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SOME RESULTS ON 3-DIMENSIONAL FINSLER MANIFOLDS

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ABSTRACT. The special and useful Moór frame for 3-dimensional manifolds was introduced and developed by Moór and Matsumoto. For every Finslerian tensor, Matsumoto was assigned three scalars $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$ which are called the main scalars of metric. In this paper, we show that there is not non-Riemannian 3-dimensional (α,β)-metric with $\mathcal{I}=0$ and $\mathcal{J}=0$. Then, we characterize weakly Landsberg 3-dimensional Finsler metrics. Finally, we characterize 3-dimensional weakly Berwald metrics and Finsler metrics with almost vanishing **H**-curvature.

Keywords: Moór frame, (α,β)-metric, weakly Landsberg metric, weakly Berwald metric, **H**-curvature.[1]

1. INTRODUCTION

Let (M,F) be a Finsler manifold. The second and third order derivatives of $\frac{1}{2}F_x^2$ at a non-zero vector $y\in T_xM_0$ are the fundamental form $\mathbf{g}_y=g_{ij}(y)dx^i\otimes dx^j$ and the Cartan torsion $\mathbf{C}_y=C_{ijk}(y)dx^i\otimes dx^j\otimes dx^k$, respectively. Taking a trace of Cartan torsion yields the mean Cartan torsion $\mathbf{I}_y=I_i(y)\,dx^i$.

In [6], Moór constructed an intrinsic orthonormal frame field (ℓ^i,m^i,n^i) on 3-dimensional Finsler manifolds which was a generalization of the Berwald frame of two-dimensional Finsler manifolds. The first vector $\ell^i=g^{ij}F_{y^j}$ is the normalized supporting element and the second one m^i is taken as the normalized torsion vector. Indeed, m^i is the unit vector along mean Cartan torsion $I^i=g^{ij}I_j$, i.e., $m^i:=I^i/||\mathbf{I}||$, where $g^{ij}=(g_{ij})^{-1}$ and $||\mathbf{I}||:=\sqrt{I_iI^i}$. The third element n^i is a unit vector orthogonal to the vectors ℓ^i and m^i. Then, Matsumoto gave a systematic description of a general theory of 3-dimensional Finsler spaces based on Moór's frame [5]. In this frame, the Cartan torsion of a Finsler metric F is written as follows

$$FC_{ijk}=\mathcal{H}m_im_jm_k-\mathcal{J}\Big\{m_im_jn_k+m_jm_kn_i+m_km_in_j-n_in_jn_k\Big\}+\mathcal{I}\Big\{n_in_jm_k+n_jn_km_i+n_in_km_j\Big\},$$

where $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$ are called the main scalars of F.

Theorem 1.1. *There is not non-Riemannian (α,β)-metric on a 3-dimensional manifold M such that $\mathcal{I}=0$ and $\mathcal{J}=0$.*

A Finsler metric F is called a Berwald, Landsberg and weakly Landsberg metric if $C_{ijk|s}=0$, $L_{ijk}:=C_{ijk|s}y^s=0$ and $J_i:=g^{jk}L_{jki}=0$, respectively, where " | " denotes the h-covariant derivative with respect to the Berwald connection of F. In [5], Matsumoto proved that a 3-dimensional non-Riemannian Berwald metric is characterized by the fact that the h–connection vector h_i vanishes and the main scalars $\mathcal{H},\mathcal{I},\mathcal{J}$ are h-covariant constant. He also characterized 3-dimensional Landsberg metric in terms of h-curvature vector h_i and the main scalars of F. In this paper, we characterize 3-dimensional weakly Landsbergian Finsler metrics as follows.

Theorem 1.2. *Let (M, F) be a 3-dimensional Finsler manifold. Then F is a weakly Landsberg metric if and only if $\mathcal{H}' + \mathcal{I}' = 0$ and $h_0 = 0$ hold, where $\mathcal{H}' := \mathcal{H}_{|s}y^s$, $\mathcal{I}' := \mathcal{I}_{|s}y^s$ and $h_0 := h_i y^i$.*

Every Berwald metric is a Landsberg metric and weakly Berwald metric. But the converse may not be holds [2]. For 2-dimensional Finsler manifolds the converse of this fact holds. It is interesting to find the condition under which every 3-dimensional Landsberg metric with vanishing mean Berwald curvature reduces to a Berwald metric. As a result of Theorem 1.2, we get the following.

Corollary 1.1. *Let (M, F) be 3-dimensional Finsler manifold. Suppose that F has horizontally constant main scalars $\mathcal{I}$ and $\mathcal{J}$. Then F is a Landsberg metric with vanishing mean Berwald curvature if and only if it is a Berwald metric.*

Finally, we give a characterization of 3-dimensional weakly Berwald Finsler metrics as follows.

Theorem 1.3. *Let (M, F) be 3-dimensional Finsler manifold. Then F is a weakly Berwald metric if and only if $E_{ij}m^i m^j = E_{ij}n^i n^j = E_{ij}m^i n^j = 0$.*

There are many connections in Finsler geometry [3][4][11][13]. Throughout this paper, we use the Berwald connection on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2. Preliminaries

A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on $TM_0 := TM \setminus \{0\}$; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM; (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that **C=0** if and only if F is Riemannian.

For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$ and $u = u^i \frac{\partial}{\partial x^i}|_x$. By Diecke Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$ [10].

The horizontal covariant derivative of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^i v^j w^k$, where

$$L_{ijk} := C_{ijk|s}y^s,$$

$u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$ [9].

The horizontal covariant derivatives of mean Cartan torsion $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := I_{i|s}y^s.$$

A Finsler metric is called a weakly Landsberg metric if $\mathbf{J} = 0$ [12].

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} g^{il} \left[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right], \quad y \in T_x M.$$

$\mathbf{G}$ is called the spray associated to (M, F). For a tangent vector $y \in T_x M_0$, define $\mathbf{B}_y : T_x M \otimes T_x M \otimes T_x M \to T_x M$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature. Then, F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.
Define the mean of Berwald curvature by $\mathbf{E}_y : T_x M \otimes T_x M \to \mathbb{R}$, where

$$\mathbf{E}_y(u, v) := \frac{1}{2} \sum_{i=1}^{n} g^{ij}(y) g_y \Big(\mathbf{B}_y(u, v, e_i), e_j \Big). \tag{2.1}$$

The family $\mathbf{E} = \{\mathbf{E}_y\}_{y \in TM \setminus \{0\}}$ is called the *mean Berwald curvature* or *E-curvature*. In local coordinates, $\mathbf{E}_y(u, v) := E_{ij}(y) u^i v^j$, where

$$E_{ij} := \frac{1}{2} B^m_{\ mij}.$$

By definition, $\mathbf{E}_y(u, v)$ is symmetric in u and v and $\mathbf{E}_y(y, v) = 0$. $\mathbf{E}$ is called the mean Berwald curvature. F is called a weakly Berwald metric if $\mathbf{E} = 0$.

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. For this aim, we need the following.

Lemma 3.1. *Let (M, F) be a 3-dimensional non-Riemannian Finsler manifold. Then the Cartan torsion of F is given by following*

$$C_{ijk} = \Big\{ a_i h_{jk} + a_j h_{ki} + a_k h_{ij} \Big\} + \Big\{ b_i I_j I_k + b_j I_i I_k + b_k I_i I_j \Big\}, \tag{3.1}$$

where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are scalar functions on TM.

Proof. For 3-dimensional Finsler manifolds, we have

$$g_{ij} = \ell_i \ell_j + m_i m_j + n_i n_j.$$

Thus

$$g^{ij} = \ell^i \ell^j + m^i m^j + n^i n^j.$$

Then the Cartan torsion of F is written as follows

$$FC_{ijk} = \mathcal{H} m_i m_j m_k - \mathcal{J} \Big\{ m_i m_j n_k + m_j m_k n_i + m_k m_i n_j - n_i n_j n_k \Big\} + \mathcal{I} \Big\{ n_i n_j m_k + n_j n_k m_i + n_i n_k m_j \Big\}, \tag{3.2}$$

where $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{J}$ are called the main scalars of F. Thus

$$F I_k = (\mathcal{H} + \mathcal{I}) m_k. \tag{3.3}$$

Contracting (3.3) with g^{mk} yields

$$F I^k = (\mathcal{H} + \mathcal{I}) m^k. \tag{3.4}$$

(3.3)×(3.4) yields

$$\mathcal{H} + \mathcal{I} = F||\mathbf{I}||, \tag{3.5}$$

where $||\mathbf{I}|| := \sqrt{I_m I^m}$. Then by assumption, we get $\mathcal{H} + \mathcal{I} \neq 0$.
The angular metric is given by

$$h_{ij} = m_i m_j + n_i n_j. \tag{3.6}$$

By considering (3.5) and (3.6), one can rewrite (3.2) as (3.1), where

$$a_i := \frac{1}{3F}\Big[3\mathcal{I} m_i + \mathcal{J} n_i\Big], \quad b_i := \frac{F}{3(\mathcal{H}+\mathcal{I})^2}\Big[(\mathcal{H} - 3\mathcal{I})m_i - 4\mathcal{J} n_i\Big]. \tag{3.7}$$

It is easy to see that $a_i y^i = 0$ and $b_i y^i = 0$. This completes the proof. □

Proof of Theorem 1.1: Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian (α, β)-metric on a 3-dimensional manifold M. Suppose that F satisfies $\mathcal{I} = 0$ and $\mathcal{J} = 0$. In this case, by (3.1) we have $a_i = 0$. By Lemma 3.1, we get

$$C_{ijk} = b_i I_j I_k + b_j I_i I_k + b_k I_i I_j. \tag{3.8}$$

Using (3.7) and (3.5) imply that

$$b_k = \frac{1}{3||\mathbf{I}||^2} I_k. \tag{3.9}$$

By (3.8) and (3.9), we have

$$C_{ijk} = \frac{1}{||\mathbf{I}||^2} I_i I_j I_k. \tag{3.10}$$

On the other hand, it is proved that every non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$ is semi-C-reducible [14]. More precisely, the Cartan tensor of F is given by following

$$C_{ijk} = \frac{P}{1+n}\Big\{h_{ij} I_k + h_{jk} I_i + h_{ki} I_j\Big\} + \frac{1-P}{||\mathbf{I}||^2} I_i I_j I_k, \tag{3.11}$$

where

$$P := \frac{n+1}{\mathfrak{a}\mathcal{A}}\Big[s\phi\phi'' - \phi'(\phi - s\phi')\Big]. \tag{3.12}$$

By (3.10) and (3.11), we get $P = 0$. According to (3.12), $P = 0$ if and only if ϕ satisfies following

$$s(\phi\phi'' + \phi'\phi') - \phi\phi' = 0. \tag{3.13}$$

By solving (3.13), we get

$$\phi = \sqrt{c_1 s^2 + c_2},$$

where c_1 and c_2 are two real constant. In this case, $F = \alpha\phi(s)$, $s = \beta/\alpha$, reduces to a Riemannian metric which contradicts with our assumptions. □

4. Proof of Theorem 1.2

Proof of Theorem 1.2: The horizontal derivation of Moór frame are giving by following

$$\ell_{i|j} = 0, \quad m_{i|j} = h_j n_i, \quad n_{i|j} = -h_j m_i,$$

where h_i are called the h-connection vectors. Thus

$$m'_i := m_{i|j} y^j = h_0 n_i, \quad n'_i := n_{i|j} y^j = -h_0 m_i,$$

where $h_0 := h_i y^i$.
According to Lemma 3.1, F is considered as a non-Riemannian metric. Then, by (3.5) it follows that the main scalars of F satisfies $\mathcal{H} + \mathcal{I} \neq 0$.
Now, by taking a horizontal derivation of (3.3), we get

$$FJ_k = (\mathcal{H}' + \mathcal{I}')m_k + (\mathcal{H} + \mathcal{I})h_0 n_k. \tag{4.1}$$

Let F be a weakly Landsberg metric. Then, by contracting (4.1) with m^k and n^k, we get

$$\mathcal{H}' + \mathcal{I}' = 0, \quad \text{and} \quad h_0 = 0. \tag{4.2}$$

The converse is trivial. □

Lemma 4.1. *Let (M, F) be a 3-dimensional non-Riemannian Finsler manifold. Then F is a relatively isotropic mean Landsberg metric if and only if the following hold*

$$\mathcal{H}' + \mathcal{I}' = cF(\mathcal{H} + \mathcal{I}), \quad h_0 = 0, \tag{4.3}$$

where $c = c(x)$ is a scalar function on M.

Proof. By assumption, (4.1) reduces to following

$$(\mathcal{H}' + \mathcal{I}')m_k + (\mathcal{H} + \mathcal{I})h_0 n_k = cF(\mathcal{H} + \mathcal{I})m_k. \tag{4.4}$$

(4.4) is equal to following

$$(\mathcal{H}' + \mathcal{I}') = cF(\mathcal{H} + \mathcal{I}), \tag{4.5}$$

$$(\mathcal{H} + \mathcal{I})h_0 = 0. \tag{4.6}$$

Since $\mathcal{H} + \mathcal{I} \neq 0$, then we get (4.3). □

In page 86 of [8], the following lemma was obtained.

Lemma 4.2. ([8]) The Berwald and mean Berwald curvatures of F are given by following

$$B^i{}_{jkl} = g^{im}\Big\{C_{mjl|k} + C_{mkl|j} - C_{jkl|m} + L_{mjk,l}\Big\}, \tag{4.7}$$

$$E_{ij} = \frac{1}{2}\Big\{I_{j|i} + J_{i,j}\Big\}. \tag{4.8}$$

Every Berwald metric is a Landsberg metric with vanishing mean Berwald curvature. For 2-dimensional Finsler manifolds the converse of this fact holds. We find the condition that every 3-dimensional Landsberg metric with vanishing mean Berwald curvature reduces to a Berwald metric.

Lemma 4.3. *Let (M, F) be 3-dimensional Landsberg manifold. Then F is weakly Berwald metric if and only if the following hold*

$$\mathcal{H}_{|j} + \mathcal{I}_{|j} = 0, \quad h_j = 0. \tag{4.9}$$

Proof. Let F be a Landsberg metric. By (4.8), we get

$$E_{ij} = \frac{1}{2} I_{j|i}. \tag{4.10}$$

Thus F is weakly Berwald metric if and only if $I_{i|j} = 0$. Taking a horizontal derivation of (3.3) implies that

$$I_{i|j} = \frac{1}{F}\Big[(\mathcal{H}_{|j} + \mathcal{I}_{|j})m_i + (\mathcal{H} + \mathcal{I})h_j n_i\Big]. \tag{4.11}$$

By (4.11), we get (4.9). □

Proof of Corollary 1.1: Let F be a weakly Berwald metric with vanishing Landsberg curvature. By (3.1), one can obtain the following

$$C_{ijk|s} = \Big\{a_{i|s}h_{jk} + a_{j|s}h_{ki} + a_{k|s}h_{ij}\Big\} + \Big\{b_{i|s}I_j I_k + b_{j|s}I_i I_k + b_{k|s}I_i I_j\Big\} \tag{4.12}$$

and by (4.9) we get

$$a_{i|s} = \frac{1}{3F}\Big[3\mathcal{I}_{|s}m_i + \mathcal{J}_{|s}n_i\Big], \quad b_{i|s} = \frac{-4}{3F||\mathbf{I}||^2}\Big[\mathcal{I}_{|s}m_i + \mathcal{J}_{|s}n_i\Big]. \tag{4.13}$$

Putting (4.13) in (4.12) yields

$$\begin{aligned} C_{ijk|s} &= \frac{\sigma_{(i,j,k)}}{3F}\Big[3\mathcal{I}_{|s}m_i + \mathcal{J}_{|s}n_i\Big]h_{jk} - \frac{4\sigma_{(i,j,k)}}{3F||\mathbf{I}||^2}\Big[\mathcal{I}_{|s}m_i + \mathcal{J}_{|s}n_i\Big]I_jI_k \\ &= \frac{\sigma_{(i,j,k)}}{3F||\mathbf{I}||^2}\Big[(3||\mathbf{I}||^2h_{jk} - 4I_jI_k)\mathcal{I}_{|s}m_i + (||\mathbf{I}||^2h_{jk} - 4I_jI_k)\mathcal{J}_{|s}n_i\Big], \end{aligned} \tag{4.14}$$

where $\sigma_{(i,j,k)}$ denotes the permutation in indexes i, j and k. By assumption, we have $\mathcal{I}_{|s} = \mathcal{J}_{|s} = 0$. By putting these relations in (4.14), it follows that F is a Berwald metric. □

5. Proof of Theorem 1.3

In this section, we will prove a generalized version of Theorem 1.3. Indeed, we study Finsler metrics with isotropic mean Berwald curvature. A Finsler metric F on a manifold M is called of isotropic mean Berwald curvature if the following holds

$$E_{ij} = \frac{n+1}{2}cF^{-1}h_{ij}, \tag{5.1}$$

where $c = c(x)$ is a scalar function on M. In this case, F is called a isotropic mean Berwald metric.

Theorem 5.1. *Let (M, F) be 3-dimensional Finsler manifold. Then F is a isotropic mean Berwald metric (5.1) if and only if the following hold*

$$E_{ij}m^im^j = E_{ij}n^in^j = \frac{2c}{F}, \quad E_{ij}m^in^j = 0. \tag{5.2}$$

Proof. Since E_{ij} is symmetric and $E_{ij}\ell^i = 0$, then with respect to Moór frame we have that

$$E_{ij} = Am_im_j + B(m_in_j + n_im_j) + Cn_in_j,$$

where

$$A := E_{ij}m^im^j, \quad B := E_{ij}m^in^j, \quad C := E_{ij}n^in^j.$$

As $h_{ij} = m_im_j + n_in_j$, then F a isotropic mean Berwald metric if

$$Am_im_j + B(m_in_j + n_im_j) + Cn_in_j = 2cF^{-1}(m_im_j + n_in_j). \tag{5.3}$$

Contracting (5.3) with m^im^j, n^in^j and m^in^j yield (5.2). □

Proof of Theorem 1.3: By Theorem 5.1, we get the proof. □

In [1], Akbar-Zadeh considered a non-Riemannian quantity $\mathbf{H}$ which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. This is a positively homogeneous scalar function of degree zero on the slit tangent bundle. Akbar-Zadeh proved that for a Finsler metric of scalar flag curvature, the flag curvature is a scalar function on the manifold if and only if $\mathbf{H} = \mathbf{0}$. It is remarkable that, the quantity $\mathbf{H}_y = H_{ij}dx^i \otimes dx^j$ is defined as the covariant derivative of $\mathbf{E}$ along geodesics, where $H_{ij} := E_{ij|m}y^m$. A Finsler metric F is called of almost vanishing $\mathbf{H}$-curvature if

$$\mathbf{H} = \frac{n+1}{2}F^{-1}\theta\,\mathbf{h}, \tag{5.4}$$

where $\theta := \theta_i(x)y^i$ is a 1-form on M and $\mathbf{h} = h_{ij}dx^i \otimes dx^j$ is the angular metric. In [7], Najafi-Shen and the first author generalized the above Akbar-Zadeh theorem and prove that a Finsler metric has almost isotropic flag curvature $\mathbf{K} = 3\theta/F + \sigma$ if and only if it has almost vanishing

H-curvature (5.4) for a scalar function $\sigma = \sigma(x)$ and a 1-form $\theta = \theta_i(x)dx^i$ on M. By the same argument used in Theorem 5.1, one can get the following.

Theorem 5.2. *Let (M, F) be 3-dimensional Finsler manifold. Then F has almost vanishing* **H***-curvature (5.4) if and only if the following hold*

$$H_{ij}m^im^j = H_{ij}n^in^j = \frac{2\theta}{F}, \quad H_{ij}m^in^j = 0. \tag{5.5}$$

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