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SOLVING PROBLEMS EXPERIMENTALLY WITH DYNAMIC GEOMETRY SOFTWARE

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ABSTRACT. We show through an example of the process of solution of a geometric problem, that Dynamic Geometry Software can be used to perform experiments that produce visual and numerical data to formulate conjectures about the solution, and to find theoretical arguments to construct a formal proof of these conjectures. We call this practice Experimental Geometry, and claim that it has important implications in the practice of geometry and its teaching.

Keywords: Symmetric lateral triangle, experimental geometry, proof, dynamic geometry software and DGPad.

1. INTRODUCTION

The appearance of technologies along human development has caused several shifts in the way human beings communicate: from orality to handwriting, from handwriting to printing, and from printing to electronic processing and representation of information [4]. These shifts have changed social practices, the way we think, the ways of literacy learning, the work patterns, and our ways of living, and undoubtedly, the new technologies of communication and information have changed human life in great extent. Doing mathematics and its teaching is not an exception.

Francis Guthrie conjectured the four color theorem in 1852. Since then, a formal proof has not been found. In 1976, Kenneth Appel and Wolfgang Haken proposed a computer-assisted proof of the theorem. A computer program reduced to 1,936 configurations the infinitude of possible maps, taking thousand computer hours to check them one by one [5]. However, because part of the proof consisted of an exhaustive analysis of many discrete cases by a computer, some mathematicians do not accept it [6]. Even though, the attempt to prove theorems computationally give rise, among other things, to a new activity which is called experimental mathematics [9]. Although many questions are raised about the role and validity of the use of computers in mathematical activities, especially its use in proving theorems, it is now undeniable its role as a technology of discovery and as a way of collecting experimental evidence which allow to state conjectures.

This paper pretends to illustrate how a dynamic geometry software (DGS)[1] can be used to perform experiments with the purpose of solving Euclidean geometry problems. This practice, which we call Experimental Geometry, has three characteristics: 1) it uses the potential of dynamic geometry software to differentiate between exact geometric constructions and approximated geometric constructions, 2) it performs experiments to collect enough data which provide information in order to formulate conjectures about the solution of a problem and 3) it finds theoretical arguments towards the proof of these conjectures. This late point is very important, because it states that computers can be used heuristically not only to formulate conjectures, but also to find theoretical connections that can be used to formally proof these conjectures. We do

[1] We use DGPad, a free DGS developed in France, see `http://dgpad.net/`

not pretend that calculations made in a computer constitute formal proofs, but that data gathered in a computer based experiment can give insights about theoretical relations which can be used to construct formal proofs of conjectures.

In order to illustrate the above, we will show how we used DGS to determine the locus of points of the plane such that their symmetric lateral triangles with respect to a fixed triangle is of a given area. In fact, what we are solving is a geometric equation defined by a geometric construction which depends on base points. More precisely, given a geometric construction $GC(A_1, A_2, ..., A_n, P)$ which depends on parameters $A_1, A_2, ..., A_n$ (fixed points of the plane) and an arbitrary point P of the plane, we want to determine the points P (the locus) which preserve certain characteristic of the construction, i.e., we want to solve the equation

$$ch(GC(A_1, A_2, ..., A_n, P)) = const.$$

In our particular case, GC is the symmetric lateral triangle of P with respect to the fixed triangle ABC, $SLT(A, B, C, P)$ and ch is its area. This gives us the equation

$$area(SLT(A, B, C, P)) = k.$$

which is solved by a locus of points P. This problem is equivalent to Steinerâs theorem about the pedal triangle [1] since the pedal triangle and the symmetric-lateral triangle are homothetic. There are already proofs of this last theorem but we want to illustrate how DGS can be used in conjecturing and formalizing processes.

We perform experiments with approximated constructions to formulate conjectures about the solution, we use theory to produce exact construction of this locus, and finally we perform experiments to look for theoretical connections between data of this construction and the property demanded in the problem.

1.1. **The symmetric-lateral triangle.** In the plane, let ABC be a fixed triangle with sides a(BC), b(AC) and c(AB), and let P be an arbitrary point. SLT(A,B,C,P) , is the triangle $P_a P_b P_c$ where P_l is the symmetric point of P with respect to the line l, i.e., $SLT(A, B, C, P) = P_a P_b P_c$.

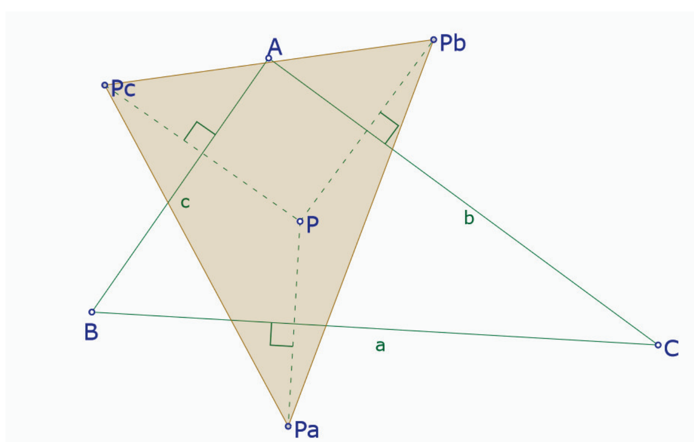

FIGURE 1. SLT of P with respect to the ABC triangle

2. Symmetric-lateral triangle of given area

Problem:
We want to find the locus of points P such that the area of SLT(A,B,C,P) is constant, i.e., we want to solve the equation

$$area(SLT(A, B, C, P)) = k.$$

We will solve this problem using Experimental Geometry: performing experiments with DGPad to formulate conjectures, verify them, and collect theoretical arguments towards the proof of conjectures. Unlike working with drawings of geometric figures on a paper, using DGS one can interact with dynamic geometric constructions on the computer screen and observe the behavior of their logical relationships, not only the characteristics of some of their particular representations.

2.1. **Experimentation to formulate the conjecture.** We call âbastingâ the technique used to find points close to the locus. It consists in placing points that approximately have the condition searched, and to use this approximated construction to conjecture the shape of the locus. DGPad [2] allows to program a point to leave a trace only on positions that approximate the condition of the locus, thus to perform an âautomatic bastingâ which we describe as follows: We construct a triangle ABC, a point P and $SLT(A, B, C, P)$. Using DGPadś calculator, we construct a point P* with the same coordinates of P and we activate its trace. Then, with DGPadś blockly [3] ( ) we modify its aspect to be hidden if $area(SLT(A, B, C, P))$ is far from the wanted value. More precisely, if k is the fixed value of the area and ε is the maximum error accepted, the point P* is visible only if $|k - area(SLT(A, B, C, P))| < \varepsilon$ [4] . As P* has the same position of P, and its trace appears only where it is visible, point P will color all these positions as it is dragged across the computer screen.
The result is a cloud of points (coloured positions) that contains the locus and whose shape resembles it. An interpretation of the shape of the cloud is a conjecture. Using automatic basting we obtain the image of figure 2. This image gives us a clear enough idea to make a conjecture about the desired locus: it is a pair of circles (c_1 and c_2), concentric with the circumcircle of ABC (circum(ABC)).
However, the conjecture is not accurate enough. We must find a relation between c_1, c_2, and the reference triangle ABC in order to construct them. The feeling that c_1 and c_2 are concentric circles with circum(ABC) drives us to compute the power of points on them with respect to circum(ABC), because the locus of points with the same power with respect to a given circle is formed by two circles concentric with the reference circle as can be deduced from definition of circle power [8] . As can be seen in Figure 3, the points E near c_1 and F near c_2 have almost the same power with respect to circum(ABC): $power(circum(ABC), E) \simeq power(circum(ABC), E)$[5].

To confirm this conjecture we need to build exactly that locus, for which we will modify the original problem slightly: Given a triangle ABC and a point P on the plane, find the locus

[2] we create a web page of the article for anyone interested in the management of DGPad and how dynamic geometry software works, on this page explain more explicitly the processes that are discussed in the article: `http://spedgs.atwebpages.com`

[3] Blockly is a programming interface in DGPad that enables to modify properties of constructed objects depending on given conditions, among other possibilities.

[4] In fact, you can choose k and ε arbitrarily; the littler ε is, the more âpreciseâ the trace of the locus. The value of $area(SLT(A, B, C, P))$ is calculated automatically by DGPad and is the value of the polygon $SLT(A, B, C, P)$.

[5] We have drawn points E and F near the trace, lines AE and AF, intersection points Ae and Af between the lines and the circumcircle. Finally, with DGPad calculator we have computed $d(E, A) * d(E, Ae)$ which is the power of E wrt the circumcircle, and $d(F, A) * d(F, Af)$ which is the power of F wrt the circumcircle.

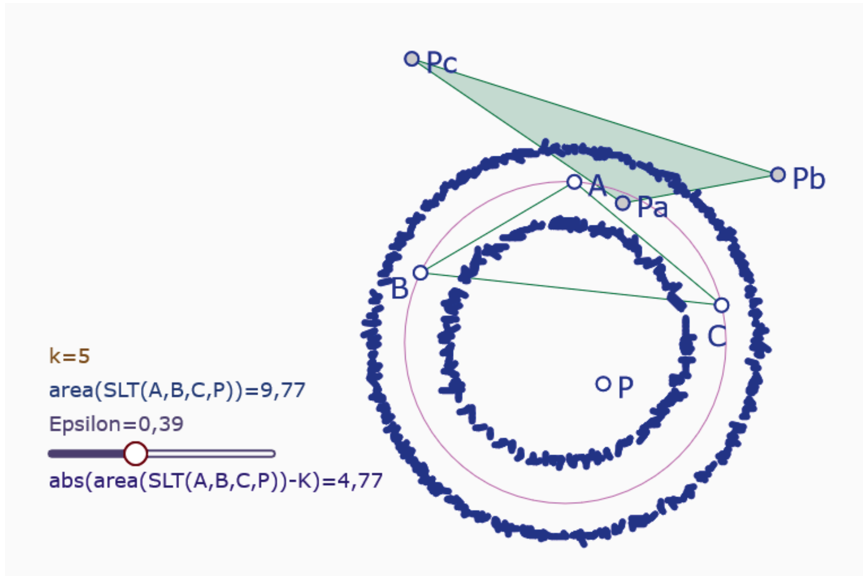

FIGURE 2. Automatic basting for k=5

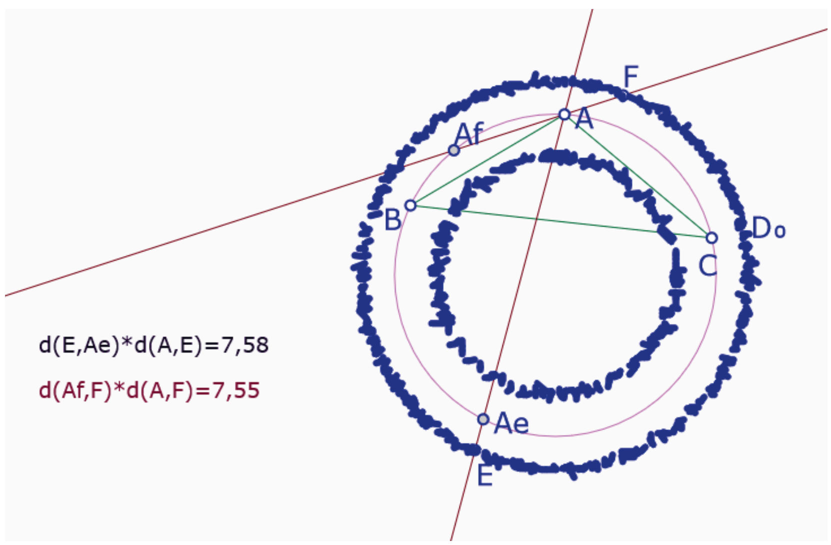

FIGURE 3. Experimental verification of the conjecture: solution points P have same power wrt circum(ABC)

of points D such that $area(SLT(A,B,C,D)) = area(SLT(A,B,C,P))$. According to our conjecture, that set of points D should be the set of points such that $power(circum(ABC),D) = power(circum(ABC),P)$.

2.1.1. *Constructing the locus.* We know from Weisstein that all points Q such that power(circum(ABC),Q)=k are on one or two circles concentric with circum(ABC) . As we know point P, one of these circles is the one which passes by P. How to construct the other one?

The number power(circle(O,r),P), that we will write π, is characterized by the equation $\pi = |r^2 - d^2|$ where $d = PO$ (Michael N. Fried.(2010)). From this result we have that $d = \sqrt{r^2 + \pi}$ if $d \geq r$ and $d = \sqrt{r^2 - \pi}$ if $d < r$. So, if $\pi > r^2$ there is only one solution for d, corresponding to the first equation. But, when $\pi < r^2$, there are two solutions of d, corresponding to the two equations. We can interpret the two equations as corresponding to Pythagoras theorem for a right triangle: the first one where d is the hypotenuse, the second one where r is the hypotenuse.

So if r is the hypotenuse of the triangle, we can construct the segment PQ corresponding to π with a perpendicular to OP by P, that cuts circum(ABC) at Q. If d is the hypotenuse of the triangle, we find a second point Pâ constructing a segment QPâ perpendicular to OQ through Q, and such that $QP' = PQ = \pi$. The circle (O,Pâ) is the second solution.

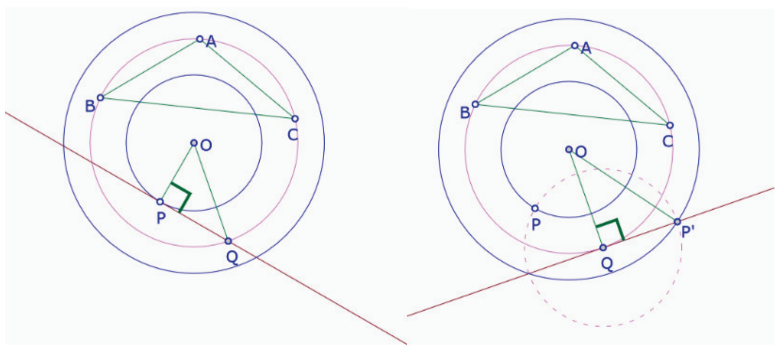

FIGURE 4. First and second solution

We make this construction for a given triangle ABC, and two points H and Hâ on each circle; then we compare area(SLT(A,B,C,H) and area(SLT(A,B,C,Hâ) for different positions of H and Hâ, and verify that are equal (Figure 5).

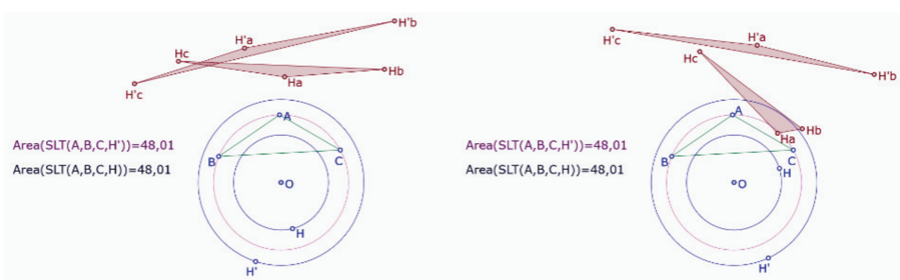

FIGURE 5. conjecture verification

In this way, we have verified experimentally that all the points that have the same power with respect to the circumcircle of ABC have symmetric-lateral triangles of equal area. However, mathematical rigor requires us to make a formal proof of that conjecture. That is why we proceed to use DGPad to look for theoretical arguments to support the conjecture.

2.2. **Experimentation for proving.** Now that we know how to construct the locus of points P, such that $area(SLT(A, B, C, P))$ is constant, we continue with the experimentation focused on the proof; in other words, we use DGPad to find relations between the locus and the theory that enable us to prove the conjecture.
First letâs state some important relationships between the symmetric lateral triangle and the reference triangle ABC:

Lemma 2.1. *Given a triangle ABC, any point P and $P_aP_bP_c = SLT(A, B, C, P)$ then:*

$$\measuredangle P_aCP_b = 2\measuredangle ACB$$

Proof. We know that the composition of two axial symmetries of secant axes is a rotation whose center is the intersection of the two axes and whose amplitude is twice the angle between the two axes [2]. Since P_a and P are symmetric with respect to BC, and P_b and P are symmetric with respect to AC, we can say that P_a is the image of P_b by a composition of axial symmetries of axes AC and BC, and therefore P_a is rotation of P_b around C with an angle $2\measuredangle ACB$. □

Similarly, we can say that P_b is rotation of P_c around A with an angle $2\measuredangle CAB$ and P_c is rotation of P_a around B with an angle $2\measuredangle ABC$.

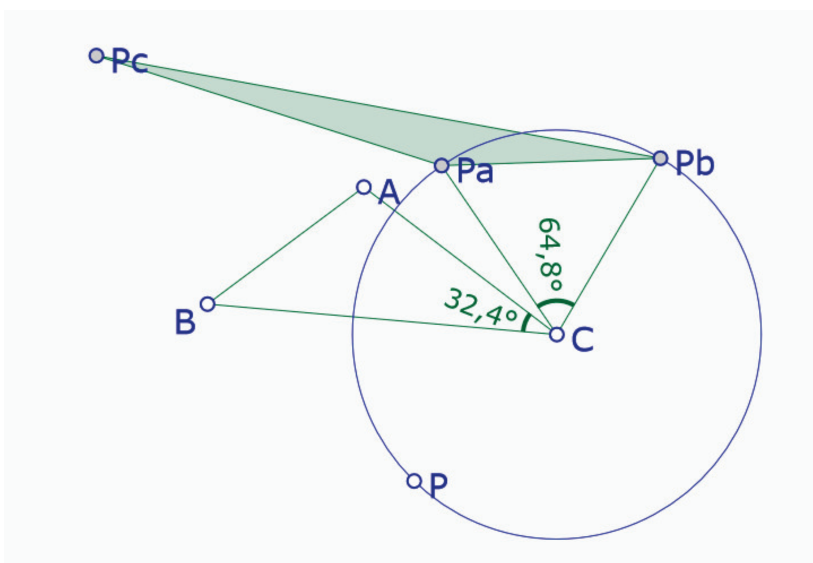

FIGURE 6. $\measuredangle PaCPb = 2\measuredangle ACB$

Lemma 2.2. *Given a triangle ABC and any point P, $P_aP_bP_c = SLT(A, B, C, P)$ then:*

$$P_aP_b = 2PC * \sin(ACB)$$

Proof. by lemma 2.1 the triangle P_aP_bC is an isosceles triangle thus,

$$\begin{aligned}\frac{P_aP_b}{2} &= P_aC * \sin(\measuredangle \frac{P_aCP_b}{2}) \\ &= P_aC * \sin(\measuredangle ACB) \\ &= PC * \sin(\measuredangle ACB) \\ P_aP_b &= 2PC * \sin(ACB)\end{aligned}$$

□

Equally, $P_bP_c = 2PA * \sin(\measuredangle BAC)$ and $P_aP_c = 2PB * \sin(\measuredangle ABC)$.
Now, we need to find relationships between power(circum(ABC),P) and area(SLT(A,B,C,P)). In order to find these relationships, we construct ABC, its circumcircle, P, its SLT and the points $P_{a'}$, $P_{b'}$, $P_{c'}$ such that $PA * P_{a'} = PB * P_{b'} = PC * P_{c'}$ [8] (Figure 7).

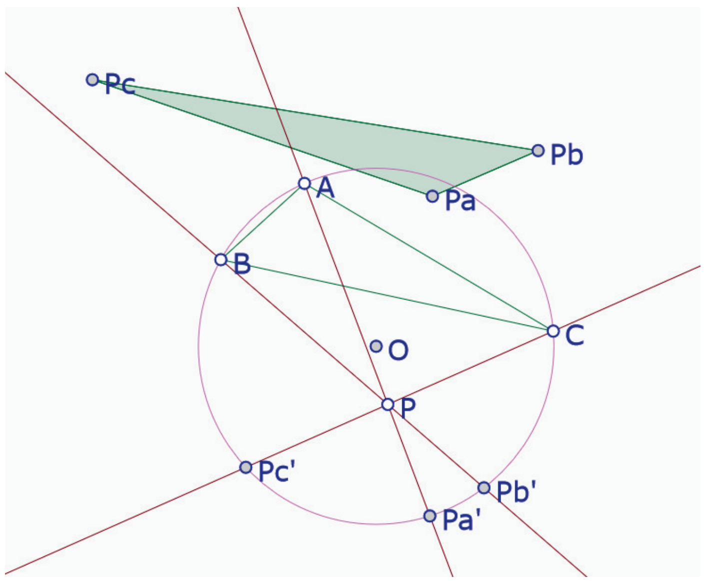

FIGURE 7. relating power of P and Area(SLT(A,B,C,P))

Observing this construction we conjecture that $\Delta P_aP_bP_c \sim \Delta P_{a'}P_{b'}P_{c'}$ (see figure 8).

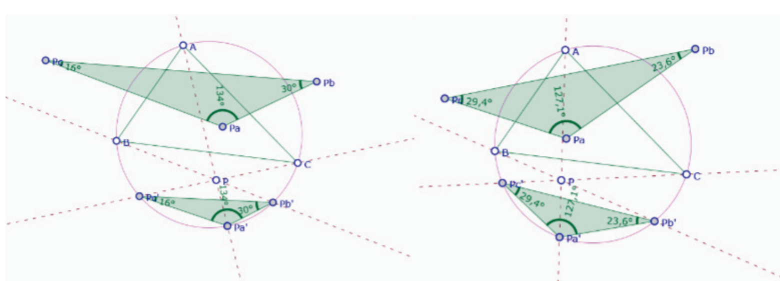

FIGURE 8. $\Delta P_aP_bP_c \sim \Delta P_{a'}P_{b'}P_{c'}$

Lemma 2.3. *Given a triangle ABC and any point P, the triangle $P_{a'}P_{b'}P_{c'}$ formed by the points of intersection of the lines AP, BP, CP with the circumcircle respectively is similar to SLT(A,B,C,P).*

Proof. Since P, P_b and P_a are on a circle of center C and radius PC (since P_aP_bC is isosceles- see figure 6), then by the central angle theorem

$$\measuredangle PP_bP_a = \frac{\measuredangle PCP_a}{2}$$

and by Lemma 2.1

$$\begin{aligned} \measuredangle PP_bP_a &= \frac{\measuredangle PCP_a}{2} \\ &= \measuredangle PCB \end{aligned}$$

In the same way since P, P_c and P_b are on a circle with center A and radius PA, then

$$\begin{aligned} \measuredangle PP_bP_c &= \frac{\measuredangle PAP_c}{2} \\ &= \measuredangle PAB \end{aligned}$$

Now, we can prove that the corresponding angles are equal, considering two cases of figure:

(1) If P is on the opposite half-plane of A with respect to line BC (see figure 9), then by the central angle theorem

$$\begin{aligned} \measuredangle P_{a'}AB &= \measuredangle PAB \\ &= \measuredangle P_{a'}CB \end{aligned}$$

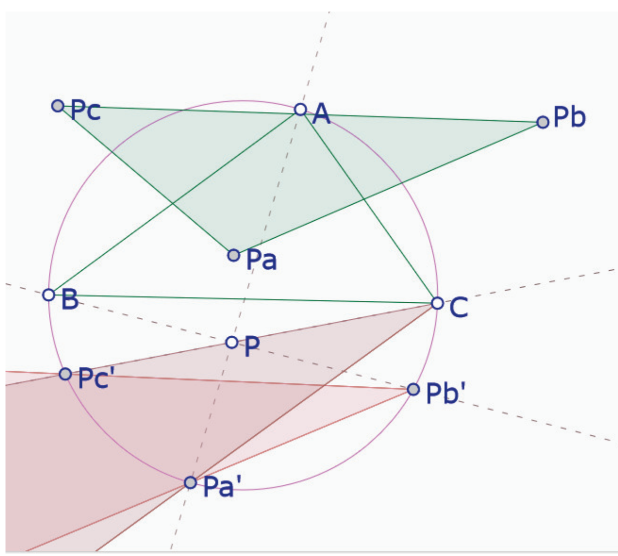

FIGURE 9. P in the opposite half-plane of A

Using the precedent results we have :

$$\begin{aligned}\measuredangle P_{a'}P_{b'}P_{c'} &= \measuredangle P_{a'}CP_{c'}\\ &= |\measuredangle P_{a'}CB - \measuredangle P_{c'}CB|\\ &= |\measuredangle PAB - \measuredangle PCB|\\ &= |\measuredangle PP_bP_a - \measuredangle PP_bP_c|\\ &= \measuredangle P_aP_bP_c\end{aligned}$$

in a similar way we can verify that:

$$\measuredangle P_{c'}P_{a'}P_{b'} = \measuredangle P_cP_aP_b$$

and

$$\measuredangle P_{b'}P_{c'}P_{a'} = \measuredangle P_bP_cP_a$$

(2) If P is in the same half-plane of A with respect to line BC (see figure 10), we have in the last step that:

$$\begin{aligned}\measuredangle P_{a'}P_{b'}P_{c'} &= \measuredangle P_{a'}CP_{c'}\\ &= \measuredangle P_{a'}CB + \measuredangle P_{c'}CB\\ &= \measuredangle PAB + \measuredangle PCB\\ &= \measuredangle PP_bP_a + \measuredangle PP_bP_c\\ &= \measuredangle P_aP_bP_c\end{aligned}$$

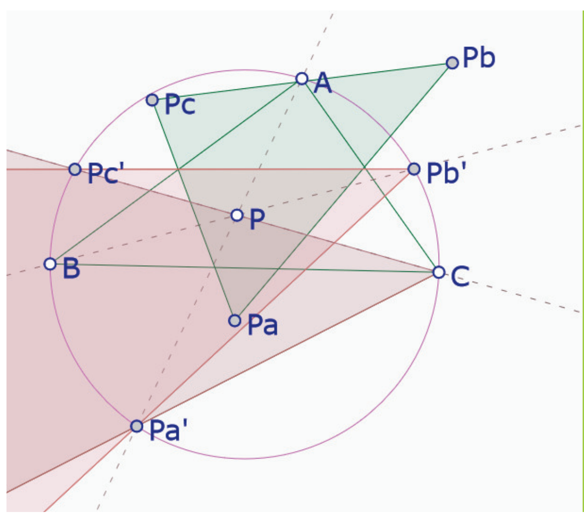

FIGURE 10. P in the same half plane of A

In conclusion, the triangles $P_{a'}P_{b'}P_{c'}$ and $P_aP_bP_c$ are similar.

□

Having a triangle similar to SLT(A,B,C,P) we could hope it would have constant area and thus prove our conjecture. When verifying experimentally we find that the area of $P_{a'}P_{b'}P_{c'}$ is not constant (see Figure 11)

However, we can try to construct a triangle whose sides have PB and PC measures and whose area is proportional to area(TSL(A,B,C,P)). If this triangle is related with power(circum(ABC),P), and its area is constant, we will have a way to prove our conjecture.

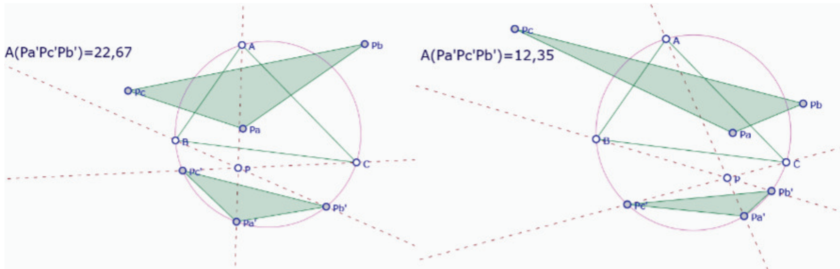

FIGURE 11. Experimental Verification of conjecture area $P_{a'}P_{b'}P_{c'}$ is constant

We know that area(TSL(A,B,C,P)) can be calculated with the formula:

$$\frac{1}{2} * P_bP_c * P_bP_a * \sin(\measuredangle P_aP_bP_c)$$

In addition, we know that P_bP_a is proportional to PC and P_bP_c is proportional to PA as shown in Lemma 2. So if we build a triangle PCQ (see figure 12) with PA=PQ and $\measuredangle QPC = \measuredangle P_aP_bP_c$, then $area(PCQ) = k * area(P_aP_bP_c)$ for some constant k.

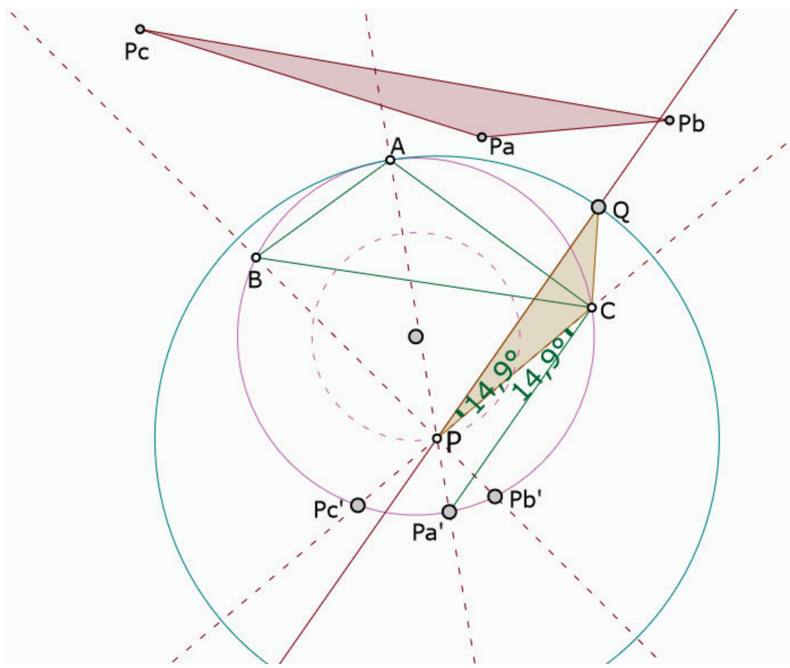

FIGURE 12. Construction of triangle PCQ proportional to $P_aP_bP_c$

To reproduce $\measuredangle P_aP_bP_c$ from CP, we draw a parallel to $CP_{a'}$ by P. In effect,

$$\measuredangle PCP_{a'} = \measuredangle Pc'CPa' = \measuredangle P_{c'}P_{b'}P_{a'}$$

But as

$$\Delta P_{a'}P_{b'}P_{c'} \sim \Delta P_aP_bP_c$$

then

$$\measuredangle PCP_{a'} = \measuredangle P_cP_bP_a$$

and by alternate internal angles between parallels $\measuredangle QPC = \measuredangle PCP_{a'}$. To ensure PA=PQ, we construct Q as an intersection between the parallel and the circle of center P passing through A.

So $area(PCQ) = k * area(P_aP_bP_c)$. But the height of the triangle PQC is proportional to the segment $P_{a'}P$ (since they are between parallels); therefore $area(PCQ) = k * (PQ * P_{a'})$. But $PQ = PA$ (by construction), so $area(PQC) = k * (AP * PP_{a'})$ where $AP * PP_{a'} = power(circum(ABC, P)$. Therefore we can say that if P is on a circle concentric with the circumcircle of ABC so that power(circum(ABC),P) is constant, then the area of PQC is constant. But since that area is proportional to the area of $P_aP_bP_c$, then we can conclude that area(SLT(A,B,C,P) will also be constant.
□

Finally, we want to determine exactly how this power π of P can be expressed with respect to the circum(ABC) in terms of the SLT(A,B,C,P). We only need to determine the proportionality of the $P_{a'}P$ segment with the height h of the PQC triangle and this is:

$$P_{a'}P = \frac{PC * sin(\measuredangle P_aP_bP_c)}{sin(\measuredangle ABC)}$$

as the reader can verify, so we have :

$$\begin{aligned}
\pi &= d(A,P) * d(P,P_{a'}) \\
&= AP * \frac{PC * sin(\measuredangle P_aP_bP_c)}{sin(\measuredangle ABC)} \\
&= \frac{AP * PC * sin(\measuredangle P_aP_bP_c)}{sin(\measuredangle ABC)} \\
&= \frac{AP_b * CP_a * sin(\measuredangle P_aP_bP_c)}{sin(\measuredangle ABC)} \\
&= \frac{AP_b * CP_a * sin(\measuredangle P_aP_bP_c) * sin(\measuredangle BAC) * sin(\measuredangle ACB)}{sin(\measuredangle ABC)} * \frac{1}{sin(\measuredangle BAC) * sin(\measuredangle ACB)} \\
&= \frac{area(SLT(A,B,C,P)))}{2 * sin(\measuredangle BAC) * sin(\measuredangle ACB) * sin(\measuredangle ABC)}
\end{aligned}$$

We have proved the following theorem:

Theorem 2.4. *The locus of points P such that $area(SLT(A,B,C,P)) = \alpha$ is the set of points P such power(circum(ABC),P) is:*

$$\pi = \frac{\alpha}{sin(\measuredangle BAC) * sin(\measuredangle ACB) * sin(\measuredangle ABC)}$$

Conclusions:

We have shown examples of an experimental geometry practice: using the DGPad software to formulate, verify and proof conjectures regarding the locus of points P such that SLT(A,B,C,P) has a constant area.

The possibility of doing both approximate and exact constructions, and verify experimentally conjectures makes the DGS a powerful tool for solving geometry problems. The software can not only be used to formulate and verify conjectures but also to look for theoretical arguments that help connecting deductively the data of the problem with the statements of the conjectures, that is to produce a formal proof of them.

We hope to contribute in this way to clarify the role of the DGS in a practice of solving geometry problems; in fact, working with DGS, is not just interacting with drawings. DGS produce dynamic figures you can drag to identify the invariants that represent geometric objects (theoretical objects) and their properties. We call Experimental Geometry the practice that uses DGS in this way to solve geometric problems.

Experimental Geometry can be taken as a reference in the teaching of geometry, as a practice that allows students to develop visualization, conjecture and deduction, having the advantage of producing greater conviction about the reasoning and results that can be verified experimentally in every moment.

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