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ON COMPLETELY SOLVABLE LIE FOLIATIONS

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ABSTRACT. In this paper we generalize the Haefliger theorem on completly solvable Lie foliations. We prove that every completely solvable Lie foliation on a compact manifold is the inverse image of a homogeneous foliation. Every manifold in this paper is compact and our Lie group G is connected and simply connected.

1. INTRODUCTION

The foliation theory is a branch of geometry which has risen in the second half of the XX-th century on a joint of ordinary differential equations and the differential topology [18]. Basic works on the foliation theory belong to G. Ehresmann [5], A. Haefliger [8], G. Lamoureux [15], R. Langevin [16], G. Reeb [19] and H. Rosenberg [21]. Important contribution to foliation theory was made by R. Herman [11, 12], T. Inaba [13], Ph. Tondeur [22].
Lie foliations have been studied by several authors (cf. [1, 2, 3, 4, 6, 10, 14]. To each Lie foliation are associated two Lie algebras, the Lie algebra $\mathcal{G}$ of the Lie group on which the foliation is modeled and the structural Lie algebra $\mathcal{H}$. The latter algebra is the Lie algebra of the Lie foliation $\mathcal{F}$ restricted to the closure of any one of its leaves. In particular, it is a subalgebra of $\mathcal{G}$. We remark that although $\mathcal{H}$ is canonically associated to $\mathcal{F}$, $\mathcal{G}$ is not. Thus two problems are naturally posed: the realization problem and the change problem. For more details on the realization and change problem, we refer to [10] and references therein.
An important class of Lie foliations is the so-called homogeneous Lie foliation, obtained in the following ways: G and H two connected, simply connected, Lie groups, Γ a cocompact discrete subgroup of H, and φ a surjective morphism of Lie groups of H in G; the classes on the left of H modulo $\ker\varphi$ reprojected on H/Γ are the leaves of a G-foliation of H/Γ whose holonomy morphism is the restriction of φ to Γ and the developing application is φ. Given a $\mathcal{F}-$ Lie foliation, in which condition $\mathcal{F}$ is inverse image of a homogeneous Lie foliation ? Since the holonomy group of a Lie G-foliation reflects the transverse structure, this question is closely related to the following: Given a Lie group G, which subgroups are feasible as holonomy group of a G-foliation. When G is nilpotent and simply connected, Haefliger [9] showed that any Γ subgroup of finite type is a holonomy group of a homogeneous G-foliation on a compact manifold. In particular, nilpotent Lie foliations on compact manifolds are all, inverse image of homogeneous Lie foliations. In the case where the Lie group is solvable, connected and simply connected (not necessarily nilpotent), Meigniez [17] showed that there are Lie foliations which are not inverse image of a homogeneous Lie foliations.

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2. Preliminaries

A codimension n foliation $\mathcal{F}$ on a $(n+m)$-manifold M is given by an open cover $\{U_i\}_{i\in I}$ and submersions $f_i : U_i \longrightarrow T$ over an n-dimensional manifold T and, for $U_i \cap U_j \neq \emptyset$, a diffeomorphism $\gamma_{ij} : f_i(U_i \cap U_j) \longrightarrow f_j(U_i \cap U_j)$ such that $f_j = \gamma_{ij} \circ f_i$. We say that $\{U_i, f_i, T, \gamma_{ij}\}$ is a foliated cocycle defining $\mathcal{F}$.

A transverse structure to $\mathcal{F}$ is a geometric structure on T invariant by the local diffeomorphisms γ_{ij}. We say that $\mathcal{F}$ is a Lie G-foliation, if T is a Lie group G and γ_{ij} are restrictions of left translations on G.
Such foliation can also be defined by a 1-form ω on M with values in the Lie algebra $\mathcal{G}$ such that:

(1) $\omega_x : T_x M \longrightarrow \mathcal{G}$ is surjective for every $x \in M$
(2) $d\omega + \frac{1}{2}[\omega, \omega] = 0$.

We shall use the following structure theorem:

Theorem 2.1. [6] *Let G be the connected and simply connected Lie group with Lie algebra $\mathcal{G}$. Then $\mathcal{F}$ is a Lie $\mathcal{G}$-foliation on a compact manifold M if and only if there exist a homomorphism $h : \pi_1(M) \longrightarrow G$ and a covering $p : \widetilde{M} \to M$ such that:*

(1) *There exist a locally trivial fibration $D : \widetilde{M} \longrightarrow G$ equivariant under the action of $\pi_1(M)$, where* $\mathrm{Aut}(p) \cong \mathrm{Im}h$.
(2) *The fibres of D are the leaves of the lifted foliation $\widetilde{\mathcal{F}} = p^*\mathcal{F}$ of $\mathcal{F}$.*

Condition (1) means that if $g_\gamma = h([\gamma])$ and $\widehat{\gamma}$ is the corresponding element to $[\gamma]$ in $\mathrm{Aut}(p)$, then

$$D(\widehat{\gamma}(x)) = g_\gamma \cdot D(x), \quad \forall x \in \widehat{M}.$$

The subgroup $\Gamma = \mathrm{Im}h$ is called the holonomy group of the foliation $\mathcal{F}$. The fibration $D : \widetilde{M} \longrightarrow G$ is called the developing map of $\mathcal{F}$. For a Lie $\mathcal{G}$-foliation the structural Lie algebra $\mathcal{H}$ is always a subalgebra of $\mathcal{G}$.
The topology of a Lie foliation of a compact manifold is trivial since the leaves are all diffeomorphic, see [7], that is why the theory of Lie foliation has developed mainly in the sense of classification.

3. Completely solvable Lie groups

Definition 3.1. A solvable Lie group G of Lie algebra $\mathcal{G}$ is said to be completely solvable if all the adjoint linear operators $ad_X, (X \in \mathcal{G})$ has only real eigenvalues.

In particular, any nilpotent Lie group is completely solvable.

Definition 3.2. [17] A Lie group G is said to be polycyclic if it has a sequence of cyclic quotient composition. If G is solvable and is of finite type with $\gamma_1, \ldots, \gamma_n$ as the generating part, then G is polycyclic if and only if the roots of γ_i are algebraic units. The roots of an element of G are the eigenvalues of its adjoint. An algebraic unit is a non-zero complex number that is an algebraic integer over $\mathbb{Z}$ as well as its inverse.

Proposition 3.3. [17] Let G be a solvable Lie group and Γ a subgroup of G of finite type and uniform. If Γ is polycyclic and dense then Γ is virtually holonomy group of homogenius Lie foliation

From the previous proposition we have the following result.

Theorem 3.4. *Let G be a completely solvable Lie group and Γ a subgroup of G of finite type and uniform. Then Γ is polycyclic and uniform if and only if Γ is virtually a holonomy group of a homogeneous G-foliation.*

We will use the following Lemma for the proof of the Theorem 3.4.

Lemma 3.5. Let G_1 and G_2 be two simply connected Lie groups and completely solvable. Let Γ be a discrete uniform subgroup in G_1. Then every continuous homomorphism: $\rho : \Gamma \longrightarrow G_2$ extends in a single way into a continuous homomorphism: $\widetilde{\rho} : G_1 \longrightarrow G_2$.

Proof. (Proof of Theorem 3.4.) Let G be a completely solvable Lie group and Γ a subgroup of G. Suppose that Γ is of finite type and uniform. Every completely solvable Lie group is solvable Lie group. Now we can use the Meigniez result, and so Γ is virtually a holonomy group of a homogeneous G-foliation.
Conversely, assuming Γ is polycyclic and uniform. Let $\varphi : \Gamma_H \longrightarrow \Gamma' \subset G$ where Γ' is a subgroup of Γ of finite index and Γ_H a cocompact lattice of completely solvable Lie group H.
Since Γ_H is faithful to H then ϕ extends to a homomorphism $\widetilde{\varphi} : H \longrightarrow \Gamma'$. And since G is completely solvable and Γ' is uniform, the continuous homomorphism: $\widetilde{\varphi}$ extends in a single way into a continuous homomorphism: $\widetilde{\widetilde{\varphi}} : H \longrightarrow G$.
Then $\ker\widetilde{\widetilde{\varphi}}$ is a homogeneous Lie G-foliation on the manifold $H_{/\Gamma_H}$. And its holonomy group is $\widetilde{\widetilde{\varphi}}(\Gamma_H) = \Gamma'$ □

Theorem 3.6. [17] *Let G be a solvable Lie group and Γ a subgroup of G of finite type. If Γ contains a subgroup that is polycyclic and uniform at the same time, then Γ is a holonomy group of G-foliation.*

Next, we generalize the Haefliger Theorem and the Theorem 3.6

Theorem 3.7. *Let G be simply connected and completely solvable Lie group, $\mathcal{F}$ a Lie G-foliation on a compact manifold M and Γ its holonomy group. If Γ is discrete then $\mathcal{F}$ is inverse image of homogeneous Lie foliation.*

We will use this Lemma for the proof of Theorem 3.7.

Lemma 3.8. Let G be a simply connected and completely solvable Lie group and Γ a closed subgroup of G. Then there exists a unique subgroup of Lie K closed and connected of G such that $\Gamma \in K$ and $K/_{\Gamma}$ is compact.

Proof. We will prove the existence by proceeding to recurrence on the dimension of G. If $\dim G = 0$ then $G = \{e\}$ and the Lemma is true. Suppose it's true for any simply connected and completely solvable Lie group of dimension lower than $\dim G$ (hypothesis of recurrence).
Let G_0 be a normal and closed subgroup of G of codimension 1. If $\Gamma \subset G_0$, then using the hypothesis of recurrence, the lemma is true. Suppose now $\Gamma \not\subseteq G_0$ and let $\Gamma_0 = \Gamma \cap G_0$ (Γ_0 is a closed subgroup of G_0). Finally suppose that $\tilde{G} = G/G_0 \cong \mathbb{R}$ and $\pi : G \longrightarrow \tilde{G}$ the canonical projection. Then we have two cases:

(1) Suppose that Γ_0 is trivial, that is, $\Gamma \cap G_0 = \{e\}$. The homomorphism $\tilde{\pi} : \Gamma \longrightarrow \tilde{\Gamma}$ is an isomorphism and $\tilde{\Gamma} \subset \tilde{G} \cong \mathbb{R}$ is abelian. Then Γ is abelian and we have the result.
(2) Suppose now Γ_0 is not trivial. From the hypothesis of recurrence there exist L_0 a unique closed connected Lie subgroup of G_0 such that $\Gamma \subset L_0$ and $L_0/_{\Gamma_0}$ is compact.
Since $L_0/_{\Gamma_0}$ is compact, there exists a compact set C of G include in G_0 such that: $L_0 = C\Gamma_0$ (1)

If we suppose L_0 is normal in G then $L_0\Gamma$ is subgroup of G and since (1) we have $L_0\Gamma = C\Gamma_0\Gamma = C\Gamma$, so $L_0\Gamma$ is close in G.
Let $G' = G/L_0$ and $\pi : G \longrightarrow G'$ the canonical surjection. Then $\Gamma' = \pi(L_0\Gamma)$ is a closed subgroup of G' and since the hypothesis of recurrence there exists a unique closed and connected subgroup K' of G' such that: $\Gamma' \subset K'$ and K'/Γ is compact. In particular there exist a compact C' of G' such that: $K' = C''\Gamma'$ (2) where $C'' = \pi(C')$. Let $K = \pi^{-1}(K')$, then K is closed and connected subgroup of G containing L_0. Now since (1) and (2) we have $K = C'C\Gamma$ and then K/Γ is compact. We also know that K/L_0 and L_0 are connected so K is connected and we have the result.
If L_0 is not normal in G then we consider the normalisator $N_G(L_0)$ of L_0 in G. Since G_0 is normal in G there exist $\gamma \in \Gamma$ such that $\gamma\Gamma_0\gamma^{-1} \subset G_0 \cap \Gamma = \Gamma_0$ and so Γ_0 is normal in Γ.
We also have for every $\gamma \in \Gamma$, $\gamma L_0 \gamma^{-1}$ is a closed and connected subgroup of G_0.
And $\gamma L_0\gamma^{-1}\Gamma_0$ is compact, so using the uniqueness of L_0, we have $\gamma L_0 \gamma^{-1} = L_0$ and $\Gamma \subset N_G(L_0)$.

We have the result using the hypothesis of recurrence.
Uniqueness:
Let K_1 and K_2 be two closed and connected subgroup of G such that K_1/Γ and K_2/Γ are compacts. W e know that $K_i/K_1 \cap K_2$ (with $i = 1, 2$) is compact. G is simply connexe and solvable then $K_i/K_1 \cap K_2$ is diffeomorphic to $\mathbb{R}^d$.
$\dim \mathbb{R}^d = d = \dim K_i - \dim K_1 \cap K_2$. And $d = 0$ imply $\dim K_i = \dim K_1 \cap K_2$ and now $K_i = K_1 \cap K_2$ so $K_1 = K_2$. □

Proof. (Proof the Theorem 3.7.) Let $\mathcal{F}$ be Lie G-foliation on a compact manifold M and Γ its holonomy group.
Suppose that Γ a discret subgroup of G. The holonomy group Γ of such foliation $\mathcal{F}$ is a uniform subgroup of the finite type of the completely solvable Lie group, simply connected G defining the transverse structure of $\mathcal{F}$.
Using the previous lemma there exists a unique subgroup of Lie K closed and connected of G such that $\Gamma \in K$ and $K/_\Gamma$ is compact. So Γ is a lattice in K.
The canonical injection $i : \Gamma \longrightarrow G$ extends in a single way into a surjective homomorphism such that the following diagram is commutative

$$\begin{array}{ccc} \Gamma & \xrightarrow{j} & K \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle h} \\ & & G \end{array}$$

$$i = h \circ j$$

We have the exact short sequence $1 \longrightarrow F \longrightarrow K \longrightarrow G \longrightarrow 1$ which define a homogeneous and classifying Lie foliation $\mathcal{F}'$ on the manifold $K_{/\Gamma}$.
The holonomy group of $\mathcal{F}'$ is $h(\Gamma) = h \circ j(\Gamma) \approx \Gamma$. Then $\mathcal{F}$ is inverse image of $\mathcal{F}'$ [9]. □

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