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SMOOTH BODIES OF CONSTANT WIDTH IN $\mathbb{E}^2$

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ABSTRACT. This paper is concerned with another possible construction of two-dimensional bodies of constant width in Euclidean plane. We derive a new parametrization of their boundary curves and give a connection to the standard support function. Considering this parametrization, we also obtain an area formula and show some examples in the sequel. At the end, we take a step back to the Reuleaux polygons in an attempt to represent them holding on to our approach, although they are not smooth and are outside the scope of the title of the paper.

1. INTRODUCTION

Let $K \subset \mathbb{E}^2$ be a compact convex body (set), and D its diameter. We say the body K is of *constant width* D if it has exactly one pair of points (refer to them as opposite or antipodal points) at distance D in all directions. This definition implies that K is to be strictly convex too. The bodies of constant width have been studied by many authors, but their properties are still subject to analysis. In brief, we only mention a few of their basic features. For instance, the Barbier's theorem states that any curve of constant width D has length of $D\pi$. Certainly, the boundary of our body K is a curve of constant width and we denote it by ∂K, as usual. A simple example of a curve of constant width is the circle, but there exist infinitely many others which are commonly named as *orbiforms*. Another widely known result refers to the area and claims that among all bodies of a given constant width, the disk and the Reuleaux triangle have the largest and (Blaschke–Lebesgue theorem) the smallest area respectively. Further, the only centrally symmetric body of constant width is the disk, and so on. In the sequel, we denote the area of K by $A(K)$ and its perimeter by $L(K)$.

The main focus herein is on the bodies of constant width which boundaries are smooth curves of at least class C^1. In spite of the fact that our construction is based on a union of two (or more) curves of class C^2, the resulting curves might not have C^2 continuity at the two (or more) join points. However, regardless of this classification, since C^1 continuity is preserved, we simply say K is a smooth body of constant width.

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2. PARAMETRIZATION

Now, let's start speculating. Suppose we have some body K of constant width which diameter is D, and a line segment, let say $\overline{AB}$, of the same length. By the definition above, and due to the strict convexity, we are able to fit the whole $\overline{AB}$ into the interior of K, except the two endpoints A and B which will belong to the boundary of K. What is more important is that we are restricted to do this placement in a unique way for every single direction and hence, we conclude ∂K could be well–defined by the position of $\overline{AB}$, taking the direction as a parameter. Moreover, for any plane body of constant width which boundary is a curve of class C^2, it is proven that the sum of the radii of curvature at any two opposite points (in our case A and B) equals to the diameter D. This fact leads us to an idea to start a rotation of $\overline{AB}$ without leaving the interior of K. Although the center of this rotation F will vary, it must lie somewhere on $\overline{AB}$ in every moment (follows from the mentioned sum of the radii of curvature and the properties of the distance as a metric function). After a rotation of $\overline{AB}$ by π, we arrive at the same direction which we started from and therefore, its position must match the starting one. The trajectories of the points A and B bring us back the boundary of K. Since we were in the interior of K all the time, and the points A and B were on its boundary, we covered the whole area $A(K)$ as well with this rotation. The conclusion is that we can perform exactly the same movement of $\overline{AB}$ even if K is not present. All we need to know is how the center of the rotation F changes when we change the direction of $\overline{AB}$. Recall that we are dealing with a smooth ∂K, which means its derivative exists everywhere. Finally, let's fix a coordinate system and place K such that one of its diameters, say $\overline{AB}$, lies on the x-axis, and the point A is at the origin, i.e. $A = (0,0)$ and $B = (D,0)$, as shown in Figure 1.

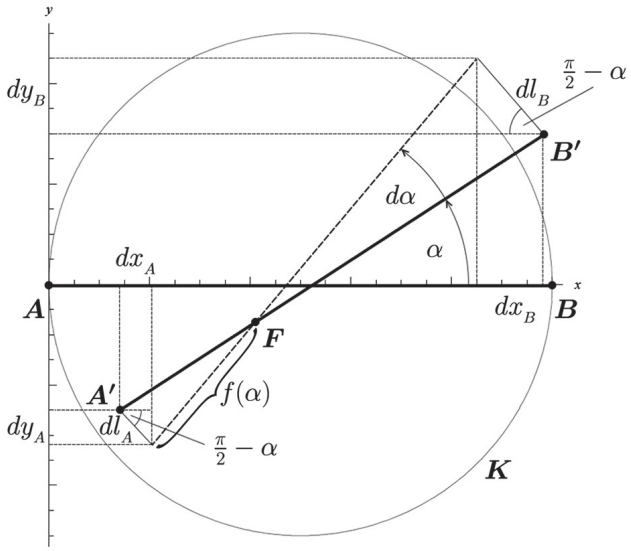

FIGURE 1. Rotation of the diameter $\overline{AB}$.

After a rotation by α, $\overline{AB}$ arrives at the position $\overline{A'B'}$. Denote the points A' and B' by their coordinates (x_A, y_A) and (x_B, y_B), where all x_A, y_A, x_B, y_B are some functions of the parameter α.

At this position, we are to continue with a rotation which center must be exactly F, as previously explained. Following the point F, we define the function

$$f : [0, \pi] \to [0, D] \quad \text{by} \quad f(\alpha) = d(A', F)\,,$$

where d is the standard metric in $\mathbb{E}^2$ (distance). The smoothness of ∂K (it is of class C^1) implies continuity of f almost everywhere on $[0, \pi]$. By definition, the function f is also non-negative, bounded by D and therefore, it is measurable. While having this in mind, in order to simplify the analysis, for now we assume f is continuous everywhere on $[0, \pi]$. Considering Figure 1, let dl_A and dl_B be positive lengths, and $d\alpha$ small enough. Then, we may write

$$\begin{aligned} dl_A &= f(\alpha)\, d\alpha\,, \\ dl_B &= (D - f(\alpha))\, d\alpha\,. \end{aligned}$$

Under this assumptions, the Barbier's theorem follows immediately,

$$L(K) = \int_0^{\pi} (dl_A + dl_B) = D \int_0^{\pi} d\alpha = D\pi\,.$$

We track the points A and B separately, and Figure 1 shows that $y_A(\alpha)$ is negative, $y_B(\alpha)$ is positive on the interval $(0, \pi)$, while the differential functions dx_A and dx_B are positive and negative on the same interval respectively. Thus, the differentials of the coordinates of A, B are:

$$\begin{aligned} dx_A &= dl_A \cos(\frac{\pi}{2} - \alpha) = dl_A \sin\alpha = f(\alpha)\sin\alpha\, d\alpha\,; \\ dy_A &= -dl_A \sin(\frac{\pi}{2} - \alpha) = -dl_A \cos\alpha = -f(\alpha)\cos\alpha\, d\alpha\,; \\ dx_B &= -dl_B \cos(\frac{\pi}{2} - \alpha) = -dl_B \sin\alpha = -(D - f(\alpha))\sin\alpha\, d\alpha\,; \\ dy_B &= dl_B \sin(\frac{\pi}{2} - \alpha) = dl_B \cos\alpha = (D - f(\alpha))\cos\alpha\, d\alpha\,. \end{aligned} \tag{2.1}$$

Taking in account the starting values $x_A(0) = y_A(0) = y_B(0) = x_B(\pi) = y_A(\pi) = y_B(\pi) = 0$, and $x_A(\pi) = x_B(0) = D$, the parametric equations of the coordinates of the points A and B are:

$$\begin{aligned} x_A(\alpha) &= \int_0^{\alpha} f(t)\sin t\, dt\,; \\ y_A(\alpha) &= -\int_0^{\alpha} f(t)\cos t\, dt\,; \\ x_B(\alpha) &= D - \int_0^{\alpha} (D - f(t))\sin t\, dt\,; \\ y_B(\alpha) &= \int_0^{\alpha} (D - f(t))\cos t\, dt\,. \end{aligned} \tag{2.2}$$

The graph constructed by the union of the points (x_A, y_A) and (x_B, y_B) coincide with ∂K or, if we are more formal, $\partial K = \big\{\big(x_A, y_A\big) : \alpha \in [0, \pi]\big\} \bigcup \big\{\big(x_B, y_B\big) : \alpha \in [0, \pi]\big\}$. Notice that the x-axis splits this union into two curves such that each of them shows the trajectory of A and B separately. At $\alpha = \pi$, the two points interchange their starting coordinates and probably this is the moment when we give up on achieving C^2 continuity. Namely, it is easy to observe

that only in case when $f(0) = f(\pi) = \frac{D}{2}$, the starting and the ending curvature match each other at the join points. Having this in mind, since in the start we picked out the diameter $\overline{AB}$ arbitrarily, if we reparametrize the same curve, we infer that the function f does not even have to be continuous necessarily. Anyway, what is important is that when the points A and B finish their journey, after integrating over $[0, \pi]$, by (2.1) we conclude

$$\begin{aligned} \int_0^\pi dx_A = x_A(\pi) - x_A(0) = D \quad &\Longleftrightarrow \quad \int_0^\pi f(\alpha)\sin\alpha\, d\alpha = D\,, \\ \int_0^\pi dy_A = y_A(\pi) - y_A(0) = 0 \quad &\Longleftrightarrow \quad \int_0^\pi f(\alpha)\cos\alpha\, d\alpha = 0\,. \end{aligned} \tag{2.3}$$

In fact, the last two relations are constraints on f which must hold for any smooth curve of constant width. These constraints obtain the curve to be closed. From this point, if we try to get back step by step aiming to reconstruct ∂K, we will find we have no other obstacle except one which hinder us to guarantee the smoothness of the curve. Let's suppose we are equipped with a non-negative continuous function f on $[0, \pi]$, bounded by the length of the segment we rotate, say D, and assume the equations (2.3) hold too. In order to eliminate prospective singular points on the curve (2.2), we are looking for the candidates. By use of (2.1), we set the next conditions:

$$\begin{aligned} x_A'(\alpha) &= f(\alpha)\sin\alpha = 0\,; \\ y_A'(\alpha) &= -f(\alpha)\cos\alpha = 0\,; \\ x_B'(\alpha) &= -(D - f(\alpha))\sin\alpha = 0\,; \\ y_B'(\alpha) &= (D - f(\alpha))\cos\alpha = 0\,. \end{aligned}$$

Because $\sin\alpha$ and $\cos\alpha$ can not be zero simultaneously, it turns out that either $f(\alpha) = 0$ or $f(\alpha) = D$, which means the set $\{\alpha : f(\alpha) \in \{0, D\}\}$ is allowed to be at most countable. Since the parametric coordinates given by (2.2) depend on functions of the upper bounds of the integrals on the right side, their derivatives exist everywhere and, if the mentioned set is empty, the curve is *regular*. The definition of f implies $\int_0^\pi f(\alpha)\sin\alpha\, d\alpha > 0$ to hold too. Finally, let the function f be discontinuous at finitely many points on $[0, \pi]$. Clearly, these points of discontinuity do not affect the values of the integrals and hence, we afford ourselves to generalize the conclusions by setting a weaker constraint on f, as stated in the following theorem.

Theorem 2.1. *Let $f : [0, \pi] \to \mathbb{R}$ be piecewise continuous and D positive real number. Let:*

(i) $\int_0^\pi f(\alpha)\cos\alpha\, d\alpha = 0$;

(ii) $0 \le f(\alpha) \le \int_0^\pi f(\alpha)\sin\alpha\, d\alpha = D$;

(iii) *The measure of the set $\{\alpha : f(\alpha) \in \{0, D\}\}$ is zero.*

Then, (2.2) *is a parametrization of a curve of constant width D of class C^1.*

Remark. If the function f is differentiable on $(0, \pi)$, then the condition $\int_0^\pi f(\alpha)\cos\alpha\, d\alpha = 0$ is equivalent to $\int_0^\pi f'(\alpha)\sin\alpha\, d\alpha = 0$ which follows immediately after integration by parts.

Definition 2.1. Let $f : [0, \pi] \to \mathbb{R}$ be a non-negative, piecewise continuous function such that

(i) $\int_0^\pi f(\alpha)\cos\alpha\, d\alpha = 0$,

(ii) $0 \not\equiv f \leq \int_0^\pi f(\alpha)\sin\alpha\, d\alpha$.

We say f is a *center function* or *c-function of magnitude* D, where $D = \int_0^\pi f(\alpha)\sin\alpha\, d\alpha$.

Definition 2.2. If f is a c-function and if the measure of the set $\{\alpha : f(\alpha) \in \{0, D\}\}$ is zero, we say f is a *proper c-function*. If the c-function is not proper, then we say, it is *improper*.

Definition 2.3. If we refer to the parametrization (2.2), we say the body K is *generated* by the c-function f or, the c-function f *generates* the body K. The set $\Omega = \{f : f \text{ is a } c\text{-function}\}$.

Remark. Perhaps, it would be better to state these definitions before Theorem 2.1 but, for the purpose of pointing out the sense, we introduce them at this point. Certainly, the statement in Theorem 2.1 sounds much simpler now: Let $f \in \Omega$ be proper of magnitude D. Then, f generates a curve of constant width D of class C^1. Anyway, it's a simplification in the following.

Proposition 2.1. *The function f is a c-function of magnitude D if and only if the function $g = D - f$ is a c-function of magnitude D. Moreover, f is proper if and only if g is proper.*

Proof. A straightforward calculation obtains:

$$\begin{aligned}
0 \leq f(\alpha) \leq D \quad &\Longleftrightarrow \quad 0 \leq D - f(\alpha) \leq D\,;\\
\int_0^\pi f(\alpha)\cos\alpha\, d\alpha = 0 \quad &\Longleftrightarrow \quad \int_0^\pi \big(D - f(\alpha)\big)\cos\alpha\, d\alpha = 0\,;\\
\int_0^\pi f(\alpha)\sin\alpha\, d\alpha = D \quad &\Longleftrightarrow \quad \int_0^\pi \big(D - f(\alpha)\big)\sin\alpha\, d\alpha = D\,;\\
f(\alpha) \in \{0, D\} \quad &\Longleftrightarrow \quad D - f(\alpha) \in \{0, D\}\,.
\end{aligned}$$

These simple equivalences confirm the claim. □

Proposition 2.2. *Let f be a c-function of magnitude D. Then, the following inequalities hold:*

$$\frac{\pi}{3} \leq \frac{1}{D}\int_0^\pi f(\alpha)\, d\alpha \leq \frac{2\pi}{3}\,.$$

Proof. In [6], Lemma 2 shows that for every measurable function G on the interval $[0, \pi]$ such that $0 \leq G \leq 1$ and $\int_0^\pi G(\alpha)\sin\alpha\, d\alpha = 1$, the inequality $\int_0^\pi G(\alpha)\, d\alpha \leq \frac{2\pi}{3}$ holds. In our case, since f is of magnitude D (clearly, f is measurable), the right-hand side inequality follows after multiplying by D the both sides. To prove the other inequality, invoke Proposition 2.1:

$$\frac{1}{D}\int_0^\pi (D - f(\alpha))\, d\alpha \leq \frac{2\pi}{3}\,; \quad \pi - \frac{1}{D}\int_0^\pi f(\alpha)\, d\alpha \leq \frac{2\pi}{3}\,.$$

Note: If f generates a Reuleaux triangle, either the left or the right equality is achieved. □

Corollary 2.1. *Let the two endpoints A and B of a given diameter of K split ∂K into curves of arc lengths $\widehat{AB_1}$ and $\widehat{AB_2}$ such that $\widehat{AB_1} \leq \widehat{AB_2}$. Then, the next two inequalities hold:*

$$1 \leq \frac{\widehat{AB_2}}{\widehat{AB_1}} \leq 2\,.$$

Proof. Let f be the c-function which generates K. Having $L(K) = D\pi = \widehat{AB_1} + \widehat{AB_2}$ and $\widehat{AB_1} = \int_0^\pi f(\alpha)\,d\alpha$, it is $\widehat{AB_2} = D\pi - \int_0^\pi f(\alpha)\,d\alpha$ and, by Proposition 2.2 we easily conclude

$$1 \leq \frac{\widehat{AB_2}}{\widehat{AB_1}} = \frac{D\pi}{\int_0^\pi f(\alpha)\,d\alpha} - 1 \leq \frac{D\pi}{\frac{1}{3}D\pi} - 1 = 2\,.$$

Note: Evidently, if f generates a Reuleaux triangle then the right equality is achieved. □

Theorem 2.2. *Let K_1 and K_2 be bodies generated by the c-functions f_1 and f_2 of magnitudes D_1 and D_2 respectively and, let $\lambda, \mu \in \mathbb{R}_0^+$. Then, $f = \lambda f_1 + \mu f_2$ is a c-function of magnitude $D = \lambda D_1 + \mu D_2$ which generates a constant width body given by the sum $K = \lambda K_1 + \mu K_2$.*

Proof. By the construction, obviously the function f satisfies all conditions of Definition 2.1:

$$\int_0^\pi (\lambda f_1(\alpha) + \mu f_2(\alpha))\cos\alpha\,d\alpha = \lambda\int_0^\pi f_1(\alpha)\cos\alpha\,d\alpha + \mu\int_0^\pi f_2(\alpha)\cos\alpha\,d\alpha = 0\,;$$
$$\int_0^\pi (\lambda f_1(\alpha) + \mu f_2(\alpha))\sin\alpha\,d\alpha = \lambda\int_0^\pi f_1(\alpha)\sin\alpha\,d\alpha + \mu\int_0^\pi f_2(\alpha)\sin\alpha\,d\alpha = \lambda D_1 + \mu D_2\,.$$

Since $0 \leq \lambda f_1 + \mu f_2 \leq \lambda D_1 + \mu D_2$, we conclude f is a c-function of magnitude $\lambda D_1 + \mu D_2$. Next, using an analogous labeling for the coordinates of K, K_1 and K_2 as in the text above or, if we want to be more precise, $\partial K = \{(x_A, y_A)\}\bigcup\{(x_B, y_B)\}$, $\partial K_1 = \{(x_{A_1}, y_{A_1})\}\bigcup\{(x_{B_1}, y_{B_1})\}$, $\partial K_2 = \{(x_{A_2}, y_{A_2})\}\bigcup\{(x_{B_2}, y_{B_2})\}$, with an intention to realize the connection between these three bodies while having $f = \lambda f_1 + \mu f_2$ and $D = \lambda D_1 + \mu D_2$, we rewrite the equations (2.2):

$$x_A(\alpha) = \int_0^\alpha (\lambda f_1(t) + \mu f_2(t))\sin t\,dt = \lambda x_{A_1}(\alpha) + \mu x_{A_2}(\alpha)\,;$$
$$y_A(\alpha) = -\int_0^\alpha (\lambda f_1(t) + \mu f_2(t))\cos t\,dt = \lambda y_{A_1}(\alpha) + \mu y_{A_2}(\alpha)\,;$$
$$x_B(\alpha) = \lambda D_1 + \mu D_2 - \int_0^\alpha (\lambda D_1 + \mu D_2 - \lambda f_1(t) - \mu f_2(t))\sin t\,dt = \lambda x_{B_1}(\alpha) + \mu x_{B_2}(\alpha)\,;$$
$$y_B(\alpha) = \int_0^\alpha (\lambda D_1 + \mu D_2 - \lambda f_1(t) - \mu f_2(t))\cos t\,dt = \lambda y_{B_1}(\alpha) + \mu y_{B_2}(\alpha)\,.$$

These direct calculations show that K is a Minkowski sum of the bodies λK_1 and μK_2. □

Remark. Since the parametrization (2.2) refers to a fixed coordinate system and the Minkowski vector addition depends on the relative position of the bodies, in the conclusions above we assumed the bodies are situated as in Figure 1. The same assumption applies in the further text.

3. AREA OF THE BODY

In order to calculate the area of the body K, we are going to sum up the areas below and above the x-axis, which we denote by $\underline{A}_K$ and $\overline{A}_K$ respectively. According to Figure 1 and intending to get positive values, we must change the signs of the differentials of $\underline{A}_K$ and $\overline{A}_K$. Thus, we write

$$\begin{aligned} dA(K) = -d\underline{A}_K - d\overline{A}_K &= -y_A(\alpha)\,dx_A - y_B(\alpha)\,dx_B \\ &= \left(\int_0^\alpha f(t)\cos t\,dt\right) f(\alpha)\sin\alpha\,d\alpha \\ &+ \left(\int_0^\alpha (D - f(t))\cos t\,dt\right)(D - f(\alpha))\sin\alpha\,d\alpha\,. \end{aligned}$$

This differential integrated over $[0, \pi]$ (and simplified), yields the formula for the area of K,

$$A(K) = \int_0^\pi \left(\int_0^\alpha f(t)\cos t\,dt\right)(2f(\alpha) - D)\sin\alpha\,d\alpha + D\int_0^\pi (D - f(\alpha))\sin^2\alpha\,d\alpha\,. \quad (3.1)$$

During the rotation, we can conclude that for some α, the diameter $\overline{AB}$ splits K into two parts of equal area. If we take this as a starting position of $\overline{AB}$, we infer there is a c-function $\underline{f}$ which generates the same[1] K such that $\underline{A}_K = \overline{A}_K$, and in particular for this case we have

$$\begin{aligned} A(K) = -2\int_0^\pi d\underline{A}_K &= 2\int_0^\pi \left(\int_0^\alpha \underline{f}(t)\cos t\,dt\right)\underline{f}(\alpha)\sin(\alpha)\,d\alpha \\ &= 2\left(\int_0^\alpha \underline{f}(t)\cos t\,dt\int_0^\alpha \underline{f}(t)\sin t\,dt\right)\Bigg|_0^\pi \\ &- 2\int_0^\pi \left(\int_0^\alpha \underline{f}(t)\sin t\,dt\right)\underline{f}(\alpha)\cos\alpha\,d\alpha \\ &= 0 - 2\int_0^\pi \left(\int_0^\alpha \underline{f}(t)\sin t\,dt\right)\underline{f}(\alpha)\cos\alpha\,d\alpha\,, \end{aligned}$$

$$A(K) = \int_0^\pi \left(\int_0^\alpha \underline{f}(t)\cos t\,dt\right)\underline{f}(\alpha)\sin(\alpha)\,d\alpha - \int_0^\pi \left(\int_0^\alpha \underline{f}(t)\sin t\,dt\right)\underline{f}(\alpha)\cos\alpha\,d\alpha\,. \quad (3.2)$$

In the above, we performed integration by parts and used the fact that $\int_0^\pi \underline{f}(\alpha)\cos\alpha\,d\alpha = 0$. Now, let K be the body from Theorem 2.2, i.e. $K = \lambda K_1 + \mu K_2$. By Minkowski, the area of K is given by $A(K) = A(\lambda K_1) + 2A(\lambda K_1, \mu K_2) + A(\mu K_2)$ where $A(\lambda K_1, \mu K_2)$ is the mixed area of the bodies λK_1 and μK_2. Applying the basic properties of the homothety and the mixed area, the same equation can be rewritten as $A(K) = \lambda^2 A(K_1) + 2\lambda\mu A(K_1, K_2) + \mu^2 A(K_2)$. If we use the labels and the assumptions from Theorem 2.2, by (3.1) we calculate the area of the body K,

$$\begin{aligned} A(K) = &\int_0^\pi \left(\int_0^\alpha \big(\lambda f_1(t) + \mu f_2(t)\big)\cos t\,dt\right)\Big(2\big(\lambda f_1(\alpha) + \mu f_2(\alpha)\big) - \lambda D_1 - \mu D_2\Big)\sin\alpha\,d\alpha \\ &+ \big(\lambda D_1 + \mu D_2\big)\int_0^\pi \Big(\lambda D_1 + \mu D_2 - \lambda f_1(\alpha) - \mu f_2(\alpha)\Big)\sin^2\alpha\,d\alpha\,. \end{aligned}$$

In order to simplify the equation, denote $I_1(\alpha) = \int_0^\alpha f_1(t)\cos t\,dt$ and $I_2(\alpha) = \int_0^\alpha f_2(t)\cos t\,dt$.

[1] We say the bodies are *same* if they are congruent up to a some rigid motion.

$$\begin{aligned} A(K) &= \lambda^2 A(K_1) + \mu^2 A(K_2) \\ &\quad + \lambda\mu \int_0^\pi \Big(I_1(\alpha)(2f_2(\alpha) - D_2) + I_2(\alpha)(2f_1(\alpha) - D_1) \Big) \sin\alpha \, d\alpha \\ &\quad + \lambda\mu \int_0^\pi \Big(2D_1D_2 - D_1 f_2(\alpha) - D_2 f_1(\alpha) \Big) \sin^2\alpha \, d\alpha \,. \end{aligned} \tag{3.3}$$

The last equation shows that the mixed area is a half of the sum of the last two terms,

$$\begin{aligned} A(\lambda K_1, \mu K_2) &= \frac{\lambda\mu}{2} \int_0^\pi \Big(I_1(\alpha)(2f_2(\alpha) - D_2) + I_2(\alpha)(2f_1(\alpha) - D_1) \Big) \sin\alpha \, d\alpha \\ &\quad + \frac{\lambda\mu}{2} \int_0^\pi \Big(2D_1D_2 - D_1 f_2(\alpha) - D_2 f_1(\alpha) \Big) \sin^2\alpha \, d\alpha \,. \end{aligned} \tag{3.4}$$

In general, the mixed area is thoroughly researched topic in convex and differential geometry. Here we list only a few of its basic properties. For example, if S_1, S_2 and S_3 are some compact convex sets in the plane, then for the mixed area of these bodies the following relations are valid:

$$\begin{aligned} &A(S_1, S_2) \geq 0 \,; \\ &A(S_1, S_1) = A(S_1) \,; \\ &A(S_1, S_2) = A(S_2, S_1) \,; \\ &S_1 \subset S_2 \Longrightarrow A(S_1, S_3) \leq A(S_2, S_3) \,; \\ &A(\lambda S_1 + \mu S_2) = \lambda^2 A(S_1) + 2\lambda\mu A(S_1, S_2) + \mu^2 A(S_2) \,. \end{aligned}$$

4. Support function parametrization

A curve of constant width is usually defined by its support function. Focusing on smooth curves, in our two-dimensional case it would be a function of the same parameter α which measures the distance from the origin to the tangent lines to ∂K. Namely, for every direction α, the body K has one pair of tangents perpendicular to the given direction. The distance from the origin to one of these tangents given in terms of the parameter α is the desired support function h of K. Clearly, the envelope of this family of tangents is ∂K. Both of the functions f and h define the same body K in two different ways, thus a natural question is what is the relation between the two functions. The direction of the tangent to the point (x_A, y_A) and its equation are the next:

$$\begin{gathered} \frac{y_A'(\alpha)}{x_A'(\alpha)} = -\frac{f(\alpha)\cos\alpha}{f(\alpha)\sin\alpha} = -\cot\alpha \,; \\ y - y_A(\alpha) = -\cot\alpha \,(x - x_A(\alpha)) \,; \\ \cos\alpha \, x + \sin\alpha \, y - \cos\alpha \, x_A(\alpha) - \sin\alpha \, y_A(\alpha) = 0 \,. \end{gathered}$$

The distance between the origin and this tangent line is

$$d = \frac{|0 + 0 - \cos\alpha \, x_A(\alpha) - \sin\alpha \, y_A(\alpha)|}{\sqrt{\cos^2\alpha + \sin^2\alpha}} = |\cos\alpha \, x_A(\alpha) + \sin\alpha \, y_A(\alpha)| \,.$$

We see that d depends on the values of x_A and y_A which depend on the function f. On the other hand, we remember that f is defined on $[0,\pi]$ and hence, for $\alpha \in [0,\pi]$, the support function is

$$h(\alpha) = \left| \cos\alpha \int_0^\alpha f(t)\sin t\,dt - \sin\alpha \int_0^\alpha f(t)\cos t\,dt \right|.$$

The direction of the tangent to the point (x_B, y_B) and its equation are the next:

$$\frac{y_B'(\alpha)}{x_B'(\alpha)} = -\frac{(D-f(\alpha))\cos\alpha}{(D-f(\alpha))\sin\alpha} = -\cot\alpha\,,$$

$$y - y_B(\alpha) = -\cot\alpha\,(x - x_B(\alpha))\,;$$
$$\cos\alpha\, x + \sin\alpha\, y - \cos\alpha\, x_B(\alpha) - \sin\alpha\, y_B(\alpha) = 0\,.$$

The distance between the origin and this tangent line is

$$d = \frac{|0+0-\cos\alpha\, x_B(\alpha) - \sin\alpha\, y_B(\alpha)|}{\sqrt{\cos^2\alpha + \sin^2\alpha}} = |\cos\alpha\, x_B(\alpha) + \sin\alpha\, y_B(\alpha)|\,.$$

Commonly, the support function h is defined to be 2π periodic and, due to the constant width of K, it is evident (if K contains the origin) that $h(\alpha) + h(\alpha+\pi) = D$ applies for every α. Taking into account that (x_B, y_B) is opposite to (x_A, y_A), in order to extend the definition of h on $(\pi, 2\pi]$ while we operate on $(0,\pi]$, we shift the argument and write $h(\alpha+\pi) = d$, i.e.

$$\begin{aligned} h(\alpha+\pi) &= \left| \cos\alpha \left(D - \int_0^\alpha (D-f(t))\sin t\,dt \right) + \sin\alpha \int_0^\alpha (D-f(t))\cos t\,dt \right| \\ &= \left| D + \cos\alpha \int_0^\alpha f(t)\sin t\,dt - \sin\alpha \int_0^\alpha f(t)\cos t\,dt \right|. \end{aligned}$$

If we use the substitution $\alpha + \pi = \alpha'$, we have

$$\begin{aligned} h(\alpha') &= \left| D + \cos(\alpha'-\pi) \int_0^{\alpha'-\pi} f(t)\sin t\,dt - \sin(\alpha'-\pi) \int_0^{\alpha'-\pi} f(t)\cos t\,dt \right| \\ &= \left| D - \cos\alpha' \int_0^{\alpha'-\pi} f(t)\sin t\,dt + \sin\alpha' \int_0^{\alpha'-\pi} f(t)\cos t\,dt \right|. \end{aligned}$$

Thus, the function f remains defined, and finally we obtain:

$$h(\alpha) = \begin{cases} \left| \cos\alpha \int_0^\alpha f(t)\sin t\,dt - \sin\alpha \int_0^\alpha f(t)\cos t\,dt \right| : \alpha \in [0,\pi]\,; \\ \left| D - \cos\alpha \int_0^{\alpha-\pi} f(t)\sin t\,dt + \sin\alpha \int_0^{\alpha-\pi} f(t)\cos t\,dt \right| : \alpha \in (\pi, 2\pi]\,. \end{cases} \tag{4.1}$$

Similar to f, the support function h is non-negative, bounded by D and continuous on the whole interval $[0, 2\pi]$. Having $h(0) = h(2\pi)$, we can make it to be 2π periodic. A simple example of a support function is given in Example 5.2 in the next section.

5. EXAMPLES

Example 5.1. Let $f(\alpha) = \frac{1}{2}$. The function f is a c-function of magnitude 1. By (2.2) we find

$$\begin{aligned} x_A(\alpha) &= -\frac{1}{2}\cos\alpha + \frac{1}{2}\,, \\ y_A(\alpha) &= -\frac{1}{2}\sin\alpha\,, \\ x_B(\alpha) &= \frac{1}{2}\cos\alpha + \frac{1}{2}\,, \\ y_B(\alpha) &= \frac{1}{2}\sin\alpha\,. \end{aligned}$$

These are parametric equations of a circle with center at $(\frac{1}{2}, 0)$ and $D = 1$. By (3.1) we calculate

$$A(K) = 0 + \frac{1}{2}\int_0^\pi \sin^2\alpha\, d\alpha = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4}\,.$$

This circle is symmetric with respect to the x-axis. Thus, we can calculate the area by (3.2) too,

$$A(K) = \frac{1}{4}\int_0^\pi \sin^2\alpha\, d\alpha + \frac{1}{4}\int_0^\pi \cos^2\alpha\, d\alpha - \frac{1}{4}\int_0^\pi \cos\alpha\, d\alpha = \frac{1}{4}\int_0^\pi d\alpha - 0 = \frac{\pi}{4}\,.$$

Example 5.2. Let $f(\alpha) = \sin\alpha$. We can verify easily that f is a c-function of magnitude $\frac{\pi}{2}$. The conditions proposed in Theorem 2.1 are satisfied, and by (2.2) we get the parametric curve

$$\begin{aligned} x_A(\alpha) &= \frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha\,, \\ y_A(\alpha) &= -\frac{1}{2}\sin^2\alpha\,, \\ x_B(\alpha) &= \frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha + \frac{\pi}{2}\cos\alpha\,, \\ y_B(\alpha) &= -\frac{1}{2}\sin^2\alpha + \frac{\pi}{2}\sin\alpha\,. \end{aligned}$$

The graph $\{(x_A, y_A)\} \cup \{(x_B, y_B)\}$ is a curve of constant width $\frac{\pi}{2}$ and it is shown in Figure 2. According to (3.1) (clearly (3.2) is not applicable in this case), the area enclosed by this curve is

$$A(K) = \int_0^\pi \frac{1}{2}\sin^2\alpha\left(2\sin\alpha - \frac{\pi}{2}\right)\sin\alpha\, d\alpha + \frac{\pi}{2}\int_0^\pi \left(\frac{\pi}{2} - \sin\alpha\right)\sin^2\alpha\, d\alpha = \frac{\pi(\pi^2 - 5)}{8}\,.$$

We can find the equations of the support function of the curve as well. By use of (4.1) we obtain

$$h(\alpha) = \begin{cases} \dfrac{1}{2}\left|\alpha\cos\alpha - \sin\alpha\right| : \alpha \in [0, \pi]\,; \\ \dfrac{1}{2}\left|\pi - (\alpha - \pi)\cos\alpha + \sin\alpha\right| : \alpha \in (\pi, 2\pi]\,. \end{cases}$$

The body of constant width 1 which is homothetic to the one here has an area of $\frac{\pi^2 - 5}{2\pi} \approx 0.7750$.

Example 5.3. Let $f(\alpha) = \sin^2 \alpha$. The function f is a c-function of magnitude $\frac{4}{3}$ and it generates a body of constant width $\frac{4}{3}$. Applying (2.2), we represent this curve by the following equations:

$$\begin{aligned} x_A(\alpha) &= -\frac{3}{4}\cos\alpha + \frac{1}{12}\cos 3\alpha + \frac{2}{3}\,; \\ y_A(\alpha) &= -\frac{1}{3}\sin^3\alpha\,; \\ x_B(\alpha) &= \frac{7}{12}\cos\alpha + \frac{1}{12}\cos 3\alpha + \frac{2}{3}\,; \\ y_B(\alpha) &= \frac{4}{3}\sin\alpha - \frac{1}{3}\sin^3\alpha\,. \end{aligned}$$

The graph of the curve is shown in Figure 3. Employing (3.1) we find the curve encloses area of

$$A(K) = \int_0^{\pi} \frac{1}{3}\sin^3\alpha\left(2\sin^2\alpha - \frac{4}{3}\right)\sin\alpha\,d\alpha + \frac{4}{3}\int_0^{\pi}\left(\frac{4}{3} - \sin^2\alpha\right)\sin^2\alpha\,d\alpha = \frac{31\pi}{72}\,.$$

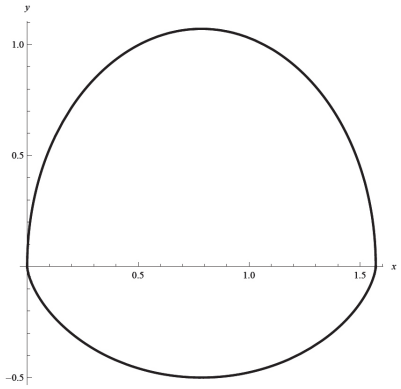

FIGURE 2. $f(\alpha) = \sin\alpha$; $D = \frac{\pi}{2}$.

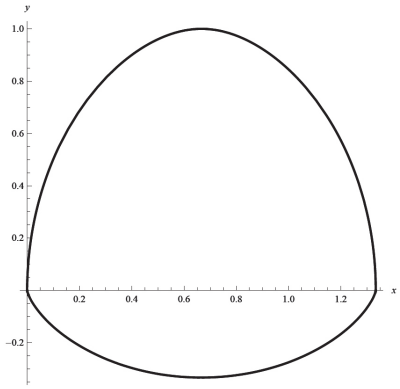

FIGURE 3. $f(\alpha) = \sin^2\alpha$; $D = \frac{4}{3}$.

The homothetic body of constant width 1 is of area $\frac{31\pi}{128} \approx 0.7608$, hence this body has smaller area than the body in Example 5.2. As we expected, the areas of these bodies do not exceed the area of the unit circle, but they are greater than the area of the Reuleaux triangle of diameter 1.

Example 5.4. As we can perceive, the bodies in the previous examples have an axis of symmetry at $x = \frac{\pi}{2}$, so next we give an example of a body which has not, i.e. is asymmetrical. It is simple to check out that the function $f(\alpha) = \cos\alpha + \frac{\alpha^2}{4}$ is a proper c-function of magnitude $\frac{\pi^2}{4} - 1$. Again, Theorem 2.1 works and hence, by (2.2) we arrive at the parametric equations as follows:

$$
\begin{aligned}
x_A(\alpha) &= \frac{1}{4}(2-\alpha^2)\cos\alpha + \frac{1}{2}(\alpha+\sin\alpha)\sin\alpha - \frac{1}{2}\,;\\
y_A(\alpha) &= -\frac{1}{4}(-2+\alpha^2)\sin\alpha - \frac{1}{2}(\alpha+\sin\alpha)\cos\alpha - \frac{\alpha}{2}\,;\\
x_B(\alpha) &= \frac{1}{4}(\pi^2-2-\alpha^2)\cos\alpha + \frac{1}{2}(\alpha+\sin\alpha)\sin\alpha - \frac{1}{2}\,;\\
y_B(\alpha) &= \frac{1}{4}(\pi^2-2-\alpha^2)\sin\alpha - \frac{1}{2}(\alpha+\sin\alpha)\cos\alpha - \frac{\alpha}{2}\,.
\end{aligned}
$$

This curve is shown in Figure 4. If we enter (3.1) in Wolfram Mathematica, the program yields

$$A(K) = \frac{4\pi^5 - 37\pi^3 + 84\pi}{192} + \frac{2\pi^5 + 105\pi^3 - 1260\pi}{960} \approx 1.6802;\ A(K) \approx 0.7803\, D^2\,.$$

Example 5.5. We were closely related to the sine and cosine functions defining f until now. This example shows that we are able to construct a constant width curve without use of trigonometric functions. The c-function $f(\alpha) = \left(\alpha - \frac{\pi}{2}\right)^2 + 2$ is of magnitude $\frac{\pi^2}{2}$, and the generated curve is plotted in Figure 5. The area of this body is

$$A(K) = \frac{5\pi^5}{48} - \frac{\pi^5}{120} - \frac{16\pi^3}{48} \approx 18.9915;\ A(K) \approx 0.7799\, D^2\,.$$

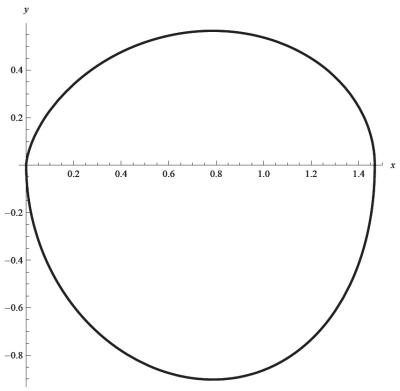

FIGURE 4. $f(\alpha) = \cos\alpha + \frac{\alpha^2}{4}$; $D = \frac{\pi^2}{4} - 1$.

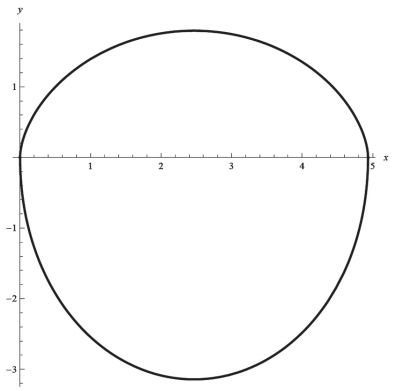

FIGURE 5. $f(\alpha) = \left(\alpha - \frac{\pi}{2}\right)^2 + 2$; $D = \frac{\pi^2}{2}$.

Example 5.6. In this example (Figures 6 through 9) we list the graphs of several c-functions and their appropriated constant width curves.

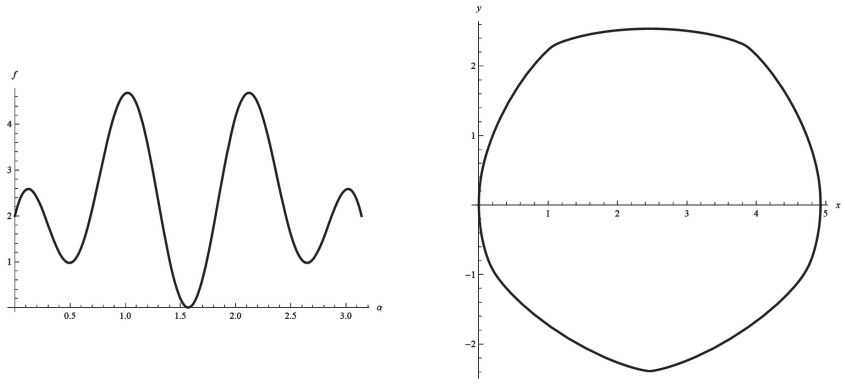

FIGURE 6. $f(\alpha) = 2\cos 6\alpha - 8\sin^2\alpha + 10\sin\alpha$; $D = 5\pi - \frac{1132}{105}$; $A(K) \approx 0.7772\, D^2$.

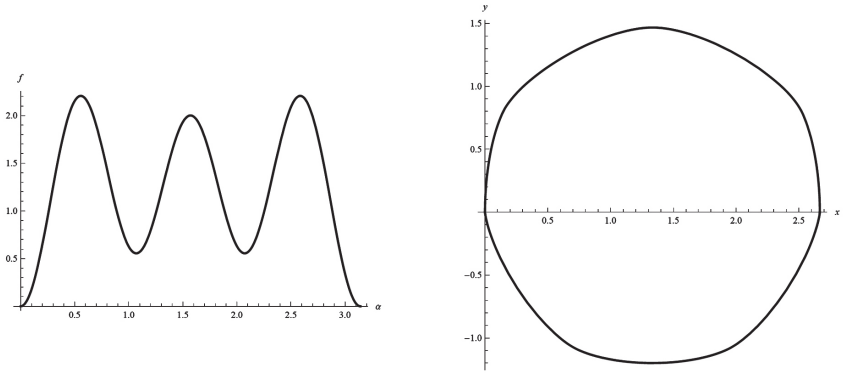

FIGURE 7. $f(\alpha) = 28\sin^6\alpha - 44\sin^4\alpha + 18\sin^2\alpha$; $D = \frac{8}{3}$; $A(K) \approx 0.7816\, D^2$.

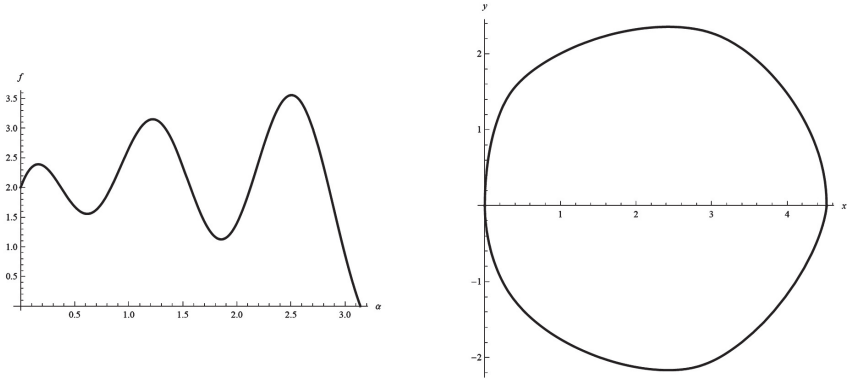

FIGURE 8. $f(\alpha) = \cos 5\alpha - 4\sin^2\alpha + 5\sin\alpha + 1$; $D = \frac{5\pi}{2} - \frac{10}{3}$; $A(K) \approx 0.7817\, D^2$.

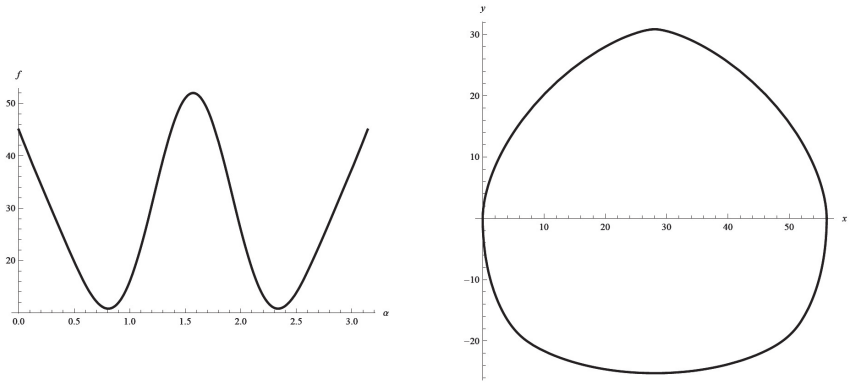

FIGURE 9. $f(\alpha) = 114\sin^6\alpha - 66\sin^4\alpha + 14\sin^2\alpha - 55\sin\alpha + 45$; $D = \frac{14962}{105} - \frac{55\pi}{2}$; $A(K) \approx 0.7721\, D^2$.

Example 5.7. At the end of this section, Figure 10 shows an example of a constant width curve when the function f is discontinuous. Actually, this is the curve from Example 5.2 rotated by $\frac{\pi}{2}$, i.e. parametrized with a different c-function:

$$f(\alpha) = \begin{cases} \cos\alpha : \alpha \in [0, \dfrac{\pi}{2}); \\ \dfrac{\pi}{2} + \cos\alpha : \alpha \in [\dfrac{\pi}{2}, \pi]. \end{cases}$$

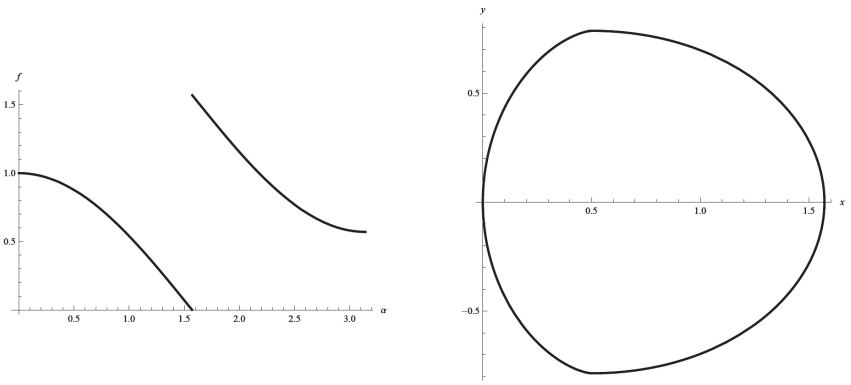

FIGURE 10. It is enough the c-function f to be continuous almost everywhere on $[0, \pi]$.

6. REULEAUX POLYGONS

Let us take a step back to the widely explored Reuleaux polygons. Although these polygons are out of the scope of our main topic, in brief we present some connections for completeness. We are familiar that the boundaries of the Reuleaux polygons are not smooth curves since they are built up of finite number of circular arcs. However, it is well known that any smooth body of constant width can be approximated up to the desired accuracy by some Reuleaux polygon. This convenience helps us in the attempt to describe the Reuleaux polygons by our c-functions, assuming that the set of the points of discontinuity of f is finite. We can partition the interval $[0, \pi]$ into subintervals on which a restriction of f remains continuous. Technically, it means we would be able to perform the integration over each subinterval separately where f is to be some constant, having in mind that actually we are dealing with an integration in the Lebesgue's style. In other words, we skip the integration over a set of measure zero and hence, a slightly modified Theorem 2.1 could be applicable on the Reuleaux polygons as well. Because these polygons are not the main matter herein, let us restrict the survey to the regular Reuleaux polygons. Without getting into the details, our goal is to provide a parametrization of the boundary of any regular Reuleaux polygon by use of improper c-functions.

Example 6.1. Let us consider the Reuleaux triangle. Assuming $D = 1$, we define it as follows:

$$f(\alpha) = \begin{cases} 0 : \alpha \in \left[0, \dfrac{\pi}{3}\right) ; \\ 1 : \alpha \in \left[\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right) ; \\ 0 : \alpha \in \left[\dfrac{2\pi}{3}, \pi\right] . \end{cases}$$

As we noticed, we can calculate the values of the integrals regardless of the discontinuity of f:

$$\int_0^{\pi} f(\alpha)\cos\alpha\, d\alpha = \int_0^{\frac{\pi}{3}} 0\cdot\cos\alpha\, d\alpha + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 1\cdot\cos\alpha\, d\alpha + \int_{\frac{2\pi}{3}}^{\pi} 0\cdot\cos\alpha\, d\alpha = 0\,;$$

$$\int_0^{\pi} f(\alpha)\sin\alpha\, d\alpha = \int_0^{\frac{\pi}{3}} 0\cdot\sin\alpha\, d\alpha + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 1\cdot\sin\alpha\, d\alpha + \int_{\frac{2\pi}{3}}^{\pi} 0\cdot\sin\alpha\, d\alpha = 1\,.$$

The first two conditions (i) and (ii) of Theorem 2.1 are satisfied, but (iii), which provides the curve to be smooth, is not. The Reuleaux triangle has three singular points. Clearly, they exist on a set which measure μ_L is greater than zero and by definition, f is an improper c-function. Before we parametrize the coordinates, the area of the Reuleaux triangle by the formula (3.1) is

$$\begin{aligned} A(K) &= \int_0^{\frac{\pi}{3}} 0\cdot d\alpha + \int_0^{\frac{\pi}{3}} 1^2\cdot\sin^2\alpha\, d\alpha \\ &\quad + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left(1\cdot\sin\alpha - 1\cdot\frac{\sqrt{3}}{2}\right)\cdot 1\cdot\sin\alpha\, d\alpha + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 0\cdot d\alpha \\ &\quad + \int_{\frac{2\pi}{3}}^{\pi} 0\cdot d\alpha + \int_{\frac{2\pi}{3}}^{\pi} 1^2\cdot\sin^2\alpha\, d\alpha \\ &= \int_0^{\pi} \sin^2\alpha\, d\alpha - \frac{\sqrt{3}}{2}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin\alpha\, d\alpha = \frac{\pi - \sqrt{3}}{2}\,. \end{aligned}$$

As we expected, this method of calculating is not suitable especially in case when the Reuleaux polygon has many vertices, but it is still possible. If we speak in general, any regular Reuleaux n-gon can be defined by an improper c-function such that it returns 0 or 1 on each of the n subintervals into which $[0, \pi]$ is partitioned.

Let n be an odd integer, and $n \geq 3$. The regular Reuleaux n-gon of diameter 1 is generated by:

$$f(\alpha) := f_k(\alpha) = \frac{1+(-1)^k}{2} : \alpha \in \left[\frac{(k-1)\pi}{n}, \frac{k\pi}{n}\right) \; ; k = 1, ..., n\,. \tag{6.1}$$

In order to be convinced of its existence, it remains to confirm that f is an improper c-function:

$$\mu_L\left(\{\alpha : f(\alpha) \in \{0,1\}\}\right) = \pi\,(> 0)\;;$$

$$\begin{aligned}
\int_0^{\pi} f(\alpha)\cos\alpha\, d\alpha &= \sum_{k=1}^{n} \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} \frac{1+(-1)^k}{2}\cos\alpha\, d\alpha \\
&= \frac{1}{2}\int_0^{\pi}\cos\alpha\, d\alpha + \frac{1}{2}\sum_{k=1}^{n}\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}(-1)^k\cos\alpha\, d\alpha \\
&= 0 + \frac{1}{2}\sum_{k=1}^{n}(-1)^k\sin\alpha\Big|_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} \\
&= \sum_{k=1}^{n-1}(-1)^k\sin\frac{k\pi}{n} = 0\,;
\end{aligned}$$

$$\begin{aligned}
\int_0^{\pi} f(\alpha)\sin\alpha\, d\alpha &= \sum_{k=1}^{n} \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} \frac{1+(-1)^k}{2}\sin\alpha\, d\alpha \\
&= \frac{1}{2}\int_0^{\pi}\sin\alpha\, d\alpha + \frac{1}{2}\sum_{k=1}^{n}\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}(-1)^k\sin\alpha\, d\alpha \\
&= 1 + \frac{1}{2}\sum_{k=1}^{n}(-1)^{k-1}\cos\alpha\Big|_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} \\
&= \sum_{k=1}^{n-1}(-1)^{k-1}\cos\frac{k\pi}{n} = 1\,.
\end{aligned}$$

In accordance with (2.2), we can deduce the parametric equations of any k^{th} edge of the polygon:

$$\begin{aligned}
x_k(\alpha) &= 1 + (-1)^{k-1}\cos\alpha - \sum_{i=1}^{k}(-1)^{i-1}\cos\frac{(i-1)\pi}{n}\,; \\
y_k(\alpha) &= (-1)^{k-1}\sin\alpha - \sum_{i=1}^{k}(-1)^{i-1}\sin\frac{(i-1)\pi}{n}\,.
\end{aligned} \tag{6.2}$$

The boundary of the regular Reuleaux n-gon (Figure 11) is a union of all n edges given by (6.2),

$$\partial K = \bigcup_{k=1}^{n} \left\{ (x_k, y_k) : \alpha \in \left[\frac{(k-1)\pi}{n}, \frac{k\pi}{n} \right) \right\}.$$

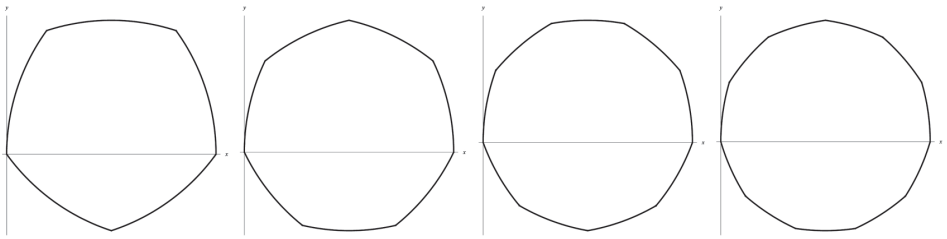

FIGURE 11. Regular Reuleaux 5-gon, 7-gon, 9-gon, 11-gon.

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