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THE ARBELOS IN WASAN GEOMETRY, TAMURA'S PROBLEM

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Abstract. Generalizing a sangaku problem involving an arbelos proposed by Tamura, we show the existence of several congruent non-Archimedean circles.

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1. Introduction

The mathematics in Japan developed in Edo era is called Wasan. Geometry is one of the main parts of this mathematics, in which figures involving tangent circles were mainly considered. When Wasan people, mostly they were amateurs, got a nice discovery or solution of a problem, they wrote the results on a wooden board called a sangaku, which was dedicated to a shrine or a temple, where the sangaku was hung under a roof. It was also a means to publish a discovery or to propose a problem. Most sangaku problems were geometric and the figures were drawn beautifully in color.

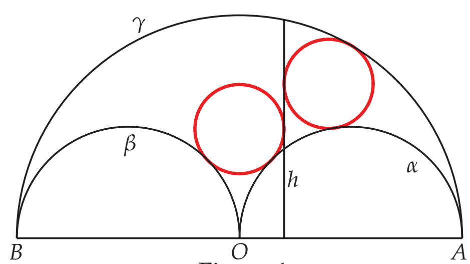

Figure 1.

The figure arbelos was also considered in this geometry in a unique way. Wasan mathematician were not interested in Archimedean circles, which appeared only as twin circles of Archimedes, but they considered non-Archimedean congruent circles in the case two inner semicircles of the arbelos being congruent. In this paper we consider such a problem. Let us consider an arbelos formed by three semicircles α, β and γ with diameters AO, BO and AB, respectively for a point O on the segment AB. We consider the following problem in the sangaku hung in Saitama in 1898 proposed by Tamura (田村金太郎治重) [4] (see Figure 1).

Problem 1. *Let h be a perpendicular to AB intersecting α. If α and β are congruent and the circle touching h and α and β externally and the incircle of the curvilinear triangle made by α, γ and h have common radius r, show $|AB| = 10r$.*

By generalizing the problem, we show the existence of two pairs of three congruent circles for the arbelos. For more recent study on the arbelos in Wasan geometry see [2].

2. Generalization

We now consider the case in which the semicircles α and β are not always congruent. Let a and b be the radii of α and β, respectively. We consider with a rectangular coordinate system with origin O such that the farthest point on α from AB has coordinates (a, a). Let δ be the incircle of the arbelos, which has center with coordinates $(ab(b-a)/d, 2abc/d)$ and radius abc/d, where $d = a^2 + ab + b^2$. The point of tangency of α and δ is denoted by T, which has coordinates

$$(x_t, y_t) = \left(\frac{2ab^2}{b^2+c^2}, \frac{2abc}{b^2+c^2}\right),$$

where $c = a + b$. Problem 1 is generalized as follows (see Figure 2).

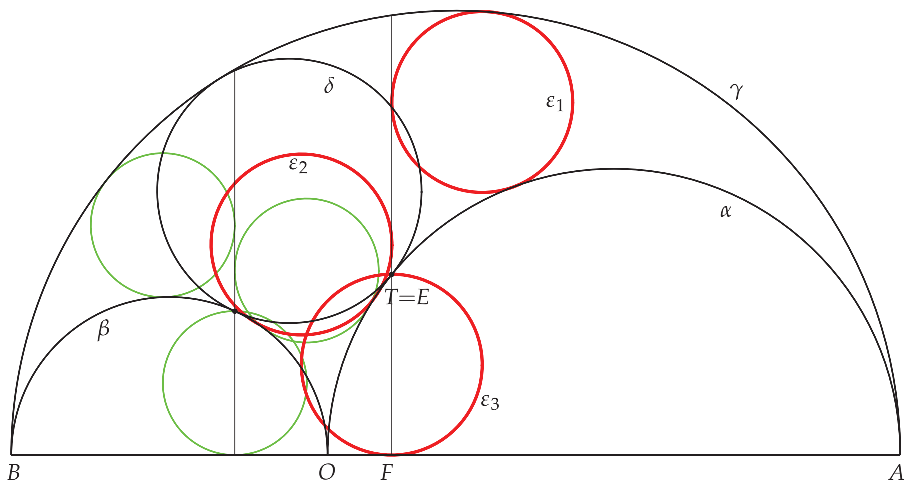

Figure 2.

Theorem 2.1. *For a point $E \neq A, O$ on the semicircle α, let F be the foot of perpendicular from E to AB. The incircle of the curvilinear triangle formed by α, γ and the line EF is denoted by ε_1. The circle touching α and β externally and the line EF is denoted by ε_2. The circle of diameter EF is denoted by ε_3. Let $r_i > 0$ be the radius of ε_i. The following statements are equivalent.*
(i) *The three circles ε_1, ε_2 and ε_3 are congruent.*
(ii) *Two of the circles ε_1, ε_2 and ε_3 are congruent.*
(iii) *The point E coincides with T.*
In this event the common radius equals $y_t/2$.

Proof. Let $k = |FO|$. Since the radius of ε_1 is proportional to the distance between its center and the radical axis of α and γ [1, p. 108], we get $r_1/(2a-k) = b/(2c)$. This is equivalent to

$$r_1 = \frac{(2a-k)b}{2c}. \tag{1}$$

From the two right triangles formed by the center of ε_2, the foot of perpendicular from this center to AB and one of the centers of α and β, we get $(a+r_2)^2 - (a-k+r_2)^2 = (b+r_2)^2 - (b+k-r_2)^2$. This is equivalent to

$$r_2 = \frac{ck}{2b}. \tag{2}$$

The power of the point F with respect to α equals $-k(2a-k) = -(2r_3)^2$, i.e.,

$$r_3 = \frac{\sqrt{(2a-k)k}}{2}. \tag{3}$$

From the equations (1), (2) and (3), we get that each of (i) and (ii) is equivalent to

$$k = x_t, \tag{4}$$

which is equivalent to (iii). □

Circles of radius ab/c are said to be Archimedean. Theorem 2.1 shows the existence of three congruent non-Archimedean circles of radius $y_t/2$. Exchanging the roles of α and β we get another three congruent non-Archimedean circles of radius $abc/(a^2 + c^2)$, which are denoted by green circles in Figure 2.

3. The other case

In the last section we consider the circle ε_1, which touches α and γ and a perpendicular to AB from the same side as A. In this section we consider the case in which ε_1 touches a perpendicular from the side opposite to A. Let Q be the point of tangency of γ and δ, which has coordinates $(2(b-a)j, 2(a+b)j)$, where $j = ab/(a^2 + b^2)$ (see Figure 3) [3]. Let P (resp. R) be the point of intersection of AQ and α (resp. BQ and β), which has coordinates

$$(2bj, 2aj) \ \text{(resp. } (-2aj, 2bj)).$$

Notice that $OPQR$ is a square .

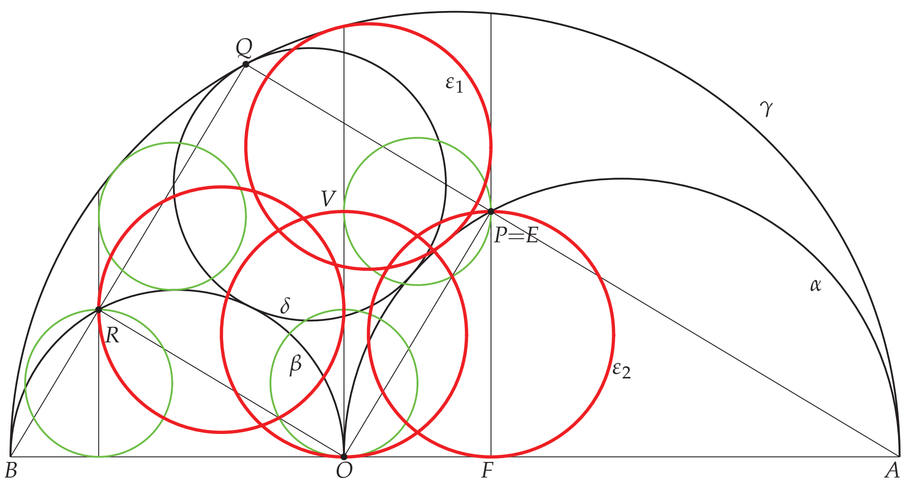

Figure 3.

Theorem 3.1. *For a point $E \neq A$ on the semicircle α, let F be the foot of perpendicular from E to AB. The circle touching α externally γ internally and the line EF from the side opposite to A is denoted by ε_1. The circle of diameter EF is denoted by ε_2. Then the circles ε_1 and ε_2 are congruent if and only if the point E coincides with P. In this event the common radius equals aj.*

Proof. Let $k = |FO|$ and let r_i be the radius of ε_i. From $r/(2a - k + 2r_1) = b/(2c)$, we get

$$r_1 = b - \frac{bk}{2a}. \tag{5}$$

Considering the power of the point F with respect to α, we have

$$r_2 = \frac{\sqrt{(2a-k)k}}{2}. \tag{6}$$

From (5) and (6) we get that $r_1 = r_2$ if and only in $k = 2bj$, which is equivalent to (iii). □

Exchanging the roles of α and β we get another two congruent non-Archimedean circles of radius jb, which are denoted by green circles in Figure 3. We call the radical axis of α and β the axis. The coordinates of R shows that the minimal circle passing through R and touching the axis has radius aj. Also the minimal circle passing through P and touching the axis has radius bj. Let V be the point of tangency of this circle and the axis. Then the circle of diameter OV has also radius aj.

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