, is given by $\gamma(t) = expt(X_1 + \lambda X_2)expt((1-\lambda)X_2), t \in \mathbb{R}$ where $X_1, X_2$ are the projections of $X$ on $\mathfrak{m}_1 = T_o\mathbb{C}H^n$ and $\mathfrak{m}_2 = T_0S^1$ respectively.

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