Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.8, (2019), Issue 1, pp.1-11

EQUILATERAL TRIANGLES AND ARCHIMEDEAN CIRCLES RELATED TO AN ARBELOS

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ABSTRACT. In this article, we construct the Archimedean circles from one or some equilateral triangles related to arbelos. And its converse, we also construct an another equilateral triangle from Yiu's circle.

1. INTRODUCTION

We consider an arbelos with greater semi-circle (O) of radius r and smaller semicircles (O_1) and (O_2) of radii r_1 and r_2 respectively. The semi-circles (O_1) and (O) meet at A, (O_2) and (O) at B, (O_1) and (O_2) at C. The line through C perpendicular to AB meets (O) at D, and (O') is the Midway semi-circle with diameter O_1O_2, all on the same side AB. Circles with radius $t := \frac{r_1 r_2}{r}$ are called Archimedean. They are congruent to the Archimedean twin circles [1, 2].
We shall make use of the Yiu's circle (that is the circle (A_{26}) in [3], is also the circle (K_3) in [4], or the circle touches γ internally, and touch $\alpha[1]$ and $\beta[1]$ externally at Corollary 3 in [5]), with center Y (See figure 1).

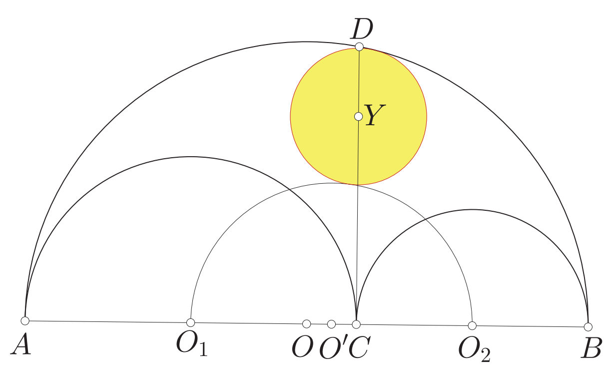

FIGURE 1. Yiu's circle

Key words and phrases. Arbelos, Archimedean circle, equilateral triangle, Yiu's circle.
AMS 2010 Mathematics Subject Classification:51M04.

Proposition 1.1. (Yiu [3], [4]). *The circle (Y) tangent to both semi-circles (O) and (O') and its center on CD is Archimedean.*

For three non-collinear points P, Q and R, we use $(PQ), P(Q), (PQR)$ to denote the circle with diameter PQ, the circle with center P passing through Q, and the circumcircle of triangle PQR.

2. Preliminaries

2.1. **Archimedean circles are constructed from one or some equilateral triangles releted to an arbelos.** In this subsection, we refer to some new Archimedean circles as well as pairs of new Archimedean circles in an arbelos.

Theorem 2.1. *Consider an equilateral triangle $A_1B_1C_1$ such that the vertex C_1 lies on the ray CD and sides C_1A_1 and C_1B_1 are all tangent to the semi-circle (O) at A_1 and B_1, respectively. Line passing through C perpendicular to A_1B_1 at E and meet the semi-circle (O) at F. Then the circles (CE) and (EF) are Archimedean.*

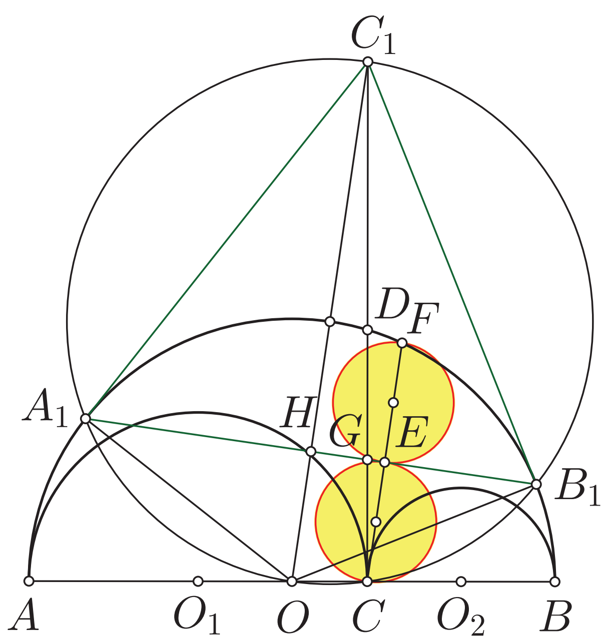

Figure 2

Proof. (See figure 2). We easily see that $O_1O = O_2C = r_2$, $O_2O = O_1C = r_1$, $CO = |r_1 - r_2|$, $O_1O_2 = r_1 + r_2 = r$.
Since $\angle C_1CO = \angle C_1A_1O = \angle C_1B_1O = 90^\circ$, it follows that C_1O is the diameter of (CA_1B_1).
Line A_1B_1 meets C_1C, C_1O at G, H, respectively. Since C_1A_1 and C_1B_1 are tangent lines of (O), OC_1 is perpendicular to A_1B_1 at H.

Since $OA_1 = OB_1 = r$ and triangle $A_1B_1C_1$ is equilateral, $C_1O = 2r$, $A_1B_1 = B_1C_1 = C_1A_1 = \sqrt{3}r$, $C_1H = \frac{3}{2}r$ and $OH = C_1O - C_1H = \frac{1}{2}r$.
Applying the Pythagoras' theorem, we have

$$\begin{aligned} C_1C^2 &= C_1O^2 - OC^2 = 4r^2 - (r_1 - r_2)^2 = 4(r_1 + r_2)^2 - (r_1 - r_2)^2 \\ &= 3r_1^2 + 3r_2^2 + 10r_1r_2. \end{aligned}$$

It follows $C_1C = \sqrt{3r_1^2 + 3r_2^2 + 10r_1r_2}$.
Since two right triangles C_1GH and C_1OC are similar,

$$\frac{C_1G}{C_1O} = \frac{C_1H}{C_1C} \Longrightarrow C_1G = \frac{C_1H}{C_1C}C_1O = \frac{3r^2}{\sqrt{3r_1^2 + 3r_2^2 + 10r_1r_2}}.$$

Hence,

$$\begin{aligned} CG = C_1C - C_1G &= \sqrt{3r_1^2 + 3r_2^2 + 10r_1r_2} - \frac{3r^2}{\sqrt{3r_1^2 + 3r_2^2 + 10r_1r_2}} \\ &= \sqrt{3r_1^2 + 3r_2^2 + 10r_1r_2} - \frac{3(r_1 + r_2)^2}{\sqrt{3r_1^2 + 3r_2^2 + 10r_1r_2}} = \frac{4r_1r_2}{\sqrt{3r_1^2 + 3r_2^2 + 10r_1r_2}}. \end{aligned}$$

Applying the Thales' theorem with note that CE is parallel to C_1O, we have

$$\frac{CE}{C_1H} = \frac{CG}{C_1G} \Longrightarrow CE = \frac{CG}{C_1G}C_1H = \frac{\frac{4r_1r_2}{\sqrt{3r_1^2+3r_2^2+10r_1r_2}}}{\frac{3(r_1+r_2)^2}{\sqrt{3r_1^2+3r_2^2+10r_1r_2}}}\frac{3}{2}(r_1 + r_2) = 2t.$$

This thing means that (CE) is Archimedean.
On the other hand, we easily see that the circumcenter of circle $(A_1B_1C_1)$ lies on the circle (O), note that two circles (O) and $(A_1B_1C_1)$ have the same radius r and O belongs to $(A_1B_1C_1)$, they are symmetric across the line A_1B_1. Hence, under this symmetry A_1B_1, C goes into the point F. It follows $EF = CE = 2t$. This means that (EF) is also Archimedean. □

Theorem 2.2. *Consider the equilateral triangle $A_2B_2C_2$ such that the vertex C_2 lies on the ray CD and sides C_2A_2 and C_2B_2 are tangent to the Midway semi-circle (O') at A_2 and B_2, respectively. Line A_2B_2 meets the circle $A(O_1)$ at K_1 and L_1, and the circle $B(O_2)$ at K_2 and L_2. The tangent lines at K_1 and L_1 of $A(O_1)$ meet at T_1, and tangent lines at K_2 and L_2 of $B(O_2)$ meet at T_2. Then the incircles of triangles $K_1L_1T_1$ and $K_2L_2T_2$ are Archimedean.*

Proof. (See figure 3). Let K, L be the projections from C, O onto A_2B_2.
Since $\overrightarrow{CA} = 2\overrightarrow{CO_1}$ and $\overrightarrow{CB} = 2\overrightarrow{CO_2}$, the homothety $\mathcal{H}_C^2$ with center C, ratio 2 transforms (O') into (O).
Under this transformation $\mathcal{H}_C^2$, the points A_2, B_2, C_2, K go into the points A_1, B_1, C_1, E. It follows that K is the mid-point of CE and A_1B_1 is parallel to A_2B_2. Note that OL is perpendicular to A_2B_2, OH is perpendicular to A_1B_1. It follows that OL and OH are coincident or L belongs to OH.

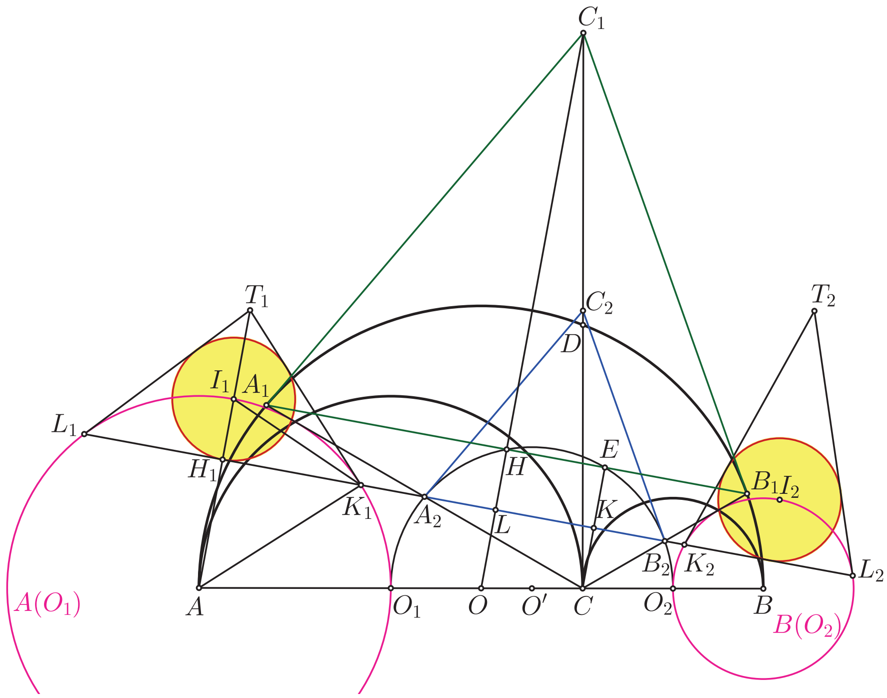

FIGURE 3

Using the theorem 2.1, we have $CE = 2t$ and $OH = \frac{r}{2}$.
It follows $HL = KE = CK = t$. Hence

$$OL = OH - HL = \frac{r}{2} - t = \frac{r_1 + r_2}{2} - \frac{r_1 r_2}{r_1 + r_2} = \frac{r_1^2 + r_2^2}{2(r_1 + r_2)}.$$

Segment AT_1 meets the circle $A(O_1)$ and the chord K_1L_1 at I_1 and H_1, respectively. Then

$$\begin{aligned}\angle I_1K_1T_1 &= 90^\circ - \angle AK_1I_1 = 90^\circ - \frac{1}{2}(180^\circ - \angle I_1AK_1)\\ &= \frac{1}{2}\angle I_1AK_1 = \frac{1}{2}\angle H_1AK_1 = \frac{1}{2}\angle H_1K_1T_1.\end{aligned}$$

It follows that I_1K_1 is the interior bisector of angle $T_1K_1L_1$. Since I_1 lies on the bisector of angle $K_1T_1L_1$, I_1 is the incenter of triangle $K_1L_1T_1$. Furthermore, the inradius of triangle $K_1L_1T_1$ is $I_1H_1 = AI_1 - AH_1 = r_1 - AH_1$.
Without loss of generality, suppose that $r_1 \geq r_2$ then $OC = r_1 - r_2$. Note that three lines AH_1, OL and CK are pairwise parallel, it follows

$$\begin{aligned}\frac{r_1^2 + r_2^2}{2(r_1 + r_2)} = OL &= \frac{OA}{AC}CK + \frac{OC}{CA}AH_1 = \frac{r}{2r_1}t + \frac{r_1 - r_2}{2r_1}AH_1\\ &= \frac{r_2}{2} + \frac{r_1 - r_2}{2r_1}AH_1.\end{aligned}$$

It follow

$$(r_1 - r_2)\left(\frac{AH_1}{r_1} - \frac{r_1}{r_1 + r_2}\right) = 0.$$

• If $r_1 = r_2$ then $r_1 = r_2 = 2t$ and $O \equiv C$. We easily see that A_1B_1 is parallel to AB and hence, $AH_1 = OL = OC = t$. It follows $I_1H_1 = r_1 - AH_1 = t$. Conversely, if $r_1 > r_2$ then

$$\frac{AH_1}{r_1} - \frac{r_1}{r_1 + r_2} = 0 \Longrightarrow AH_1 = \frac{r_1^2}{r_1 + r_2}.$$

And hence $I_1H_1 = r_1 - AH_1 = r_1 - \frac{r_1^2}{r_1+r_2} = t$.
This thing means that, we always have $I_1H_1 = t$ in all of cases. It follows that the incircle of triangle $K_1L_1T_1$ is Archimedean.
Similarly, the incircle of triangle $K_2L_2T_2$ is also Archimedean. □

Theorem 2.3. *Consider the equilateral triangle $O_1O_2O_3$ lying inside the semi-circle (O). Construct two lines O_3M_1 and O_3N_1 that they are tangent to (O_1) at M_1 and N_1, and two lines O_3M_2 and O_3N_2 are all tangent to (O_2) at M_2 and N_2. Then the incircles of triangles $O_3M_1N_1$ and $O_3M_2N_2$ are Archimedean.*

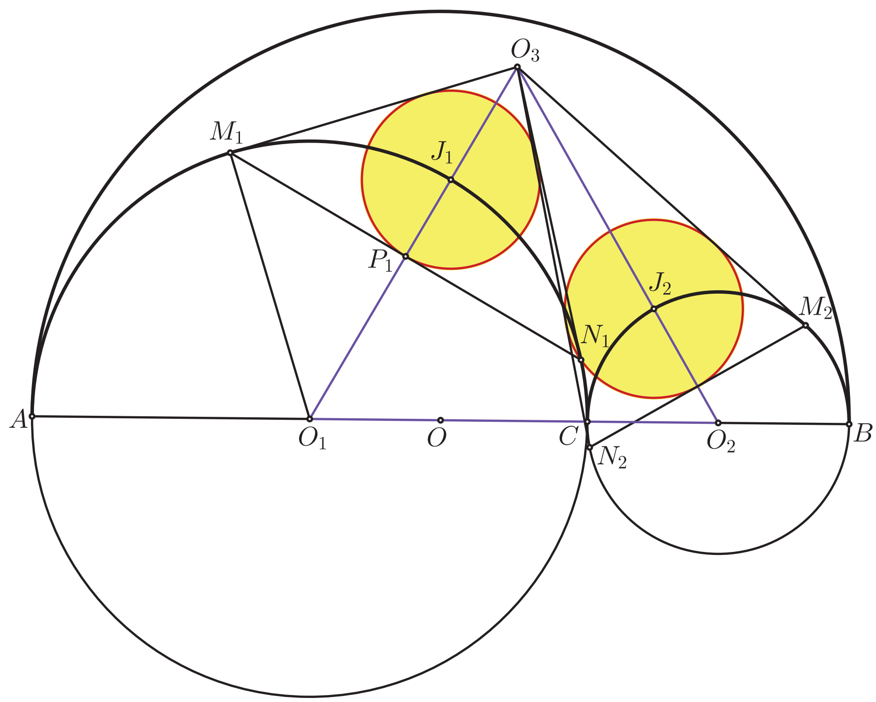

FIGURE 4

Proof. (See figure 4). Segment O_1O_3 meets the circle (O_1) and the chord M_1N_1 at J_1 and P_1, respectively. Then, we easily have that J_1 and J_1P_1 are the incenter an inradius of triangle $O_3M_1N_1$, respectively.

Since the triangle $M_1O_1O_3$ is right at M_1, M_1P_1 is perpendicular to O_1O_3 and note that $O_1O_3 = O_1O_2 = r_1 + r_2$, we get

$$O_1M_1^2 = O_1P_1.O_1O_3 \Longrightarrow O_1P_1 = \frac{O_1M_1^2}{O_1O_3} = \frac{r_1^2}{r_1 + r_2}.$$

Therefore,

$$J_1P_1 = OJ_1 - OP_1 = r_1 - \frac{r_1^2}{r_1 + r_2} = t.$$

This thing proves that the incircle of triangle $O_3M_1N_1$ is Archimedean.
Similarly, the incircle of triangle $O_3M_2N_2$ is also Archimedean. □

We known the homothety $\mathcal{H}_C^2$ transforms (O') into (O). From our exploitation, we obtain the following result.

Theorem 2.4. *Consider the equilateral triangle ABC' lying on the same side for AB as arbelos. Lines $C'Y_1$ and $C'Z_1$ are tangent to $A(C)$ at Y_1 and Z_1; lines $C'Y_2$ and $C'Z_2$ are tangent to $B(C)$ at Y_2 and Z_2. Segment AC' meets the circle $A(C)$ at G_1, and Y_1Z_1 at D_1; segment BC' meets $B(C)$ at G_2, and Y_2Z_2 at D_2. Let E_1, F_1 be the projections from G_1 onto $C'Y_1, C'Z_1$ respectively, and E_2, F_2 be the projections from G_2 onto $C'Y_2, C'Z_2$ respectively. Let Q_i, R_i $(i = 1, 2)$ be the mid-points of $C'Y_i, C'Z_i$. Then*
(a) The circles (G_iE_i), (G_iF_i) and (G_iD_i) are Archimedean.
(b) The incircles of triangles $C'Q_iR_i$, $Y_iD_iQ_i$, $Z_iD_iR_i$ and $D_iQ_iR_i$ are Archimedean.

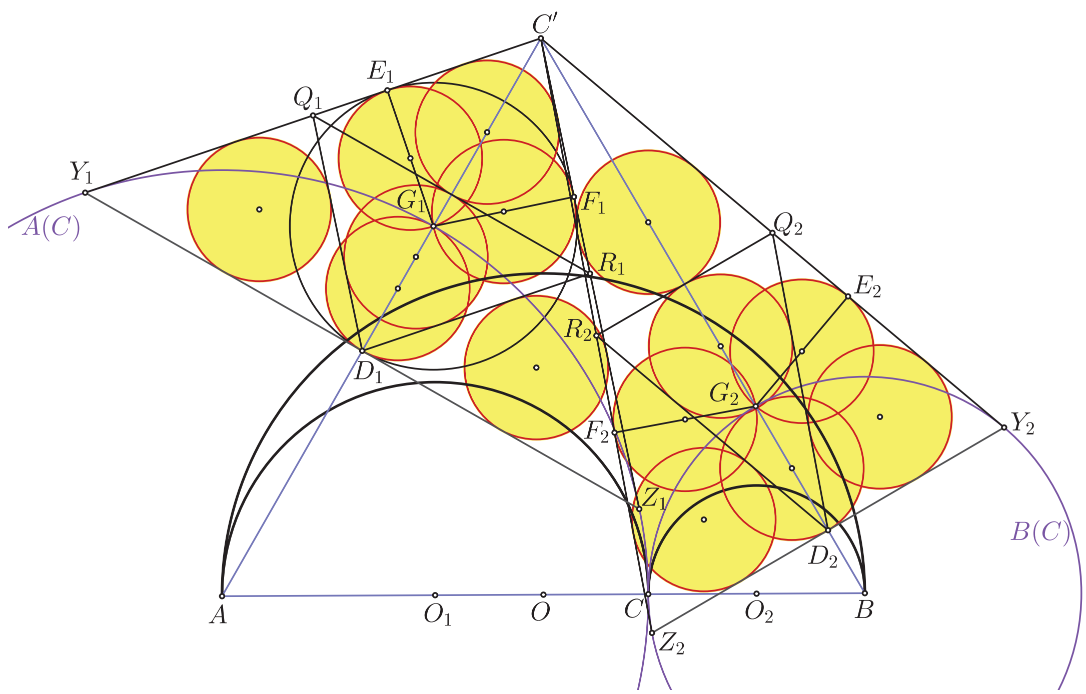

FIGURE 5

Proof. (See figure 4 and figure 5). Under the homothety $\mathcal{H}_C^2$, points O_1, O_2, O_3, M_1, N_1 go into A, B, C', Y_1, Z_1, respectively. Hence, the incircle of triangle $O_3M_1N_1$ goes into the incircle of triangle $C'Y_1Z_1$ under the homothety $\mathcal{H}_C^2$. Since the theorem 2.3, the inradius of triangle $O_3M_1N_1$ is equal to t. It follows that the inradius of triangle $C'Y_1Z_1$ is equal to $2t$.
We easily see that G_1, G_1D_1 be incentre and inradius of triangle $C'Y_1Z_1$, respectively. It follows $G_1D_1 = GE_1 = GF_1 = 2t$ and hence (G_1D_1), (G_1E_1) and (G_1F_1) are Archimedean. Similarly, (G_2D_2), (G_2E_2) and (G_2F_2) are also Archimedean. The part (a) is proved.
We easily see that pair of triangles $C'Q_1R_1$, $Q_1Y_1D_1$, $R_1D_1Z_1$ and $D_1Q_1R_1$ are equal to one another and they are all similar to triangle $C'Y_1Z_1$ with the ratio $\frac{1}{2}$. Hence, the inradii of these triangles are equal to one another and equal to the half of the inradius of triangle $C'Y_1Z_1$, i.e, equal to t. It follows that the incircles of triangles $C'Q_1R_1$, $Q_1Y_1D_1$, $R_1D_1Z_1$ and $D_1Q_1R_1$ are Archimedean. Similarly for incircles of triangles $C'Q_2R_2$, $Q_2Y_2D_2$, $R_2D_2Z_2$ and $D_2Q_2R_2$. The part (b) is proved. □

Theorem 2.5. *Consider two equilateral triangles AU_1V_1 and BU_2V_2 such that two vertices U_1 and U_2 lie on the semi-circle (O), and the vertices V_1 and V_2 lie on the circles (O_1) and (O_2), respectively. Then*
(a) There exists a point P lying on AU_2 and a point Q lying on BU_1 such that triangle CPQ is equilateral with side PQ is parallel to AB and circles (CP), (PQ), (QC) are Archimedean.
(b) There exits a point R lying on AU_2 and a point S lying on BU_1 such that triangle CRS is equilateral with side RS is parallel to U_1U_2 and circles (CR), (RS), (SC) are Archimedean.

Proof. (See figure 6). Let ACA' and BCB' are the equilateral triangles on the same side AB containing the semicircles (O).
It is easy to see that the triangles AV_1C and AU_1A' are congruent by side-angle-side. Hence, $\angle AU_1A' = AV_1C = 90°$. It follows that the points B, U_1, A' are on a line perpendicular to the segment AU_1 at U_1. Similarly, A, U_2, B' are on a line perpendicular to the segment BU_2 at U_2.
(a) Let $P := CA' \cap AU_2$, and $Q := CB' \cap BU_1$. We show that CPQ is an equilateral triangle with PQ is parallel to AB and $PQ = 2t$.
Indeed, from CA' and BB' are parallel, we get CP being parallel to BB'. By Thales' theorem,

$$\frac{CP}{2r_2} = \frac{CP}{BB'} = \frac{AC}{AB} = \frac{2r_1}{2r_1 + 2r_2} = \frac{r_1}{r_1 + r_2}.$$

It follows that $CP = 2t$. Similarly, $CQ = 2t$. Hence, $CP = CQ = 2t$. Note that

$$\angle PCQ = 180° - \angle ACA' - \angle BCB' = 180° - 60° - 60° = 60°.$$

This means that CPQ is an equilateral triangle. We deduce that $\angle CPQ = 60° = \angle ACP$. This implies that PQ is parallel to AB and $PQ = CP = 2t$.
Conclusion, the equilateral triangle CPQ has a side parallel to AB, and its sides are the diameters of circles are congruent to the Archimedean twin circles.
(b) Let $M := AU_2 \cap BU_1$; R, S are the reflective points of P, Q across the line CM, respectively. We show that CRS is an equilateral triangle with RS is parallel to U_1U_2 and $RS = 2t$.

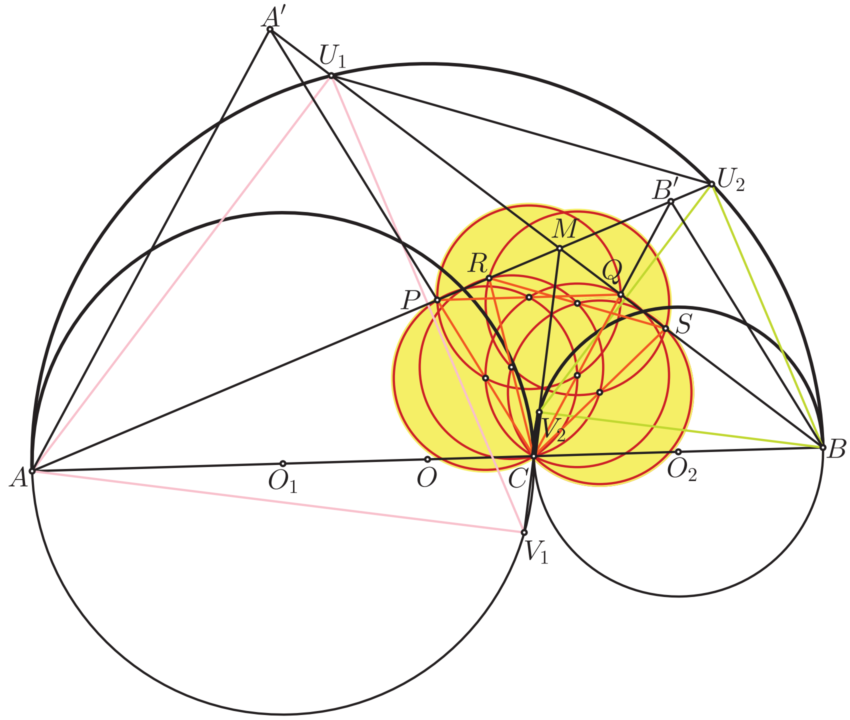

FIGURE 6

Indeed, it is easy to see that the triangles ACB' and $A'CB$ are congruent by side-angle-side. Chassing angles, we have

$$\angle MB'C = \angle AB'C = \angle A'BC = \angle MBC.$$

It follows that $BCMB'$ is the cyclic quadrilateral. Similarly, $ACMA'$ is also the cyclic quadrilateral. We deduce that $\angle CMA = \angle CA'A = \angle 60^\circ = \angle CB'B = \angle CMB$. This means that MC is the interior angle bisector of triangle ABM. By the symmetry, R belongs to AM, S belongs to BM, and CRS is the equilateral triangle with $RS = CR = CS = 2t$.
Note that PQ and AB are parallel (by part (a)), and ABU_2U_1 is the cyclic quadrilateral. Again, by the symmetry,

$$\angle MRS = \angle MQP = \angle MBA = \angle MU_2U_1.$$

This means that RS is parallel to U_1U_2.
Conclusion, the equilateral triangle CRS has a side parallel to U_1U_2, and its sides are the diameters of circles which are congruent to the Archimedean twin circles. □

Remark 2.6. When C is the midpoint AB then the triangles CPQ and CRS are coincident.

2.2. **An equilateral triangle is derived from Yiu's circle in an arbelos.**

Theorem 2.7. *The circle $O(Y)$ meets the segments CA and CB at A_0 and B_0 respectively. Let S_i $(i = 1, 2)$ be the points on $O(Y)$ and the same side of AB as the arbelos such that W_i is the* *foot of perpendicular from S_i to AB and line S_iW_i tangent to the Yiu's circle (Y), and the points A_0, S_1, S_2 and B_0 lie on $O(Y)$ in this order. Then the circles $A_0(S_1)$ and $B_0(S_2)$ intersect, and if N is one of the points of intersection, the triangle NW_1W_2 is equilateral.*

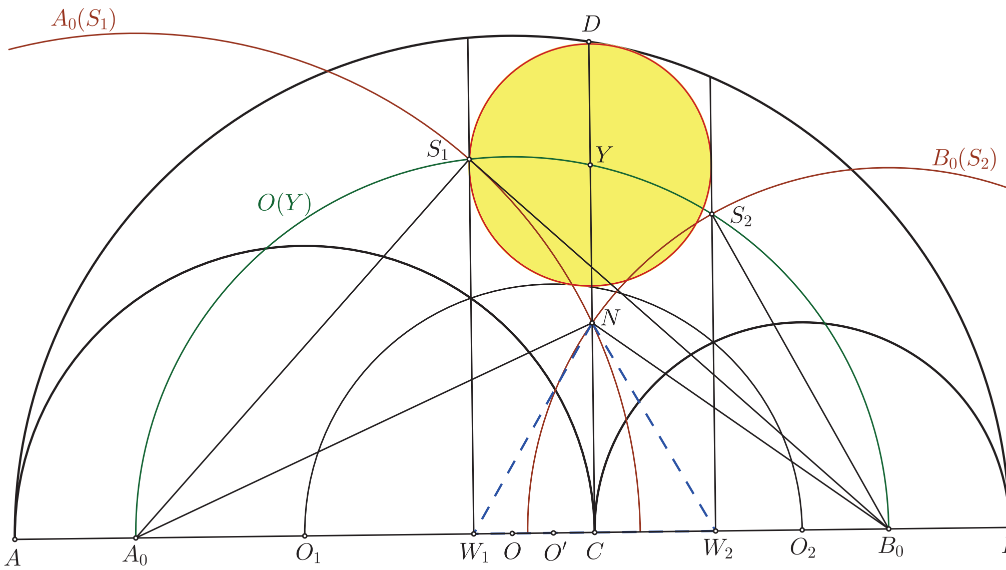

FIGURE 7

Proof. (See figure 7). To find the length of A_0S_1, note that W_1 is the foot of perpendicular from S_1 on the hypotenuse A_0B_0 of the right triangle $A_0S_1B_0$. We have $CW_1 = CW_2 = A_0A = BB_0 = t = \frac{r_1r_2}{r}$ by Proposition 1.1. It follows that

$$A_0W_1 = AC - AA_0 - CW_1 = 2r_1 - \frac{2r_1r_2}{r} = \frac{2r_1^2}{r},$$

and

$$A_0B_0 = AB - AA_0 - BB_0 = 2r - \frac{2r_1r_2}{r} = \frac{2(r_1^2 + r_2^2 + r_1r_2)}{r}.$$

Therefore,

$$S_1A_0^2 = A_0W_1.A_0B_0 = \frac{4r_1^2(r_1^2 + r_2^2 + r_1r_2)}{r^2}.$$

The same calculation shows that

$$S_2B_0^2 = \frac{4r_2^2(r_1^2 + r_2^2 + r_1r_2)}{r^2}.$$

Hence,

$$\begin{aligned}A_0S_1 + B_0S_2 = 2\sqrt{r_1^2 + r_2^2 + r_1r_2} &= 2\frac{r_1^2 + r_2^2 + r_1r_2}{\sqrt{r_1^2 + r_2^2 + r_1r_2}}\\ &> \frac{2(r_1^2 + r_2^2 + r_1r_2)}{\sqrt{r_1^2 + r_2^2 + 2r_1r_2}}\\ &= \frac{2(r_1^2 + r_2^2 + r_1r_2)}{\sqrt{(r_1 + r_2)^2}}\\ &= \frac{2(r_1^2 + r_2^2 + r_1r_2)}{r} = A_0B_0,\end{aligned}$$

and

$$\begin{aligned}|A_0S_1 - B_0S_2| &= \frac{2\,|r_1 - r_2|\sqrt{r_1^2 + r_2^2 + r_1r_2}}{r} = \frac{2\sqrt{r_1^2 + r_2^2 + r_1r_2}}{r}\sqrt{(r_1 - r_2)^2}\\ &= \frac{2\sqrt{r_1^2 + r_2^2 + r_1r_2}}{r}\sqrt{r_1^2 + r_2^2 - 2r_1r_2}\\ &< \frac{2\sqrt{r_1^2 + r_2^2 + r_1r_2}}{r}\sqrt{r_1^2 + r_2^2 + r_1r_2} = A_0B_0.\end{aligned}$$

This thing means that the segments A_0S_1, A_0B_0 and B_0S_2 are the sides of a triangle. We deduce that the circles $A_0(S_1)$ and $B_0(S_2)$ intersect. Let N be one of points of intersection.

Since $CA_0 = AC - AA_0 = 2r_1 - \frac{r_1r_2}{r} = \frac{2r_1^2 + r_1r_2}{r}$, and $CB_0 = \frac{2r_2^2 + r_1r_2}{r}$. It follows that

$$CA_0^2 - CB_0^2 = \frac{4(r_1^2 - r_2^2)(r_1^2 + r_2^2 + r_1r_2)}{r^2}.$$

On the other hand,

$$NA_0^2 - NB_0^2 = S_1A_0^2 - S_2B_0^2 = \frac{4(r_1^2 - r_2^2)(r_1^2 + r_2^2 + r_1r_2)}{r^2}.$$

We deduce that $CA_0^2 - CB_0^2 = NA_0^2 - NB_0^2$. Hence, CN perpendicular to A_0B_0 and point N is on CD.
We have

$$\begin{aligned}NC^2 = A_0N^2 - A_0C^2 = A_0S_1^2 - A_0C^2 &= \frac{4r_1^2(r_1^2 + r_2^2 + r_1r_2)}{r^2} - \left(\frac{2r_1^2 + r_1r_2}{r}\right)^2\\ &= \frac{3r_1^2r_2^2}{r^2} = 3t^2,\end{aligned}$$

and

$$W_1W_2 = CW_1 + CW_2 = 2\frac{r_1r_2}{r} = \frac{2r_1r_2}{r} = 2t.$$

It follows that $NC = \frac{\sqrt{3}}{2}W_1W_2$. Therefore the triangle NW_1W_2 is equilateral. □

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