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SYMMETRIC CONICS IN HYPERBOLIC GEOMETRY

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ABSTRACT. We prove that no conic in any Cayley–Klein model of the hyperbolic plane can be symmetric.

1. INTRODUCTION

Let $(A, B; C, D)$ denote the *cross-ratio* of the (maybe ideal) points A, B, C, D in $\mathbb{R}^n$ ($n = 1, 2, \dots$), and let $\overline{CD}$ denote the open segment of the (maybe ideal) points $C, D \in \mathbb{R}^n$. If $\mathcal{M}$ is an open, strictly convex, proper subset of $\mathbb{R}^n$ ($n = 2, 3, \dots$), then the function $d\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ defined by

$$d(A,B) = \begin{cases} 0, & \text{if } A = B, \\ \frac{1}{2}\big|\ln(A,B;C,D)\big|, & \text{if } A \neq B, \text{ where } \overline{CD} = \mathcal{M} \cap AB, \end{cases} \tag{1}$$

is a *metric* on $\mathcal{M}$ [1, page 297] which satisfies the strict triangle inequality, i.e. $d(A, B) + d(B, C) = d(A, C)$ if and only if $B \in \overline{AC}$. This function d is called the *Hilbert metric on* $\mathcal{M}$, the pair $(\mathcal{M}, d)$ is a *Hilbert geometry*, and $\mathcal{M}$ is its *domain*.

A Hilbert geometry is a model of the hyperbolic geometry of Bolyai, Lobachevskii and Gauss, if and only if its domain is an ellipsoid [1, (29.3)]. These isomorphic models of the hyperbolic geometry are called Cayley–Klein models.

In a Hilbert geometry $(\mathcal{M}, d)$ a set

(D_1) $\mathcal{C}^{\varepsilon}_{F,\mathcal{H}} := \{X \in \mathbb{R}^n : \varepsilon d(X, \mathcal{H}) = d(F, X)\}$ is called a *conic*,

where $\mathcal{H}$ is a hyperplane, the *leading hyperplane* or *directrix*, $F \notin \mathcal{H}$ is a point, the *focus*, and $\varepsilon > 0$ is a number, the *numeric eccentricity*. A conic is said to be *elliptic*, *parabolic* and *hyperbolic*, if $\varepsilon < 1$, $\varepsilon = 1$ and $\varepsilon > 1$, respectively.

In [2, Theorem 4.2 and Theorem 4.3] Kurusa proved that if even one conic is symmetric in a Minkowski plane, then the Minkowski plane is Euclidean. He conjectures that no symmetric conic may exist in Hilbert geometry.

In this article we support Kurusa's conjecture by proving in Theorem 1 that no conic in the Cayley–Klein model of the hyperbolic plane can be symmetric.

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2. Preliminaries

Points of $\mathbb{R}^n$ are denoted as $A, B, \ldots$, vectors are $\overrightarrow{AB}$ or $a, b, \ldots$, but we use these latter notations also for points if the origin is fixed. The open segment with endpoints A and B is denoted by $\overline{AB} = (A, B)$, $\overline{A}B$ is the open ray starting from A passing through B and the line through A and B is denoted by AB.

We denote the *affine ratio* of the collinear points A, B and C by $(A, B; C)$ that satisfies $(A, B; C)\overrightarrow{BC} = \overrightarrow{AC}$. The *cross ratio* of the collinear points A, B and C, D is $(A, B; C, D) = (A, B; C)/(A, B; D)$ [1, page 243].

Notations $u_\varphi = (\cos\varphi, \sin\varphi)$ and $u_\varphi^\perp := (\cos(\varphi + \pi/2), \sin(\varphi + \pi/2))$ are frequently used.

A set $\mathcal{S}$ in the hyperbolic plane is called *symmetric about a point* C, if $X \in \mathcal{S}$ if and only if $Y \in \mathcal{S}$, where $C \in \overline{XY}$ and S is the metric midpoint of the segment $\overline{XY}$.

We use the Cayley–Klein model $(\mathcal{D}, \delta)$ (likewise called the Beltrami model) in the interior of the unit circular disc $\mathcal{D}$ of the plane $\mathbb{R}^2$ with the metric given by (1) as

$$\delta(A, B) = \begin{cases} 0, & \text{if } A = B, \\ \frac{1}{2}\big|\ln(A, B; C, D)\big|, & \text{if } A \neq B, \text{ where } \overline{CD} = \mathcal{D} \cap AB. \end{cases} \tag{2}$$

Straight lines of $(\mathcal{D}, \delta)$, the *h-lines*, are the chords of $\mathcal{D}$, and segments of $(\mathcal{D}, \delta)$ are the segments of $\mathcal{D}$.

Isometries of $(\mathcal{D}, \delta)$, called *h-isometries*, are the restriction of those projectivities of the projective plane[1] $\mathbb{P}^2$ that leave $\mathcal{D}$ invariant. Any h-isometry is a product of at most three h-isometries which are restrictions of harmonic homologies, and any two non-degenerate triangles with pair-wisely equal side-lengths determine one and only one h-isometry that maps the first of these triangles onto the second one.

Let ℓ be an h-line in $(\mathcal{D}, \delta)$ and let $P \in \mathcal{H}$ be a point outside of ℓ. The point $S \in \ell$ is the *ℓ-foot of* P, if $\delta(P, X) \geq \delta(P, S)$ for every $X \in \ell$. An h-line ℓ' intersecting the h-line ℓ in a point S is said to be *h-perpendicular to* ℓ if S is an ℓ-foot of P for every $P \in \ell' \setminus \{S\}$. Notice that h-perpendicularity is invariant under isometries, because its definition is based on the metric. Further, the Euclidean line containing ℓ' is the one that connects S and the intersection of those tangents of $\mathcal{D}$ that touch $\mathcal{D}$ at the points $\partial\mathcal{D} \cap \overline{\ell}$, where $\overline{\ell}$ is the Euclidean line containing ℓ.

3. Conics in hyperbolic plane

Consider a conic $\mathcal{C}^{\varepsilon}_{F,\ell}$. Let $F^\perp$ be the foot of F on the h-line ℓ, and let C be a point on the h-line $FF^\perp$ different from $F^\perp$.

It is well known that there are h-isometries that maps C into the center O of $\mathcal{D}$. As an h-isometry $\imath$ keeps the orthogonality, Thus we can restrict without loss of generality the investigation of conics $\mathcal{C}^{\varepsilon}_{F,\ell}$ in $(\mathcal{D}, \delta)$ to those ones for which $\overline{(m, -\sqrt{1-m^2})(m, \sqrt{1-m^2})}$ is the directrix ℓ for some $m \in (-1, 0)$, the center is $O = (0, 0)$, and the focus F is $(f, 0)$, where $f \in (-1, 1) \setminus \{m\}$.

To calculate the points $P = (p, q)$ on $\mathcal{C}^{\varepsilon}_{F,\ell}$, we have to calculate $\delta(P, \ell)$ and $\delta(F, P)$, where $P = (p, q) \in \mathcal{C}^{\varepsilon}_{F,\ell}$. Observe that the line through P orthogonal to ℓ is the one that connects P to L, the intersection of the tangents of $\mathcal{D}$ at the limit points of ℓ. We clearly have $L = (-1/m, 0)$.

[1] For now, this is the affine plane $\mathbb{R}^2$ expanded with ideal points and straight line.

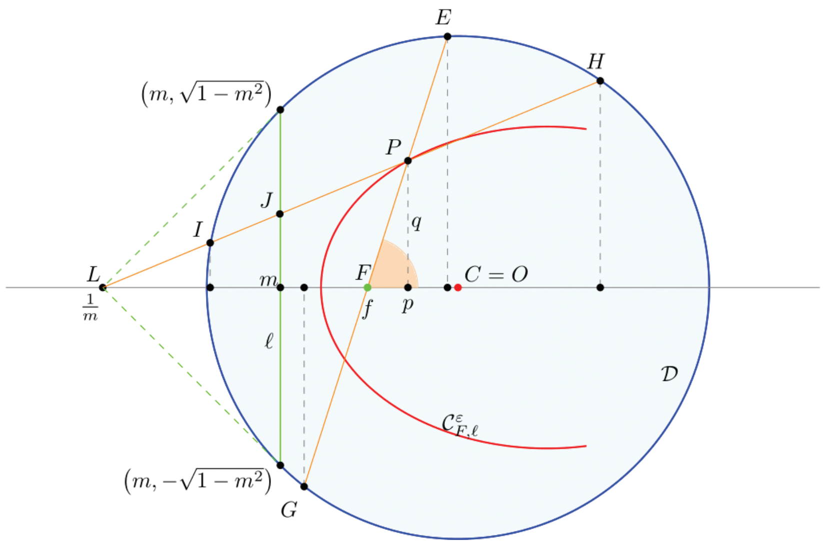

FIGURE 1. Directrix ℓ is $\overline{(m,-\sqrt{1-m^2})(m,\sqrt{1-m^2})}$, $\mathcal{C}^{\varepsilon}_{F,\ell}$ is symmetric in O, the center of the Cayley–Klein model, and the focus F is at $(f,0)$, where $f \in (-1,1) \setminus \{m\}$.

To obtain $\delta(P,\ell)$, we firstly determine the points $\{I,H\} = \{(x_\pm, y_\pm)\}$, where line LP intersects the unit circle, the border of $\mathcal{D}$. These points clearly satisfy the equations $x^2+y^2=1$ and $(x-1/m)q = y(p-1/m)$. So $(p-1/m)^2(1-x^2) = (x-1/m)^2q^2$, and we obtain $0 = x^2\big((p-\frac{1}{m})^2+q^2\big) - 2\frac{q^2}{m}x + \big(\frac{q^2}{m^2} - (p-\frac{1}{m})^2\big)$, hence

$$x_\pm = \frac{\frac{q^2}{m} \pm (p-\frac{1}{m})\sqrt{(p-\frac{1}{m})^2+q^2\frac{1-m^2}{m^2}}}{(p-\frac{1}{m})^2+q^2}, \quad y_\pm = \frac{-\frac{q}{m}(p-\frac{1}{m}) \pm q\sqrt{(p-\frac{1}{m})^2+q^2\frac{1-m^2}{m^2}}}{(p-\frac{1}{m})^2+q^2}.$$

Further, we need the coordinates of point J, where PL intersects ℓ. We clearly have $(m-1/m)q = y(p-1/m)$, hence $y = q(m-\frac{1}{m})/(p-\frac{1}{m})$. Thus

$$\begin{aligned}
\delta(P,\ell) &= \delta(P,J) \\
&= \frac{1}{2}\left|\log\left\{\frac{\big((p-\frac{1}{m})^2+q^2\big) + \frac{1}{m}(p-\frac{1}{m}) + \sqrt{(p-\frac{1}{m})^2+q^2\frac{1-m^2}{m^2}}}{\big((p-\frac{1}{m})^2+q^2\big) + \frac{1}{m}(p-\frac{1}{m}) - \sqrt{(p-\frac{1}{m})^2+q^2\frac{1-m^2}{m^2}}}\right.\right. \\
&\qquad \left.\left. : \frac{\frac{m-\frac{1}{m}}{p-\frac{1}{m}}\big((p-\frac{1}{m})^2+q^2\big) + \frac{1}{m}(p-\frac{1}{m}) + \sqrt{(p-\frac{1}{m})^2+q^2\frac{1-m^2}{m^2}}}{\frac{m-\frac{1}{m}}{p-\frac{1}{m}}\big((p-\frac{1}{m})^2+q^2\big) + \frac{1}{m}(p-\frac{1}{m}) - \sqrt{(p-\frac{1}{m})^2+q^2\frac{1-m^2}{m^2}}}\right\}\right|.
\end{aligned}$$

So

$$
\begin{aligned}
&\delta(P,\ell)\\
&=\frac{1}{2}\left|\log\left\{\frac{\left((p-\frac{1}{m})^2+q^2\right)+\left(\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}+\frac{1}{m}(p-\frac{1}{m})\right)}{\left((p-\frac{1}{m})^2+q^2\right)-\left(\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}-\frac{1}{m}(p-\frac{1}{m})\right)}\right.\right.\\
&\qquad\left.\left.:\frac{m(p-\frac{1}{m})+\frac{m-\frac{1}{m}}{p-\frac{1}{m}}q^2+\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}}{m(p-\frac{1}{m})+\frac{m-\frac{1}{m}}{p-\frac{1}{m}}q^2-\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}}\right\}\right|\\
&=\frac{1}{2}\left|\log\left\{\frac{\left((p-\frac{1}{m})^2+q^2\right)+\left(\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}+\frac{1}{m}(p-\frac{1}{m})\right)}{\left((p-\frac{1}{m})^2+q^2\right)-\left(\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}-\frac{1}{m}(p-\frac{1}{m})\right)}\times\right.\right.\\
&\qquad\left.\left.\times\frac{(p-\frac{1}{m})^2+(1-\frac{1}{m^2})q^2-\frac{1}{m}(p-\frac{1}{m})\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}}{(p-\frac{1}{m})^2+(1-\frac{1}{m^2})q^2+\frac{1}{m}(p-\frac{1}{m})\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}}\right\}\right|\\
&=\frac{1}{2}\left|\log\left\{\frac{\left((p-\frac{1}{m})^2+q^2\right)+\left(\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}+\frac{1}{m}(p-\frac{1}{m})\right)}{\left((p-\frac{1}{m})^2+q^2\right)-\left(\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}-\frac{1}{m}(p-\frac{1}{m})\right)}\times\right.\right.\\
&\qquad\left.\left.\times\frac{\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}-\frac{1}{m}(p-\frac{1}{m})}{\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}+\frac{1}{m}(p-\frac{1}{m})}\right\}\right|\\
&=\frac{1}{2}\left|\log\left\{\frac{\left((p-\frac{1}{m})^2+q^2\right)\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}+\left((p-\frac{1}{m})^2+q^2\right)(1-\frac{1}{m}p)}{\left((p-\frac{1}{m})^2+q^2\right)\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}-\left((p-\frac{1}{m})^2+q^2\right)(1-\frac{1}{m}p)}\right\}\right.\\
&=\frac{1}{2}\left|\log\left\{\frac{\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}+(1-\frac{1}{m}p)}{\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}-(1-\frac{1}{m}p)}\right\}\right|\\
&=\frac{1}{2}\left|\log\left\{\frac{\left(\sqrt{(p-\frac{1}{m})^2+q^2(1-\frac{1}{m^2})}+(1-\frac{1}{m}p)\right)^2}{(1-p^2-q^2)(\frac{1}{m^2}-1)}\right\}\right|. \qquad (3)
\end{aligned}
$$

To obtain $\delta(F,P)$, we firstly determine the points $E=(x_1,y_1)$ and $G=(x_2,y_2)$, where line FP intersects the unit circle, the border of $\mathcal{D}$. These points clearly satisfy the equations $x^2+y^2=1$ and $(x-f)q=y(p-f)$. So $(p-f)^2(1-x^2)=(x-f)^2q^2$, and we obtain $0=x^2((p-f)^2+q^2)-2fq^2x+(f^2q^2-(p-f)^2)$, hence

$$
x_\pm=\frac{fq^2\pm(p-f)\sqrt{(p-f)^2+(1-f^2)q^2}}{(p-f)^2+q^2},\quad y_\pm=\frac{-qf(p-f)\pm q\sqrt{(p-f)^2+(1-f^2)q^2}}{(p-f)^2+q^2}.
$$

Thus, we get

$$\begin{aligned}\delta(F,P) &= \frac{1}{2}\Big|\log\Big\{\frac{q^2+(p-f)^2+(f(p-f)+\sqrt{(p-f)^2+(1-f^2)q^2})}{q^2+(p-f)^2+(f(p-f)-\sqrt{(p-f)^2+(1-f^2)q^2})} \\ &\qquad : \frac{f(p-f)+\sqrt{(p-f)^2+(1-f^2)q^2}}{f(p-f)-\sqrt{(p-f)^2+(1-f^2)q^2}}\Big\}\Big| \\ &= \frac{1}{2}\Big|\log\Big\{\frac{(fp-1-\sqrt{(p-f)^2+(1-f^2)q^2})^2}{(1-f^2)(1-p^2-q^2)}\Big\}\Big|,\end{aligned} \tag{4}$$

where we have used the identities

$$\begin{aligned}&(f(p-f)+\sqrt{(p-f)^2+(1-f^2)q^2})(f(p-f)-\sqrt{(p-f)^2+(1-f^2)q^2}) \\ &\qquad = f^2(p-f)^2-(p-f)^2-(1-f^2)q^2 = -(1-f^2)(q^2+(p-f)^2), \\ &(fp-1-\sqrt{(p-f)^2+(1-f^2)q^2})(fp-1+\sqrt{(p-f)^2+(1-f^2)q^2}) \\ &\qquad = (fp-1)^2-(p-f)^2-(1-f^2)q^2 = (1-f^2)(1-p^2-q^2).\end{aligned}$$

According to (D_1) equations (3) and (4) give

$$\left(\frac{\left(\sqrt{\left(p-\frac{1}{m}\right)^2+q^2\left(1-\frac{1}{m^2}\right)}+\left(1-\frac{1}{m}p\right)\right)^2}{(1-p^2-q^2)\left(\frac{1}{m^2}-1\right)}\right)^{\epsilon} = \frac{\left(fp-1-\sqrt{q^2(1-f^2)+(p-f)^2}\right)^2}{(1-f^2)(1-q^2-p^2)}, \tag{5}$$

where $\epsilon \in \{\varepsilon, -\varepsilon\}$. Figure 2 shows how these conics look like based on (5) with $\epsilon = \varepsilon$.

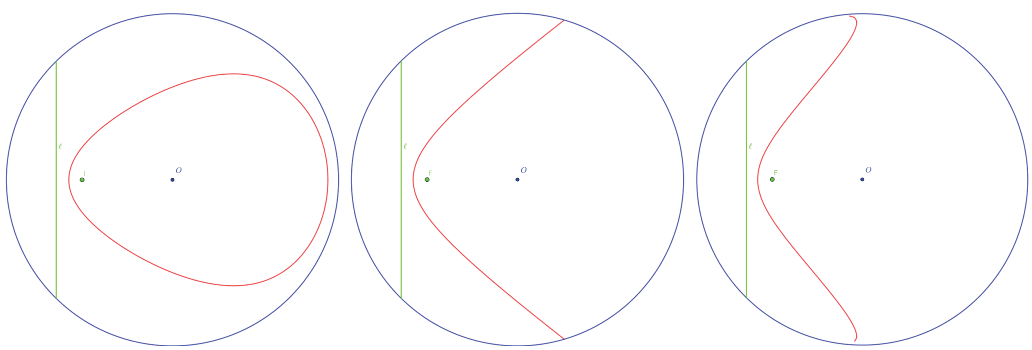

FIGURE 2. An elliptic ($\varepsilon = 0.9$), parabolic ($\varepsilon = 1$), and hyperbolic ($\varepsilon = 1.1$) conic in the Cayley–Klein model of the hyperbolic geometry.

4. SYMMETRIC CONICS IN HYPERBOLIC PLANE

For the sake of later contradiction in this section we assume from now on that

$$\text{conic } \mathcal{C}^{\varepsilon}_{F,\ell} \text{ is symmetric in a point } C.$$

Such a point of symmetry C clearly is on the h-line $FF^{\perp}$, where $F^{\perp}$ is the foot of F on the h-line ℓ. So we can restrict without loss of generality the investigation of symmetric conics $\mathcal{C}^{\varepsilon}_{F,\ell}$ in $(\mathcal{D},\delta)$ to those ones for which the directrix ℓ is $\overline{(m,-\sqrt{1-m^2})(m,\sqrt{1-m^2})}$ for some

$m \in (-1,1)$, the center is $O=(0,0)$, and the focus F is $(f,0)$, where $f \in (m,1)$. Thus we can use the formulas given in the previous section.

As the conic is symmetric in the x-axis, and it is symmetric in point O, it is symmetric about the y-axis too, so, substituting $-p$ into p, equation (5) gives

$$\left(\frac{\left(\sqrt{\left(p+\frac{1}{m}\right)^2+q^2\left(1-\frac{1}{m^2}\right)}+\left(1+\frac{1}{m}p\right)\right)^2}{(1-p^2-q^2)\left(\frac{1}{m^2}-1\right)}\right)^{\epsilon}=\frac{\left(fp+1+\sqrt{q^2(1-f^2)+(p+f)^2}\right)^2}{(1-f^2)(1-q^2-p^2)}, \tag{6}$$

Dividing (5) with this equation and taking the square root result in

$$\left(\frac{\sqrt{\left(p-\frac{1}{m}\right)^2+q^2\left(1-\frac{1}{m^2}\right)}+\left(1-\frac{1}{m}p\right)}{\sqrt{\left(p+\frac{1}{m}\right)^2+q^2\left(1-\frac{1}{m^2}\right)}+\left(1+\frac{1}{m}p\right)}\right)^{\epsilon}=\frac{fp-1-\sqrt{q^2(1-f^2)+(p-f)^2}}{fp+1+\sqrt{q^2(1-f^2)+(p+f)^2}}, \tag{7}$$

If $q=0$, then (5) gives

$$\left(\frac{\left(\frac{1}{m}-1\right)(1+p)}{\left(\frac{1}{m}+1\right)(1-p)}\right)^{\epsilon}=\frac{\left(fp-1-|p-f|\right)^2}{(1-f^2)(1-p^2)}=\begin{cases}\frac{(1-f)(1+p)}{(1+f)(1-p)}, & \text{if } p>f,\\ \frac{(1+f)(1-p)}{(1-f)(1+p)}, & \text{if } p<f.\end{cases}$$

After some rearrangement this becomes

$$\frac{1\pm f}{1\mp f}\left(\frac{1-m}{1+m}\right)^{\epsilon}=\left(\frac{1+p}{1-p}\right)^{-\epsilon\pm 1}, \text{ where } \pm 1=\frac{p-f}{|p-f|}. \tag{8}$$

If p is a solution of these equations, then the symmetry in O implies, that $-p$ is also a solution of (8).

Suppose the two solutions $\pm p$ belong to the equation with the same sign of ε. If $\pm p$ are on the same side of f, then $p=0$, a contradiction follows. If $\pm p$ are on different sides of f, then

$$\frac{1+f}{1-f}\left(\frac{1-m}{1+m}\right)^{\epsilon}=\left(\frac{1+p}{1-p}\right)^{-\epsilon+1}, \text{ and } \frac{1-f}{1+f}\left(\frac{1-m}{1+m}\right)^{\epsilon}=\left(\frac{1-p}{1+p}\right)^{-\epsilon-1}=\left(\frac{1+p}{1-p}\right)^{\epsilon+1}.$$

Division and multiplication of the first equation by the second one give

$$\frac{1+f}{1-f}=\left(\frac{1-p}{1+p}\right)^{\epsilon} \text{ and } \left(\frac{1-m}{1+m}\right)^{\epsilon}=\frac{1-p}{1+p}, \tag{9}$$

respectively. Substituting the second equation into the first one results in

$$\frac{1+f}{1-f}=\left(\frac{1-m}{1+m}\right)^{\varepsilon^2}. \tag{10}$$

Note, that the sign of ε in (9) is irrelevant regarding the solutions. If $\varepsilon=1$, the second equation gives $f=-m$, hence O is the midpoint of the segment of F and its foot on ℓ, so $p=0$, a contradiction.

Suppose the two solutions $\pm p$ belong to the equation with different signs of ε. If $\pm p$ are on the same side of f, then

$$\frac{1\pm f}{1\mp f}\left(\frac{1-m}{1+m}\right)^{\varepsilon}=\left(\frac{1+p}{1-p}\right)^{-\varepsilon\pm 1}, \text{ and } \frac{1\pm f}{1\mp f}\left(\frac{1-m}{1+m}\right)^{-\varepsilon}=\left(\frac{1-p}{1+p}\right)^{\varepsilon\pm 1}$$

follow, where $\pm 1 = \frac{p-f}{|p-f|} = \frac{-p-f}{|p+f|}$. Division and multiplication of the first equation by the second one give

$$\left(\frac{1-m}{1+m}\right)^{\pm\varepsilon} = \frac{1+p}{1-p}, \text{ and } \frac{1\pm f}{1\mp f} = \left(\frac{1+p}{1-p}\right)^{\varepsilon} \tag{11}$$

respectively. Rearranging the first equation, then substituting it into the second one results in

$$\left(\frac{1\mp m}{1\pm m}\right)^{\varepsilon} = \frac{1+p}{1-p}, \text{ and } \frac{1\pm f}{1\mp f} = \left(\frac{1\mp m}{1\pm m}\right)^{\varepsilon^2}, \tag{12}$$

respectively. If $\varepsilon = 1$, the second equation gives $f = -m$, hence O is the midpoint of the segment of F and its foot on ℓ, so $p = 0$, a contradiction.

If $\pm p$ are on different sides of f, we may suppose that $p > f$, and so $-p < f$. Then

$$\frac{1+f}{1-f}\left(\frac{1-m}{1+m}\right)^{\varepsilon} = \left(\frac{1+p}{1-p}\right)^{-\varepsilon+1}, \text{ and } \frac{1-f}{1+f}\left(\frac{1-m}{1+m}\right)^{-\varepsilon} = \left(\frac{1-p}{1+p}\right)^{\varepsilon-1}$$

follows, where $\frac{p-f}{|p-f|} = \frac{p+f}{|p+f|}$. Division and multiplication of the first equation by the second one give

$$\frac{1+f}{1-f}\left(\frac{1-m}{1+m}\right)^{\varepsilon} = 1, \text{ and } 1 = \left(\frac{1+p}{1-p}\right)^{1-\varepsilon} \tag{13}$$

respectively. This is obviously a contradiction.

Thus we have either $\varepsilon \in (0,1)$ or $\varepsilon \in (1,\infty)$, the elliptic or the hyperbolic case, respectively. Further, we have for the pair of points either (9) and (10), or (11) and (12), respectively. Moreover $\pm p$ falls in different sides of f, and in the same side of f, respectively.

Theorem 1. *No conic of the hyperbolic plane can be symmetric.*

Proof. Assume first the elliptic case, so that $\varepsilon \in (0,1)$.

Rearrangement of (7) gives

$$\left(1 + \frac{-\frac{2}{m}p}{\sqrt{\left(p+\frac{1}{m}\right)^2+q^2\left(1-\frac{1}{m^2}\right)}+\left(1+\frac{1}{m}p\right)}\right)^{\epsilon} = \frac{2fp}{fp+1+\sqrt{q^2(1-f^2)+(p+f)^2}} - 1, \tag{14}$$

This is contradictory, because $p \to 0$ causes the left-hand side to tend to 1, and the right-hand side to tend to -1.

Assume now the hyperbolic case, so that $\varepsilon > 1$.

Rearrangement of (7) gives for $p^2 + q^2 \to 1$ that

$$\left(\frac{|1-\frac{1}{m}p|+\left(1-\frac{1}{m}p\right)}{|1+\frac{1}{m}p|+\left(1+\frac{1}{m}p\right)}\right)^{\epsilon} = \frac{fp-1-|fp-1|}{fp+1+|fp+1|}, \tag{15}$$

Since $|fp| < 1$, the right-hand side vanishes, hence also the left-hand side vanishes, so $\frac{1}{m}p > 1$. This is a contradiction, because it can not be valid for both p and $-p$.

The contradictions prove the statement. □

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