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GEODESIC VECTORS OF EXPONENTIAL METRICS ON NILPOTENT LIE GROUPS OF DIMENSION FIVE

PARISA BAHMANDOUST AND DARIUSH LATIFI

ABSTRACT. In this paper we consider invariant exponential metrics which are induced by invariant Riemannian metrics and invariant vector fields on homogeneous spaces. We study geodesic vectors and investigates the set of all homogeneous geodesics on two-step nilpotent Lie groups of dimension five.

1. INTRODUCTION

Let $\mathfrak{n}$ denote a finite dimensional Lie algebra over the real numbers. The Lie algebra $\mathfrak{n}$ is called 2-step nilpotent Lie algebra if $[x,[y,z]] = 0$ for any $x,y,z \in \mathfrak{n}$. A Lie group N is said to be 2-step nilpotent if its Lie algebra $\mathfrak{n}$ is 2-step nilpotent. Throughout, N will denote a simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ having center $\mathfrak{z}$. The first general studies for 2-step nilpotent Lie groups were done by P. Eberlein [6]. There are some recent papers on the geometry of two-step nilpotent groups with left invariant metrics [12–14].
Let $\mathfrak{g}$ be a Lie algebra and G be the corresponding connected and simply connected Lie group. A metric Lie algebra $(\mathfrak{g}, \langle,\rangle)$ is a Lie algebra $\mathfrak{g}$ together with a Euclidean inner product $\langle,\rangle$ on $\mathfrak{g}$. This inner product on $\mathfrak{g}$ induces a left invariant Riemannian metric on the Lie group G in a natural way. Let $(\mathfrak{n}, \langle,\rangle)$ be a nilpotent metric Lie algebra. The corresponding nilpotent Lie group N endowed with the left invariant metric arising from $\langle,\rangle$ is a Riemannian nilmanifold. E. Wilson described a classification procedure for the isometry equivalence class of Riemannian nilmanifolds [15]. This is applied by J. Lauret for the determination of 3 and 4 dimensional Riemannian nilmanifold up to isometry and isometry group [11]. S. Homolya and O. Kowalski have classified in [8] all 5-dimensional 2-step nilpotent Riemannian nilmanfolds and their isometry groups. For this reason they classified metric Lie algebras with one, two and three dimensional center. We use their results in this paper.

An $(\alpha,\beta)-$metric is a Finsler metric of the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on a connected smooth n-dimensional manifold M and $\beta = b_i(x)y^i$ is a 1-form on M. Some interesting class of $(\alpha,\beta)-$metrics are Randers metric, Kropina metric, Matsumoto metric, and exponential metric $F = \alpha\exp(\frac{\beta}{\alpha})$. In this paper we study the geometry of simply connected two-step nilpotent Lie groups of dimension five endowed with left invariant exponential metrics. We consider homogeneous geodesics in a left

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invariant exponential metric on simply connected two-step nilpotent Lie groups of dimension five.

2. Preliminaries

Let M be a smooth n-dimensional C^∞ manifold and TM be its tangent bundle. A Finsler metric on a manifold M is a non-negative function $F : TM \longrightarrow R$ with the following properties [2]:

(1) F is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$.
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$.
(3) The $n \times n$ Hessian matrix $(g_{ij}) = (\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j})$ is positive definite at every point $(x, y) \in TM^0$.

The following bilinear symmetric form $g_y : T_xM \times T_xM \longrightarrow R$ is positive definite

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{s=t=0}.$$

Definition. Let $\mathfrak{n}$ be a Lie algebra and N is the simply connected Lie group with Lie algebra $\mathfrak{n}$. A Finsler metric $F : TN \longrightarrow [0, \infty)$ will be called left-invariant if

$$\forall a \in N, \quad \forall X \in \mathfrak{n}, \quad F((L_a)_{*e}X) = F(X),$$

where L_a is the left translation and e is the unit element of the Lie group.
The left invariant Finsler functions on TN may be identified with Minkowski norms on $\mathfrak{n}$. By left translations, for every Minkowski norm $\tilde{F}$ on $\mathfrak{n}$ we can define a left invariant Finsler metric on N

$$\forall a \in N, X_e \in \mathfrak{n}, F((L_a)_* X_e) := \tilde{F}(X_e).$$

Definition. Let $\alpha = \sqrt{\tilde{a}_{ij}(x) y^i y^j}$ be a Riemannian metric and $\beta(x, y) = b_i(x) y^i$ be a $1-$form on an $n-$dimensional manifold M. Let

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x) b_i(x) b_j(x)}. \tag{2.1}$$

Now, let the function F is defined as follows

$$F := \alpha\phi(s) \quad , \quad s = \frac{\beta}{\alpha}, \tag{2.2}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad , \quad |s| \leq b < b_0. \tag{2.3}$$

Then by lemma 1.1.2 of [3], F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (2.2) is called an $(\alpha, \beta)-$metric [1].

Let M be a smooth manifold. Suppose that $\tilde{a}$ and β are a Riemannian metric and a 1-form on M respectively. In this case we can write the exponential metric on M as follows:

$$F(x, y) = \alpha(x, y) \exp(\frac{\beta(x, y)}{\alpha(x, y)}).$$

The Riemannian metric $\tilde{a}$ induce a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field X on M such that

$$\tilde{a}(X_x, y) = \beta(x, y).$$

Therefore we can write the exponential metric $F = \alpha \exp(\frac{\beta}{\alpha})$ as follows:

$$F(x,y) = \sqrt{\tilde{a}(y,y)}\exp(\frac{\tilde{a}(X,y)}{\sqrt{\tilde{a}(y,y)}}).$$

Definition. Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra identified with the tangent space at the identity element, $\widetilde{F} : \mathfrak{g} \longrightarrow R_+$ a Minkowski norm and F the left-invariant Finsler metric induced by $\widetilde{F}$ on G. A geodesic $\gamma : R_+ \longrightarrow G$ is said to be homogeneous if there is a $Z \in \mathfrak{g}$ such that $\gamma(t) = \exp(tZ)\gamma(0)$, $t \in R_+$ holds. A tangent vector $X \in T_eG - \{0\}$ is said to be a geodesic vector if the 1-parameter subgroup $t \longrightarrow \exp(tZ)$, $t \in R_+$, is a geodesic of F [9, 10].

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\tilde{a}$ be a left-invariant Riemannian metric on G. In [9], it is proved that a vector $Y \in \mathfrak{g}$ is a geodesic vector if and only if

$$\tilde{a}(Y, [Y,Z]) = 0, \qquad \forall Z \in \mathfrak{g}. \tag{2.4}$$

For results on homogeneous geodesics in homogeneous Finsler manifolds we refer to [4, 5, 7, 10, 16, 17]. The following result proved in [10] gives a criterion for non-zero vector to be a geodesic vector in a homogeneous Finsler space.

Lemma 2.1. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let F be a left invariant Finsler metric on G. Then $Y \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if

$$g_Y(Y, [Y,Z]) = 0$$

holds for every $Z \in \mathfrak{g}$.

Next, we deduce necessary and sufficient condition for a non-zero vector in a two-step nilpotent Lie group of dimension five with left-invariant exponential Finsler metric to be a geodesic vector.

3. Lie algebras with 1-dimensional center

In this section we study simply connected two-step nilpotent Lie group of dimension five with 1-dimensional center equipped with left-invariant exponential Finsler metric. Let $\mathfrak{n}$ denotes a 5-dimensional 2-step nilpotent Lie algebra with 1-dimensional center $\mathfrak{z}$ and let N be the corresponding simply connected Lie group. We assume that $\mathfrak{n}$ is equipped with an inner product $\langle , \rangle$. Let e_5 be a unit vector in $\mathfrak{z}$ and let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$. In [8] S. Homolya and O. Kowalski showed that there exist an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{n}$ such that

$$[e_1, e_2] = \lambda e_5, \quad [e_3, e_4] = \mu e_5, \tag{3.1}$$

where $\lambda \geq \mu > 0$. Also the other commutators are zero.

Example 3.1. Let $x_1, x_2, y_1, y_2, z \in R$. H_5 , the 5-dimensional Heisenberg group is the group of matrices with the elements of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Heisenberg group is, up to isomorphism, the only two-step nilpotent Lie group with 1-dimensional center [8]. The Lie algebra $\mathfrak{h}_5$ is a 5-dimensional vector space with basis $\{X_1, X_2, Y_1, Y_2, Z\}$ and the only non-zero Lie bracket are $[X_i, Y_i] = -[Y_i, X_i] = Z$ for $i = 1, 2$. The center $\mathfrak{z}$ of $\mathfrak{h}_5$ is $\mathfrak{z} = span\{z\}$.

Let F be a left invariant exponential metric on simply connected two-step nilpotent Lie group N defined by the Riemannian metric $\tilde{a} = \tilde{\langle , \rangle}$ and the vector field $X = \Sigma_{i=1}^{5} x_i e_i$. We want to describe all geodesic vectors of (N, F).
By using the formula

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial t \partial s} F^2(y + su + tv)|_{s=t=0},$$

and some computations for the exponential metric F defined by relation

$$F(x, y) = \sqrt{\tilde{a}(y, y)} \exp(\frac{\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}}),$$

we get:

$$\begin{aligned} g_y(u, v) \quad = \quad & \exp(\frac{2\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}})\Big(\tilde{a}(u, v) + 2\tilde{a}(X, u)\tilde{a}(X, v) - \frac{\tilde{a}(X, y)\tilde{a}(y, u)\tilde{a}(y, v)}{\tilde{a}(y, y)^{\frac{3}{2}}}\Big) \\ & + \exp(\frac{2\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}})\frac{1}{\sqrt{\tilde{a}(y, y)}}\Big(\tilde{a}(X, u)\tilde{a}(y, v) + \tilde{a}(X, v)\tilde{a}(y, u) - \tilde{a}(X, y)\tilde{a}(u, v)\Big) \quad (3.2) \\ & + \exp(\frac{2\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}})\frac{2\tilde{a}(X, y)}{\tilde{a}(y, y)}\Big(\frac{\tilde{a}(X, y)\tilde{a}(y, u)\tilde{a}(y, v)}{\tilde{a}(y, y)} - \tilde{a}(y, u)\tilde{a}(X, v) - \tilde{a}(X, v)\tilde{a}(y, v)\Big). \end{aligned}$$

From equation (3.2), for all $z \in \mathfrak{n}$ we have

$$g_y(y, [y, z]) = \tilde{a}\Big(X + (\frac{\sqrt{\tilde{a}(y, y)} - \tilde{a}(X, y)}{\tilde{a}(y, y)})y, [y, z]\Big)\sqrt{\tilde{a}(y, y)} \exp(\frac{2\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}}) \quad (3.3)$$

By using lemma 2.1 and equation (3.3) a vector $y = \sum_{i=1}^{5} y_i e_i$ of $\mathfrak{n}$ is a geodesic vector if and only if

$$\tilde{a}\Big(\sum_{i=1}^{5} x_i e_i + (\frac{\sqrt{\sum y_i^2} - \sum x_i y_i}{\sum y_i^2}) \sum y_i e_i, \Big[\sum y_i e_i, e_j\Big]\Big) = 0. \quad (3.4)$$

So we get

$$
\begin{aligned}
\lambda y_1\Big(x_5+\frac{(\sqrt{\sum y_i^2}-\sum x_iy_i)y_5}{\sum y_i^2}\Big) &= 0\\
\lambda y_2\Big(x_5+\frac{(\sqrt{\sum y_i^2}-\sum x_iy_i)y_5}{\sum y_i^2}\Big) &= 0\\
\lambda y_3\Big(x_5+\frac{(\sqrt{\sum y_i^2}-\sum x_iy_i)y_5}{\sum y_i^2}\Big) &= 0 \qquad (3.5)\\
\lambda y_4\Big(x_5+\frac{(\sqrt{\sum y_i^2}-\sum x_iy_i)y_5}{\sum y_i^2}\Big) &= 0.
\end{aligned}
$$

Corollary 3.1. Let (N,F) be a Finsler space with exponential metric F defined by an invariant Riemannina metric $\tilde{a}$ and the left invariant vector field $X=\sum_{i=1}^{5}x_ie_i$ on simply connected two-step nilpotent Lie group of dimension five with one dimensional center. Then geodesic vectors depending only on x_5.

Corollary 3.2. Let (N,F) be a Finsler space with exponential metric F defined by an invariant Riemannina metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie group of dimension five with one dimensional center. Then X is a geodesic vector of $(N,\tilde{a})$ if and only if X is a geodesic vector of (N,F).

Corollary 3.3. Let F be the exponential metric defined by the invariant Riemannian metric $\tilde{a}$ and the left invariant vector field $X=\sum_{i=1}^{4}x_ie_i$ on simply connected two-step nilpotent Lie group of dimension five with one dimensional center. Then a vector $y\in\mathfrak{n}$ is a geodesic vector if and only if $y\in span\{e_1,e_2,e_3,e_4\}$ or $y=\beta e_5$ for $\beta\neq 0$.

Theorem 1. *Let (N,F) be a Finsler space with exponential metric F defined by an invariant Riemannian metric $\tilde{a}$ and an invariant vector field $X=\sum_{i=1}^{4}x_ie_i$ on simply connected two-step nilpotent Lie group of dimension five with one dimensional center. Then $y\in\mathfrak{n}$ is a geodesic vector of (N,F) if and only if y is geodesic vector of $(N,\tilde{a})$.*

Proof: From (3.1), $\tilde{a}(X,[y,e_i])=0$ for each $i=1,2,3,4,5$. Therefore from equation (3.3) we can write

$$g_y(y,[y,z])=\tilde{a}(y,[y,z])\Big(1-\frac{\tilde{a}(X,y)}{\sqrt{\tilde{a}(y,y)}}\Big)\exp\Big(\frac{2\tilde{a}(X,y)}{\sqrt{\tilde{a}(y,y)}}\Big).$$

Therefore $g_y(y,[y,z])=0$ if and only if $\tilde{a}(y,[y,z])=0$. □

4. Lie algebras with 2-dimensional center

In this section we study simply connected two-step nilpotent Lie group of dimension five with 2-dimensional center equipped with left-invariant exponential Finsler metric. Let $\mathfrak{n}$ denotes a 5-dimensional two-step nilpotent Lie algebra the center $\mathfrak{z}$ of which is two-dimensional and let N be the corresponding simply connected Lie group. We assume that $\mathfrak{n}$ is equipped with an inner product $\langle,\rangle$. In [8] S. Homolya and O. Kowalski showed that there exist an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{n}$ such that

$$[e_1, e_2] = \lambda e_4 \qquad , \qquad [e_1, e_3] = \mu e_5, \tag{4.1}$$

where $\{e_4, e_5\}$ is a basis for the center $\mathfrak{z}$, the other commutators are zero and $\lambda \geq \mu > 0$.

Let F be a left invariant exponential metric on simply connected two-step nilpotent Lie group of dimension five with two dimensional center defined by the Riemannian metric $\tilde{a}$ and the vector field $X = \sum_{i=1}^{5} x_i e_i$.
By using lemma 2.1 and equation (3.3) a vector $y = \sum y_i e_i$ of F is a geodesic vector if and only if

$$\tilde{a}\Big(\sum_{i=1}^{5} x_i e_i + (\frac{\sqrt{\sum y_i^2} - \sum x_i y_i}{\sum y_i^2}) \sum y_i e_i, \Big[\sum y_i e_i, e_j\Big]\Big) = 0, \tag{4.2}$$

for each $j = 1, 2, 3, 4, 5$. So we have

$$\begin{aligned} \lambda y_2\Big(x_4 + (\frac{\sqrt{\sum y_i^2} - \sum x_i y_i}{\sum y_i^2}) y_4\Big) + \mu y_3 (x_5 + \Big(\frac{\sqrt{\sum y_i^2} - \sum x_i y_i}{\sum y_i^2}) y_5\Big) &= 0 \\ \lambda y_1\Big(x_4 + (\frac{\sqrt{\sum y_i^2} - \sum x_i y_i}{\sum y_i^2}) y_4\Big) &= 0 \\ \lambda y_1\Big(x_5 + (\frac{\sqrt{\sum y_i^2} - \sum x_i y_i}{\sum y_i^2}) y_5\Big) &= 0 \end{aligned} \tag{4.3}$$

Corollary 4.1. Let (N, F) be a Finsler space with exponential metric F defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = \sum_{i=1}^{5} x_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with two dimensional center. Then geodesic vectors depending on λ, μ, x_4 and x_5.

Corollary 4.2. Let (N, F) be a Finsler space with exponential metric F defined by an invariant Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie group of dimension five with two dimensional center. Then X is a geodesic vector of $(N, \tilde{a})$ if and only if X is a geodesic vector of (N, F).

Theorem 2. *Let (N, F) be a Finsler space with exponential metric F defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = \sum_{i=1}^{3} x_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with two dimensional center. Then $y \in \mathfrak{n}$ is a geodesic vector of (N, F) if and only if y is a geodesic vector of $\tilde{a}$.*

Proof: Let $y = \sum_{i=1}^{5} y_i e_i \in \mathfrak{n}$. From (4.1), $\tilde{a}(X, [y, e_i]) = 0$ for each $i = 1, 2, 3, 4, 5$. Let y is a geodesic vector of $(N, \tilde{a})$ by using equation (2.4) we have $\tilde{a}(y, [y, e_i]) = 0$ for each i=1,2,3,4,5. Therefore by using (4.2), y is a geodesic vector of (N, F). Conversely let $y = \sum_{i=1}^{5} y_i e_i \in \mathfrak{n}$ is a geodesic vector of (N, F), because $\tilde{a}(X, [y, e_i]) = 0$ for each $i = 1, 2, 3, 4, 5$ by using (4.2) we have $\tilde{a}(y, [y, e_i]) = 0$. $\square$

5. Lie algebras with 3-dimensional center

In this section we study simply connected two-step nilpotent Lie group of dimension five with 3-dimensional center equipped with left-invariant exponential Finsler metric. In [8] S. Homolya and O. Kowalski showed that there exist an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{n}$ such that

$$[e_1, e_2] = \lambda e_3, \tag{5.1}$$

where $\{e_3, e_4, e_5\}$ is a basis for the center of $\mathfrak{n}$, the other commutators are zero and $\lambda > 0$.

Example 5.1. The direct sum of the Hiesenberg Lie algebra $\mathfrak{h}_3$ and the abelian Lie algebra $\mathbb{R}^2$, $\mathfrak{h}_3 \oplus \mathbb{R}^2$, is a two-step nilpotent 5-dimensional Lie algebra with 3-dimensional center. The corresponding simply connected Lie group is $H_3 \times \mathbb{R}^2$.

Let F be a left invariant exponential metric on simply connected two-step nilpotent Lie group of dimension five with 3-dimensional center defined by the Riemannian metric $\tilde{a}$ and the vector field $X = \sum_{i=1}^{5} x_i e_i$.
By using Lemma 2.1 and equation (3.3) a vector $y = \sum y_i e_i$ of F is a geodesic vector if and only if

$$\tilde{a}\Big(\sum_{i=1}^{5} x_i e_i + (\frac{\sqrt{\sum y_i^2} - \sum x_i y_i}{\sum y_i^2}) \sum y_i e_i, \Big[\sum y_i e_i, e_j\Big]\Big) = 0, \tag{5.2}$$

for each $j = 1, 2, 3, 4, 5$. So we have

$$\begin{aligned} \lambda y_1 \Big(x_3 + (\frac{\sqrt{\sum y_i^2} - \sum x_i y_i}{\sum y_i^2}) y_3\Big) &= 0 \\ \lambda y_3 \Big(x_3 + (\frac{\sqrt{\sum y_i^2} - \sum x_i y_i}{\sum y_i^2}) y_3\Big) &= 0 \end{aligned} \tag{5.3}$$

Corollary 5.1. Let (N, F) be a Finsler space with exponential metric F defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = \sum_{i=1}^{5} x_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with three dimensional center. Then geodesic vectors depending only on x_3.

Corollary 5.2. Let (N, F) be a Finsler space with exponential metric F defined by an invariant Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie group of dimension five with three dimensional center. Then X is a geodesic vector of $(N, \tilde{a})$ if and only if X is a geodesic vector of (N, F).

Theorem 3. *Let (N, F) be a Finsler space with exponential metric F defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = x_1 e_1 + x_2 e_2 + x_4 e_4 + x_5 e_5$ on simply connected two-step nilpotent Lie group of dimension five with three dimensional center. Then $y \in \mathfrak{n}$ is a geodesic vector if and only if $y \in Span\{e_3, e_4, e_5\}$ or $y \in Span\{e_1, e_2, e_4, e_5\}$.*

Corollary 5.3. Let F be the exponential metric of Berwald type on simply connected two-step nilpotent Lie group of dimension five N with three dimensional center induced by the Riemannian metric $\tilde{a}$ and the vector field X. Then its geodesic vectors are forms of $y \in Span\{e_3, e_4, e_5\}$ or $y \in Span\{e_1, e_2, e_4, e_5\}$

Proof: The Levi-Civita connection of the $(N,\tilde{a})$ can be obtained as the following

$$\begin{array}{lll} \nabla_{e_1}e_2=\frac{\lambda}{2}e_3, & \nabla_{e_2}e_1=-\frac{\lambda}{2}e_3, & \nabla_{e_1}e_3=-\frac{\lambda}{2}e_2, \\ \nabla_{e_3}e_1=-\frac{\lambda}{2}e_2, & \nabla_{e_2}e_3=\frac{\lambda}{2}e_1, & \nabla_{e_3}e_2=\frac{\lambda}{2}e_1, \end{array} \tag{5.4}$$

the other connection components are zero.
Let $X=\sum x_ie_i$ be a left invariant vector field on N which is parallel with respect to the Riemannian connection of $\tilde{a}$. By a direct computation we have

$$X=x_4e_4+x_5e_5.$$

Now by using theorem 3 the proof is completed. □

Theorem 4. *Let (N,F) be a Finsler space with exponential metric F defined by an invariant metric $\tilde{a}$ and an invariant vector field $X=x_1e_1+x_2e_2+x_4e_4+x_5e_5$ on simply connected two-step nilpotent Lie group of dimension five with three dimensional center.Then $y\in\mathfrak{n}$ is a geodesic vector of (N,F) if and only if y is a geodesic of $(N,\tilde{a})$.*

Proof: By using equation (5.2) and equation (2.4) completes the proof. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MOHAGHEGH ARDABILI, P.O.BOX. 5619911367, ARDABIL-IRAN.
E-mail address: bahmandoust.p@uma.ac.ir

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MOHAGHEGH ARDABILI, P.O.BOX. 5619911367, ARDABIL-IRAN.
E-mail address: latifi@uma.ac.ir