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ON A NORMALIZED PARALLEL VECTOR FIELD IN THE HYPERSURFACE OF FINSLER SPACES

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ABSTRACT. In the present paper, with the help of a normalized parallel Finsler hypersurface vector field, we modify the fundamental function and investigate some properties of the newly obtained space.

§1. Introduction

Let F^n be an n-dimensional Finsler space with a fundamental function $L(x,y), (y = \dot{x})$, and we shall introduce in F^n the Cartan's connection $C\Gamma = \left(F^i_{jk}, N^i_k, C^i_{jk}\right)$.
Let us consider a vector field $X^i(x)$ in F^n.This field is called parallel vector field if it satisfies the partial differential equations [3]

$$X^i_{|j} = \partial_j X^i - N^h_j \dot{\partial}_h X^i + X^h F^i_{hj} = \partial_j X^i + X^h F^i_{hj} = 0, \tag{0.1}$$

$$X^i|_j = \dot{\partial}_j X^i + X^h C^i_{hj} = X^h C^i_{hj} = 0, \tag{0.2}$$

where $\dot{\partial}_j = \frac{\partial}{\partial y^j}$, $\partial_j = \frac{\partial}{\partial x^j}$ and $X^i_{|j}$, $X^i\,|_j$ denote h- and v- covariant derivative of X^i with respect to Cartan connection $C\Gamma$.
In terms of covariant components$X_i(x)$'s, (0.1) and (0.2) are written as

$$X_{i|j} = 0, \tag{0.3}$$

$$X_i\,|_j = 0. \tag{0.4}$$

If $L(x,y)$ is a metric function of the Finsler space F^n, then a modified Finsler space $\overset{*}{F^n}$ is defined as [4]

$$\overset{*}{L}{}^2 = L^2 + \beta^2, \ \ (\beta = X_i(x)y^i \neq 0) \tag{0.5}$$

$\overset{*}{L}$ defines a new metric of M. It is said that $\overset{*}{L}$ is obtained by a β - change of the metric L, [7]. The metric tensor derived from $\overset{*}{L}$ is written as follows

$$\overset{*}{g}_{ij} = g_{ij} + X_i X_j, \ \ \overset{*}{g}{}^{ij} = g^{ij} - \frac{X^i X^j}{1 + X^2}, \tag{0.6}$$

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where X is the length of X^i w. r. to the original metric. The coefficients of Cartan's connection are written in the form [3]

$$\overset{*}{N}{}^i_j = N^i_j, \;\; \overset{*}{F}{}^i_{jk} = F^i_{jk}.$$

Study of the special vector fields in Finsler geometry are of great importace,because of their applications in biology, theorical physics and engineering. B. Bidabad and P. Joharinad, studied on Finsler spaces with a special conformal vector fields and obtained some interesting results [1]. The theory of hypersurface of Finsler space is an important field of Finsler geometry which pursues the geometers of all over to investigate . M. K. Gupta and P. N. Pandey, worked on a certain properties of Finslerian hypersurface given by $h-$vector [2]. The author studied some special hypersurface of Finsler spaces with certain $(\alpha, \beta)-$metrics ([8], [9],[10]). In this paper we shall study the properties of hypersurface of F^n and $\overset{*}{F}{}^n$**.** It has been shown that, if the vector field X_i is tangential to the hypersurface of F^n,then the normal curvature vector and second fundamental tensors of the hypersurface of both spaces are identical. Hence, if the hypersurface of $\overset{*}{F}{}^n$ is a hyperplane [5] of any one of three kinds, the hypersurface of $\overset{*}{F}{}^n$is also a hypersurface of the same kind.

§2. Induced Cartan connection

Let F^{n-1} be a hypersurface of F^ngiven by the equations

$$x^i = x^i(u^\alpha), (i = 1, 2, ..., n\, ; \alpha = 1, 2, ..., n-1). \tag{0.7}$$

In this article we shall use only the induced connection of the Cartan's connection $C\Gamma$. The induced connection $IC\Gamma = (\overset{C}{F}{}^\alpha_{\beta\gamma}, \overset{C}{N}{}^\alpha_\beta, C^\alpha_{\beta\gamma})$ induced from the Cartan's connection$C\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$is given by [5]

$$\overset{C}{F}{}^\alpha_{\beta\gamma} = B^\alpha_i (B^i_{\beta\gamma} + F^i_{jk} B^{jk}_{\beta\gamma}) + M^\alpha_\beta H_\gamma,$$

$$\overset{C}{N}{}^\alpha_\beta = B^\alpha_i (B^i_{0\beta} + N^i_j B^j_\beta),$$

$$C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk} B^{jk}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \dot{\partial}_\delta\, g_{\beta\gamma}. \tag{0.8}$$

Where $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ is the matrix of the projection factor at a point u$^\alpha$.
The relative h-and v- covariant derivatives of the projection factor $B^i_\alpha$and the normal vector $N^i$with respect to $IC\Gamma$ are given by

$$B^i_{\alpha|\beta} = B^i_{\alpha\beta} + B^j_\alpha F^i_{j\beta} - B^i_\gamma\, \overset{C}{F}{}^\gamma_{\alpha\beta} = \overset{C}{H}_{\alpha\beta}\, N^i, \tag{0.9}$$

$$B^i_\alpha\, |_\beta = B^j_\alpha C^i_{j\beta} - B^i_\gamma C^\gamma_{\alpha\beta} = M_{\alpha\beta} N^i,$$

$$N^i_{|\beta} = \partial_\beta N^i + N^j F^i_{j\beta} = -\, \overset{C}{H}{}^\alpha_\beta\, B^i_\alpha,$$

$$N^i\, |_\beta = \dot{\partial}\, N^i + N^j C^i_{j\beta} = -M^\alpha_\beta B^i_\alpha,$$

where

$$F^i_{j\beta} = F^i_{jk} B^k_\beta + C^i_{jk} N^k H_\beta \;\; , C^i_{j\beta} = C^i_{jk} B^k_\beta$$

and the quantities $\overset{C}{H}_{\alpha\beta}, M_{\alpha\beta}, H_{\beta}$ are defined as

$$H_{\beta\gamma} = N_i(\overset{C}{B^i_{\beta\gamma}} + F^i_{jk}B^{jk}_{\beta\gamma}) + M_\beta H_\gamma,$$

$$M_{\beta\gamma} = g_{\beta\delta}M^\delta_\gamma = N_i C^i_{jk}B^{jk}_{\beta\gamma}, \tag{0.10}$$

and

$$H_\beta = N_i(B^i_{o\beta} + N^i_j B^j_\beta). \tag{0.11}$$

If X_i is a vector field of F^n, then along F^{n-1} we have

$$X_{i|\beta} = X_i\big|_j B^j_\beta + X_i|_k N^k H_\beta, \tag{0.12}$$

$$X_i\big|_\beta = X_i\big|_j B^j_\beta .$$

With respect to $IC\Gamma$ the Gauss and Codazzi equations of three of Cartan's curvature Tensors R_{ijkh} , P_{ijkh} and S_{ijkh} are given by [5],

$$R_{\alpha\beta\gamma\delta} = R_{ij\gamma\delta}B^{ij}_{\alpha\beta} + (\overset{C}{H}_{\alpha\gamma}\overset{C}{H}_{\beta\delta} - \overset{C}{H}_{\alpha\delta}\overset{C}{H}_{\beta\gamma}),$$

$$P_{\alpha\beta\gamma\delta} = P_{ij\gamma\delta}B^{ij}_{\alpha\beta} - (\overset{C}{H}_{\alpha\gamma} M_{\beta\gamma} + M_{\alpha\delta}\overset{C}{H}_{\beta\gamma}),$$

$$S_{\alpha\beta\gamma\delta} = S_{ijkh}B^{ijkh}_{\alpha\beta\gamma\delta},$$

where

$$R_{ij\beta\gamma} = R_{ijkh}B^{kh}_{\beta\gamma} + P_{ijkh}(B^k_\beta H_\gamma - B^k_\gamma H_\beta)B^h,$$

$$P_{ij\beta\gamma} = P_{ijkh}B^{kh}_{\beta\gamma} + S_{ijkh}B^k H_\beta B^h_\gamma.$$

Hyperplanes of first, second and third kind are defined in [5]. We only quote the following

Lemma 1. A hypersurface F^{n-1} is a hyperplane of the first kind if and only if $H_\alpha = 0$.

Lemma 2. A hypersurface F^{n-1} is a hypersurface of the second kind with respect to the Cartan connection $C\Gamma$ if and only if $H_\alpha = 0$ and $\overset{C}{H}_{\alpha\beta} = 0$.

Lemma 3. A hypersurface F^{n-1} is a hypersurface of the third kind with respect to the connection $C\Gamma$ if and only if $H_\alpha = 0$ and $M_{\alpha\beta} = \overset{C}{H}_{\alpha\beta} = 0$.

§3. Normalized parallel vector fields in F^{n-1}

A non zero vector field X^iin a Finsler space F^n is said to be parallel if conditions (0.3) and (0.4) hold [3]. Further, the vector field X^i is said to be normalized if

$$g_{ij}X^iX^j = 1. \tag{0.13}$$

From the Ricci indentities for h- and v-covariant differentiation fields,we have:

$$X^h R_{hijk} = 0,$$

$$X^h P_{hijk} = 0,$$

$$X^h S_{hijk} = 0. \tag{0.14}$$

Theorem 1. If X_i is a parallel vector field in F^n, then the vector field $X_\alpha = X_i B^i_\alpha$ is a parallel vector field in F^{n-1} with respect to $IC\Gamma$, if and only if (i) X_i is tangential to the hypersurface or (ii) $\overset{C}{H}_{\alpha\beta} = M_{\alpha\beta} = 0$.
Proof . At each point of F^{n-1} the vector field X_i may be written as

$$X_i = X_\alpha B^\alpha_i + \rho N_i, \tag{0.15}$$

where

$$(a)\ \ X_\alpha = X_i B^i_\alpha\ \ , (b)\ \rho = X_i N^i \tag{0.16}$$

Since $\dot\partial = \dot\partial_j B^j_\beta$, $\dot\partial_\beta B^i_\alpha = 0$, from (0.16)$a$ and the fact that X_i is a function of position only, we get

$$\dot\partial_\beta X_\alpha = 0.$$

Hence X_α is a function of coordinate u^α only. Again, from the relation $B^i_\alpha B^\alpha_j = \delta^i_j - N^i N_j$and relations (0.8) and (0.16), we get

$$\begin{aligned} X_\alpha C^\alpha_{\beta\gamma} &= X_\alpha B^\alpha_i C^i_{jk} B^{jk}_{\beta\gamma} \\ &= X_h B^h_\alpha B^\alpha_i C^i_{jk} B^{jk}_{\beta\gamma} \\ &= X_h (\delta^h_i - N^h N_i)\, C^i_{jk} B^{jk}_{\beta\gamma} \\ &= X_h C^h_{jk} B^{jk}_{\beta\gamma} - \rho\, N_i C^i_{jk} B^{jk}_{\beta\gamma}. \end{aligned}$$

In view of (0.10), since $X_h C^h_{ij} = 0$, we have

$$X_\alpha C^\alpha_{\beta\gamma} = -\rho\, N_i C^i_{jk} B^{jk}_{\beta\gamma} = -\rho\, M_{\beta\gamma}.$$

Next, from (0.16)a, we get

$$X_{\alpha|\beta} = X_{i|\beta} X^i_\alpha + X_i B^i_{\alpha|\beta}.$$

Since $X_i|_j = 0$, in view of the relation $g_{\alpha\beta} = g_{ij} B^{ij}_{\alpha\beta}$ and (0.9), (0.12), (0.3), the above equation reduces to

$$X_\alpha|_\beta = B^i_\alpha (X_i|_j B^j_\beta + X_i|_j N^j H_\beta) + (X_i\, \overset{C}{H}_{\alpha\beta}\, N^i) = \rho\, \overset{C}{H}_{\alpha\beta}\,.$$

Hence the theorem is proved.
Theorem 2. If X_i is a parallel vector field in F^n, then the vector field $X_\alpha = X_i B^i_\alpha$ is also a parallel vector field in a hyperplane of third kind.
Proof. The theorem is easily proved by the help of theorem 1 and in view of Lemma 3 of section 2.
Next we consider the normalization of parallel vector field and prove the following
Theorem 3. If X_i is a normalized parallel vector field in F^n, then X_α is also a normalized parallel vector field with respect to $I\Gamma C$ of F^{n-1} if and only if X_α is tangential to F^{n-1} ; i.e. $\rho = 0$.
Proof. From (0.13) and (0.15) we get

$$g^{\alpha\beta} X_\alpha X_\beta = 1 - \rho^2.$$

Thus the vector field X_α is also a vector field in F^{n-1} if $\rho = 0$ and hence the theorem is proved.

In the following we assume that $X_i$is tangential to F^{n-1} so that

$$(a)\ X_i = X_\alpha B^\alpha_i\ ,(b)\ X^i = X^\alpha B^i_\alpha, \tag{0.17}$$

where $X^\alpha = g^{\alpha\beta}X_\beta$, and the components of the three Cartan's curvature tensor of $F^n$and $\overset{*}{F}{}^n$ are related by [3]

$$\overset{*}{R}_{ijkh} = R_{ijkh}, \tag{0.18}$$

$$\overset{*}{P}_{ijkh} = P_{ijkh},$$

$$\overset{*}{S}_{ijkh} = S_{ijkh}. \tag{0.19}$$

§4. The hypersurface of $\overset{*}{F}{}^n$

Let $\overset{*}{F}{}^n$ be a Finsler space with modified metric (0.5),where $L(x,y)$ is the metric function of F^n and X_i is a normalized parallel vector field in F^n and, hence, from (0.6) the metric tensors of the space are related by

$$(a)\ \ \overset{*}{g}_{ij} = g_{ij} + X_iX_j,\ \ (b)\ \ \overset{*}{g}{}^{ij} = g^{ij} - \frac{1}{2}X^iX^j. \tag{0.20}$$

A direct calculation based on the equation (0.1) and the above equations give

$$\overset{*}{F}{}^i_{jk} = F^i_{jk}, \tag{0.21}$$

$$\overset{*}{N}{}^i_{jk} = N^i_{jk}, \tag{0.22}$$

$$\overset{*}{C}{}^i_{jk} = C^i_{jk}, \overset{*}{C}_{ijk} = C_{ijk},$$

where $(\overset{*}{F}{}^i_{jk}, \overset{*}{N}{}^i_{jk}, \overset{*}{C}{}^i_{jk})$ are the Cartan's connection parameters of $\overset{*}{F}{}^n$ and are related by (0.18), (0.14) and (0.19). Further we prove the following

Theorem 4. If the hypersurface F^{n-1} of F^n is a hyperplane of any of three kinds, then the hypersurface $\overset{*}{F}{}^{n-1}$ and $\overset{*}{F}{}^n$ is also a hyperplane of the same kind.

Proof. Let $\overset{*}{F}{}^{n-1}$be a hypersurface of $\overset{*}{F}{}^n$ given by the equation (0.7). If $\overset{*}{g}_{\alpha\beta}$ is the induced metric tensor of $\overset{*}{F}{}^{n-1}$ then, from the relation $g_{\alpha\beta} = g_{ij}B^{ij}_{\alpha\beta}$ and (0.17) and (0.20) we get

$$(a)\ \overset{*}{g}_{\alpha\beta} = g_{\alpha\beta} + X_\alpha X_\beta, (b)\ \overset{*}{g}{}^{\alpha\beta} = g^{\alpha\beta} - \frac{1}{2}X^\alpha X^\beta. \tag{0.23}$$

From the relations [6]

$$g_{ij}(x(u), y(u,v))B^i_\alpha N^j = 0,$$
$$g_{ij}(x(u), y(u,v))N^iN^j = 1,$$

where $N^i(u,v)$ is a unit normal, and in view of equations (0.17) , (0.20) it follows that $N^i(u,v)$is also a unit normal vector to $\overset{*}{F}{}^{n-1}$. If $\overset{*}{B}{}^\alpha_i$ denote the inverse projection vector to B^i_α in $\overset{*}{F}{}^{n-1}$ then, in view of (0.20) and (0.23) and the fact that X^α is a unit vector, we get

$$\overset{*}{B}{}^\alpha_i = B^\alpha_i.$$

The second foundation h-tensor $\overset{C}{H}_{\beta\nu}$ with respect to $C\Gamma$is defined by [6] as

$$\overset{C}{H}_{\beta\nu} = N_i(B^i_{\beta\nu} + F^i_{jk}B^{jk}_{\beta\nu}) + M_\beta H_\nu,$$

where

$$M_\beta = N_i C^i_{jk} B^j_\beta N^k. \qquad (0.24)$$

From (0.10), (0.11), (0.18), (0.21), (0.22), (0.24) and the condition that $X_i N^i = 0$, it follows that

$\overset{*}{M}_{\alpha\nu} = M_{\alpha\nu}$, $\overset{*}{H}_\beta = H_\beta$, $\overset{*C}{H}_{\beta\gamma} = \overset{C}{H}_{\beta\gamma}$ and $\overset{*}{M}_\beta = M_\beta$.

In view of Lemmas 1, 2 and 3 and in view of above relations , the proof is completed.

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