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Finsler Metrics with Relatively Isotropic Stretch Curvature

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1. Introduction

The well-known Hilbert's Fourth Problem is to characterize the distance functions on an open subset in $\mathbb{R}^n$ such that straight lines are shortest paths [9]. Distance functions induced by a Finsler metrics are regarded as *smooth* ones. The Hilbert Fourth Problem in the smooth case is to characterize Finsler metrics on an open subset in $\mathbb{R}^n$ whose geodesics are straight lines. Such Finsler metrics are called *projective Finsler metrics*. G. Hamel first characterizes projective Finsler metrics by a system of PDE's [8]. Later on, A. Rapcsák extends Hamel's result to projectively equivalent Finsler metrics [13]. It is well-known that every projective Finsler metric is of scalar curvature, namely, the flag curvature $\mathbf{K}$ is a scalar function of tangent vectors. It is then natural to determine the structure of those with constant (flag) curvature. In the early 20th century, P. Funk classified all projective Finsler metrics with constant curvature on convex domains in R^2 [6] [7]. Then, R. Bryant showed that there is exactly a 2-parameter family of projectively flat Finsler metrics on $\mathbb{S}^2$ with $\mathbf{K} = 1$ and that the only reversible one is the standard Riemannian metric [4] [5]. In [21], Xu-Li has found a family of projectively flat Finsler metrics. The final solution is given by the author and Shahbazi Nia which construct a new class of projectively flat Finsler metric [19].

Let (M, F) be a Finsler manifold. The third order derivative of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is the Cartan torsion $\mathbf{C}_y$ on T_xM. The rate of change of the Cartan torsion along geodesics is said to be Landsberg curvature. Finsler metrics with vanishing Landsberg curvature are called Landsberg metrics. As a generalization of Landsberg curvature, Berwald introduced the notion of stretch curvature and denoted it by $\mathbf{\Sigma}_y$ [3]. He showed that $\mathbf{\Sigma} = 0$ if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram. Then, this curvature investigated by Matsumoto in [11]. For $y \in T_xM_0$, define the stretch curvature $\mathbf{\Sigma}_y : T_xM \otimes T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{\Sigma}_y(u, v, w, z) := \Sigma_{ijkl}(y)u^iv^jw^kz^l$, where

$$\Sigma_{ijkl} := 2(L_{ijk|l} - L_{ijl|k}), \tag{1.1}$$

where "|" denotes the horizontal derivation with respect to the Berwald connection of F. A Finsler metric is said to be a stretch metric if $\mathbf{\Sigma} = 0$. In [20], Tayebi-Sadeghi characterized the stretch (α, β)-metrics with vanishing S-curvature.

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Theorem 1.1. ([20]) *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type (α,β)-metric with vanishing S-curvature on a manifold M of dimension $n \geq 3$. Suppose that F is a stretch metric. Then one of the following holds:*

: *(i) If F is a regular metric, then it reduces to a Berwald metric;*
: *(ii) If F is an almost regular metric which is not Berwaldian, then ϕ is given by*

$$\phi = c\exp\left[\int_0^s \frac{kt + q\sqrt{b^2-t^2}}{1+kt^2+qt\sqrt{b^2-t^2}}dt\right], \tag{1.2}$$

where $c > 0$, $q > 0$ and k are real constants. In this case, F is not a Landsberg metric.

Also, in [18], Tayebi-Najafi give a description of stretch homogeneous metrics, in the special class of (α,β)-metrics and proved the following:

Theorem 1.2. ([18]) A homogeneous (α,β)-metric on a manifold M is a stretch metric if and only if it is a Berwald metric.

Let us come back to the projective Finsler metrics constructed by Funk. A Finsler metric F satisfying $F_{x^k} = FF_{y^k}$ is called a Funk metric. The standard Funk metric on the Euclidean unit ball is defined by

$$F_f(x,y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x,y>^2)}}{1-|x|^2} + \frac{<x,y>}{1-|x|^2}, \quad y \in T_xB^n(1) \simeq \mathbb{R}^n,$$

where $<,>$ and $|.|$ denote the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. By $G^i = \frac{1}{2}Fy^i$, it follows that F is has special stretch curvature $\Sigma_{ijkl} = -F(C_{ijk|l} - C_{ijl|k})$.

Definition 1.1. Let (M,F) be a Finsler manifold. Then F is called a relatively isotropic stretch metric if its stretch curvature is given by

$$\Sigma_{ijkl} = cF(C_{ijk|l} - C_{ijl|k}), \tag{1.3}$$

where $c = c(x)$ is a scalar function on M. In this case, (M,F) is called a relatively isotropic stretch manifold.

Example 1. Let us define

$$F_a(x,y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x,y>^2)}}{1-|x|^2} + \frac{<x,y>}{1-|x|^2} + \frac{<a,y>}{1+<a,x>}, \quad y \in T_x\mathbb{B}^n(1) \simeq \mathbb{R}^n,$$

where $a \in \mathbb{R}$ is a constant vector with $|a| < 1$. For $a \neq 0$, it is easy to see that F_a are locally projectively flat Finsler metrics with negative constant flag curvature. It follows that F is a relatively isotropic stretch metric with $c = -1$.

Every Landsberg metric is a stretch metric. In this paper, we find a topological condition under which a relatively isotropic stretch manifold reduces to a Landsberg manifold.

Theorem 1.3. Every complete relatively isotropic stretch manifold with bounded Landsberg curvature is a Landsberg manifold.

In [14], Szabó consider Berwald surfaces and prove a rigidity theorem: every Berwald surfaces is Riemannian or locally Minkowskian. On the other hand, the class of Berwald metrics is a very special subclass of the class of relatively isotropic stretch metrics. This motivates us to consider relatively isotropic stretch surfaces.

Theorem 1.4. Let (M,F) be a Finsler surface. Suppose that F has relatively isotropic stretch curvature such that

$$(c+\mu)\mu F - 2\mu' \neq 0,$$

where $\mu := -2I_{,1}/I$, $I = I(x,y)$ is the main scalar of F and $\mu' := \mu_{|l}y^l$. Then F is a Riemannian metric.

By definition, $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along Finslerian geodesics. A non-Riemannian Finsler metric F is called relatively isotropic Landsberg metric if it satisfies

$$\mathbf{L} = \lambda F \mathbf{C}, \tag{1.4}$$

where $\lambda = \lambda(x)$ is a scalar function on M. Izumi called these metrics by $^*P$-Finsler metrics. In [12], Okada proved that the Funk metric F_f satisfies the following equation

$$\frac{\partial F_f}{\partial x^i} = F_f \frac{\partial F_f}{\partial y^i}. \tag{1.5}$$

Okada used (1.5) to prove the fact that the Funk metric is of constant curvature $\kappa = -1/4$. Using (1.5), one can show that the Funk metric satisfies

$$\mathbf{L}_y(u,v,w) = -\frac{1}{2}F(y)\mathbf{C}_y(u,v,w) = 0. \tag{1.6}$$

This leads to study of relatively isotropic Landsberg metrics which was first considered by H. Izumi [10] [16]. In this paper, we prove the following.

Theorem 1.5. Let (M,F) be a relatively isotropic stretch manifold. Suppose that F has relatively isotropic Landsberg curvature such that

$$2\lambda' + 2\lambda^2 F - c\lambda F \neq 0,$$

where $\lambda' := \lambda_{|l}y^l$. Then F is a Riemannian metric.

In [2], Bajancu-Farran introduced a new class of Finsler metrics, called generalized Landsberg metrics. This class of Finsler metrics contains the class of Landsberg metrics as a special case. A Finsler metric F on a manifold M is called generalized Landsberg metric if its Landsberg curvature satsifies

$$L^i{}_{jl|k} - L^i{}_{jk|l} + L^i{}_{sk}L^s{}_{jl} - L^i{}_{sl}L^s{}_{jk} = 0,$$

where "|" denotes the horizontal derivation with respect to the Berwald connection of F. Indeed, F is a generalized Landsberg metric if the h-curvatures of Berwald and Chern connections coincide in the sense of Bajancu-Farran [2].

Theorem 1.6. Let (M,F) be a a relatively isotropic stretch manifold. Suppose that F has generalized Landsberg curvature. Then F is a Landsberg metric.

2. Preliminaries

Let (M,F) be an n-dimensional Finsler manifold. The fundamental tensor $\mathbf{g}_y : T_xM \otimes T_xM \to \mathbb{R}$ of F is defined by following

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y+su+tv)\Big]|_{s,t=0}, \;\; u,v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]|_{t=0}, \;\; u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. By definition, $\mathbf{C}_y$ is a symmetric trilinear form on T_xM. It is well known that, **C=0** if and only if F is Riemannian.

Given a Finsler manifold (M,F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where $G^i = G^i(x,y)$ are scalar functions on TM_0 given by

$$G^i := \frac{1}{4}g^{ij}\left\{\frac{\partial^2[F^2]}{\partial x^k \partial y^j}y^k - \frac{\partial[F^2]}{\partial x^j}\right\}, \;\; y \in T_xM. \tag{2.1}$$

The $\mathbf{G}$ is called the spray associated to (M,F). In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a non-zero vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

The quantity $\mathbf{B}$ is called the Berwald curvature. F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

For $y \in T_xM$, define the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u,v,w) := -\frac{1}{2}\mathbf{g}_y\big(\mathbf{B}_y(u,v,w),y\big).$$

F is called a Landsberg metric if $\mathbf{L}_y = 0$. By definition, every Berwald metric is a Landsberg metric. Also, the Landsberg curvature of F can be defined by following

$$\mathbf{L}_y(u,v,w) := \frac{d}{dt}\Big[\mathbf{C}_{\dot\sigma(t)}\big(U(t),V(t),W(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U(t),V(t),W(t)$ are linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$. Then the Landsberg curvature $\mathbf{L}_y$ is the rate of change of $\mathbf{C}_y$ along geodesics for any $y \in T_xM_0$.

For $y \in T_xM_0$, define the stretch curvature $\mathbf{\Sigma}_y : T_xM \otimes T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{\Sigma}_y(u,v,w,z) := \Sigma_{ijkl}(y)u^iv^jw^kz^l$, where

$$\Sigma_{ijkl} := 2(L_{ijk|l} - L_{ijl|k}), \tag{2.2}$$

and "|" denotes the horizontal derivation with respect to the Berwald connection of F. A Finsler metric is said to be a stretch metric if $\mathbf{\Sigma} = 0$.

3. Proof of Theorem 1.3

In this section, we are going to prove Theorem 1.3.

Proof of Theorem 1.3: Let p be an arbitrary point of manifold M, and $y, u, v, w \in T_pM$. Let $c : (-\infty, \infty) \to M$ is the unit speed geodesic passing from p and

$$\frac{dc}{dt}(0) = y.$$

If $U(t), V(t)$ and $W(t)$ are the parallel vector fields along c with

$$U(0) = u, \quad V(0) = v \quad W(0) = w.$$

Let us put

$$\mathbf{L}(t) = \mathbf{L}(U(t), V(t), W(t)), \quad \mathbf{L}'(t) = \mathbf{L}'_{\dot{c}}(U(t), V(t), W(t)).$$

By contracting (1.3) with y^l, we get

$$L_{ijk|l}y^l = cFC_{ijl|k}y^l = cFL_{ijk}. \tag{3.1}$$

By definition, we have the following ODE,

$$\mathbf{L}'(t) = c\mathbf{L}(t). \tag{3.2}$$

Its general solution is

$$\mathbf{L}(t) = e^{ct}\mathbf{L}(0). \tag{3.3}$$

Using $||\mathbf{L}|| < \infty$, and letting $t \to +\infty$ or $t \to -\infty$, we get

$$\mathbf{L}(0) = \mathbf{L}(u, v, w) = 0.$$

So $\mathbf{L} = 0$, i.e., (M, F) is a Landsberg manifold. □

Every compact Finsler manifold (M, F) is complete and all of tensors on M are bounded. Then, by Theorem 1.3 we get the following result.

Corollary 3.1. Every compact relatively isotropic stretch manifold is a Landsberg manifold.

4. Proof of Theorem 1.4

Proof of Theorem 1.4: Let (M, F) be a two-dimensional Finsler manifold. We refer to the Berwald's frame (ℓ^i, m^i) where $\ell^i = y^i/F(y)$, m^i is the unit vector with $\ell_i m^i = 0$ and $\ell_i = g_{ij}\ell^i$. Then the Berwald curvature is given by

$$B^i{}_{jkl} = F^{-1}(-2I_{,1}\ell^i + I_2 m^i)m_j m_k m_l,$$

where I is 0-homogeneous function called the main scalar of F and

$$I_2 = I_{,2} + I_{,1|2}.$$

See page 689 in [1]. Since the Cartan tensor of F is given by

$$C_{ijk} = F^{-1}Im_i m_j m_k,$$

then the Berwald curvature can be written as following

$$B^i{}_{jkl} = \mu C_{jkl}\ell^i + \lambda(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}), \tag{4.1}$$

where $h_{ij} := m_i m_j$ is the angular metric and

$$\mu = -\frac{2I_{,1}}{I}, \quad \lambda = \frac{I_2}{3}.$$

Contracting (4.1) with y_i implies that

$$L_{ijk} = -\frac{1}{2}\mu F C_{ijk}. \tag{4.2}$$

Thus

$$L_{ijk|l} = -\frac{1}{2}F(\mu_l C_{ijk} + \mu C_{ijk|l}). \tag{4.3}$$

Therefore

$$\Sigma_{ijkl} = -F\Big[\mu_l C_{ijk} - \mu_k C_{ijl} + \mu(C_{ijk|l} - C_{ijl|l})\Big]. \tag{4.4}$$

By (1.3) and (4.4), we get

$$(c+\mu)(C_{ijk|l} - C_{ijl|k}) = -[\mu_l C_{ijk} - \mu_k C_{ijl}]. \tag{4.5}$$

Multiplying (4.5) with y^l yields

$$(c+\mu)L_{ijk} = -\mu' C_{ijk}. \tag{4.6}$$

Putting (4.2) in (4.6) implies that

$$[(c+\mu)\mu F - 2\mu']C_{ijk} = 0. \tag{4.7}$$

By assumption, we get $\mathbf{C} = 0$ and F reduces to a Riemannian metric. □

5. Proof of Theorem 1.5

Proof of Theorem 1.5: F has relatively isotropic Landsberg curvature

$$L_{ijk} = \lambda F C_{ijk}, \tag{5.1}$$

where $\lambda = \lambda(x)$ is a scalar function on M. Taking a horizontal derivation of (5.1) along Finslerian geodesics implies that

$$\begin{aligned} L_{ijk|l}y^l &= F\big[\lambda' C_{ijk} + \lambda L_{ijk}\big] \\ &= F\big[\lambda' + \lambda^2 F\big]C_{ijk}, \end{aligned} \tag{5.2}$$

where $\lambda' := \lambda_{|l}y^l$. On the other hand, contracting (1.3) with y^l implies that

$$2L_{ijk|l}y^l = cFL_{ijk}. \tag{5.3}$$

By (5.1), (5.2) and (5.3) we get

$$\big[2\lambda' + 2\lambda^2 F - c\lambda F\big]C_{ijk} = 0. \tag{5.4}$$

By assumption, we get $\mathbf{C} = 0$ and F reduces to a Riemannian metric. □

6. Proof of Theorem 1.6

Proof of Theorem 1.6: By definition, we have

$$\Sigma_{ijkl} = cF(C_{ijk|l} - C_{ijl|k}), \tag{6.1}$$

In [17], it is proved that F is a generalized Landsberg metric if and only if the following equations hold

$$L_{isk}L^s{}_{jl} - L_{isl}L^s{}_{jk} = 0, \tag{6.2}$$

$$L_{ijl|k} - L_{ijk|l} = 0. \tag{6.3}$$

By (6.1) and (6.3), we get

$$C_{ijk|l} = C_{ijl|k}. \tag{6.4}$$

Contracting (6.4) with y^l implies that

$$L_{ijk} = 0. \tag{6.5}$$

Thus F is a Landsberg metric. □

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