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GEODESIC VECTORS OF RANDERS METRICS ON NILPOTENT LIE GROUPS OF DIMENSION FIVE

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ABSTRACT. In this paper we study the geometry of simply connected two-step nilpotent Lie groups of dimension five equipped with a left invariant Randers metric. We first study geodesic vectors and investigates the set of all homogeneous geodesics.Then we obtain some conditions for left invariant Randers metric to be of Douglas type.

1. INTRODUCTION

A connected Riemannian manifold which admits a transitive nilpotent Lie group N of isometries is called a nilmanifold [10, 12]. E. Wilson showed that for a given homogeneous nilmanifold M, there exists a unique nilpotent Lie subgroup N of $I(M)$ acting simply transitively on M, and N is normal in $I(M)$ (see [11]). In [6] J. Lauret classified, up to isometry, all homogeneous nilmanifolds of dimension 3 and 4 (not necessarily two-step nilpotent) and computed the corresponding isometry groups. He also studied, as example, the structure of specific 5-dimensional two-step nilmanifolds with 2-dimensional center. His results will be used in the present paper. Our purpose is to classify all simply connected two-step Riemannian nilmanifolds of dimension 5 and to determine their full isometry groups.

In this paper we study the geometry of simply connected two-step nilpotent Lie groups of dimension five endowed with left invariant Randers metrics. Homogeneous geodesics have important applications to mechanics. For example, the equation of motion of many systems of classical mechanics reduces to the geodesic equation in an appropriate Riemannian manifold M. In this paper, we consider homogeneous geodesics in a homogeneous Randers space on simply connected two-step nilpotent Lie groups of dimension five. Then we obtain some conditions for left invariant Randers metric to be of Douglas type.

From now we consider that N is a simply connected two-step nilpotent Lie group of dimension five and $\mathfrak{n}$ is its Lie algebra.

2. PRELIMINARIES

In this section, we recall briefly some known facts about Finsler spaces. For details, see [2]. Let M be a n-dimensional C^∞ manifold and $TM = \cup_{x \in M} T_x M$ the tangent bundle.

A Finsler metric on a manifold M is a non-negative function $F : TM \longrightarrow \mathbb{R}$ with the following properties:

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(1) F is smooth on the slit tangent bundle $TM^0 := TM \setminus 0$,
(2) $F(x, \lambda Y) = \lambda F(x, Y)$ for any $x \in M$, $Y \in T_xM$ and $\lambda > 0$,
(3) the $n \times n$ Hessian matrix $[g_{ij}] = [\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}]$ is positive definite at every point $(x, Y) \in TM^0$.
The following bilinear symmetric form $g_y : T_xM \times T_xM \longrightarrow \mathbb{R}$ is positive definite :

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(x, y + us + vt)]|_{s=t=0}.$$

By the homogeneity of F, we have

$$g_y(u, v) = g_{ij}(x, y)u^i v^j \qquad , \qquad F = \sqrt{g_{ij}(x, y)u^i v^j}.$$

In 1941, G. Randers [5] studied a very interesting class of Finsler manifolds. Let M be an n-dimensional manifold. A Randers metric is a Finsler structure F on TM that has the form

$$F(x, y) := \alpha(x, y) + \beta(x, y),$$

where

$$\alpha(x, y) := \sqrt{\tilde{a}_{ij} y^i y^j}, \qquad \beta(x, y) := \tilde{b}_i(x) y^i.$$

The $\tilde{a}_{ij}$ are the components of a Riemannian metric and the $\tilde{b}_i$ are those of a 1-form. Due to the presence of the β term, Randers metrics do not satisfy $F(x, -y) = F(x, y)$ when $\tilde{b} \neq 0$. In fact, the Finsler function of a Randers space is absolutely homogeneous if and only if it is Riemannian. Also, in order for F to be positive if and only if

$$\|\tilde{b}\| := \sqrt{\tilde{b}_i \tilde{b}^i} < 1, \qquad \text{where } \tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j.$$

The Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ induces the musical bijections between 1-forms and vector fields on M, namely $\flat : T_xM \longrightarrow T_x^*M$ and its inverse $\sharp : T_x^*M \longrightarrow T_xM$(see [2]). In the local coordinates we have

$$(X^\flat)_i = \tilde{a}_{ij} y^j, \qquad (w^\sharp)^i = \tilde{a}^{ij} w_j, \qquad \forall X \in T_xM \quad and \quad \forall w \in T_x^*M.$$

Thus a Randers metric F with Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ and 1-form $\tilde{b}$ can be showed by

$$F(x, y) = \sqrt{\tilde{a}_x(y, y)} + \tilde{a}_x(\tilde{b}^\sharp, y), \qquad \forall y \in T_xM.$$

where $\beta(x, y) = (\tilde{b}^\sharp)^\flat(y) = \tilde{a}_x(\tilde{b}^\sharp, y)$ and $\tilde{a}_x(\tilde{b}^\sharp, \tilde{b}^\sharp) < 1$.
Let π^*TM be the pull-back of the tangent bundle TM by $\pi : TM^0 \longrightarrow M$. Unlike the Levi-Civita connection in Riemannian geometry, there is no unique natural connection in the Finsler case. Among these connections on π^*TM, we choose the Chern connection whose coefficients are denoted by Γ^i_{jk}(see[2], p. 38). This connection is almost g-compatible and has no torsion. Since, in general, the Chern connection coefficients Γ^i_{jk} in natural coordinates have a directional dependence, we must define a fixed reference vector.
Let $\sigma(t)$ be a smooth regular curve in M, with velocity field T. Let $W(t) := W^i(t)\frac{\partial}{\partial x^i}$ be a vector field along σ. The expression

$$\Big[\frac{dW^i}{dt} + W^j T^k (\Gamma^i_{jk})_{(\sigma,T)}\Big]\frac{\partial}{\partial x^i}|_{\sigma(t)}$$

would have defined the covariant derivative $D_T W$ with reference vector T. A curve $\sigma(t)$, with velocity $T = \dot{\sigma}(t)$ is a Finslerian geodesic if

$$D_T\Big[\frac{T}{F(T)}\Big] = 0, \qquad \text{with reference vector } T,$$

that the constant speed geodesics are precisely the solution of

$$D_T T = 0, \qquad \text{with reference vector } T.$$

Since $T = \frac{d\sigma^i}{dt}\frac{\partial}{\partial x^i}$, this differential equations that describe constant speed geodesics are:

$$\frac{d^2\sigma^i}{dt^2} + \frac{d\sigma^j}{dt}\frac{d\sigma^k}{dt}(\Gamma^i_{jk})_{(\sigma,T)} = 0.$$

Let

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i).$$

Then it is easy to verify that $D^i_{jkl} dx^j \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$ is a well-defined tensor on $TM \setminus 0$. We call D^i_{jkl} the Douglas tensor. Douglas metric is characterized by the curvature equation D=0. We know that all Berwald spaces are Douglas spaces. But there are many non-Berwald Douglas metrics. For example, a Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is a closed 1-form.

3. Lie algebras with 1-dimensional center

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We may identify the tangent bundle TG with $G \times \mathfrak{g}$ by means of the diffeomorphism that sends (g, X) to $(L_g)_* X \in T_g G$.

Definition 3.1. A Finsler function $F : TG \longrightarrow R_+$ will be called G-invariant if F is constant on all G-orbits in $TG = G \times \mathfrak{g}$; that is, $F(g, X) = F(e, X)$ for all $g \in G$ and $X \in \mathfrak{g}$.

The G-invariant Finsler functions on TG may be identified with the Minkowski norms on $\mathfrak{g}$. If $F : TG \longrightarrow R_+$ is an G-invariant Finsler function, then we may define $\widetilde{F} : \mathfrak{g} \longrightarrow R_+$ by $\widetilde{F}(X) = F(e, X)$, where e denotes the identity in G. Conversely, if we are given a Minkowski norm $\widetilde{F} : \mathfrak{g} \longrightarrow R_+$, then $\widetilde{F}$ arises from an G-invariant Finsler function $F : TG \longrightarrow R_+$ given by $F(g, X) = \widetilde{F}(X)$ for all $(g, X) \in G \times \mathfrak{g}$ [3].

Definition 3.2. Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra identified with the tangent space at the identity element, $\widetilde{F} : \mathfrak{g} \longrightarrow R_+$ a Minkowski norm and F the left-invariant Finsler metric induced by $\widetilde{F}$ on G. A geodesic $\gamma : R_+ \longrightarrow G$ is said to be *homogeneous* if there is a $Z \in \mathfrak{g}$ such that $\gamma(t) = exp(tZ)\gamma(0)$, $t \in R_+$ holds. A tangent vector $X \in T_e G - \{0\}$ is said to be a *geodesic vector* if the 1-parameter subgroup $t \longrightarrow exp(tX)$, $t \in R_+$, is a geodesic of F.

The geodesic defined by a geodesic vector is obviously a homogeneous one. Conversely, let γ be a geodesic with $\gamma(0) = g$ which is homogeneous with respect to a 1-parameter group of left-translations, namely

$$\gamma(t) = exp(tY)g, \quad t \in R_+,$$

then a homogeneous geodesic $\widetilde{\gamma}$ is given by

$$\begin{aligned}\widetilde{\gamma}(t) &= L_g^{-1} \circ \gamma(t) = L_g^{-1} \circ R_g \circ exp(tY) \\ &= exp(Ad(g^{-1})tY).e = exp(Ad(g^{-1})tY)\widetilde{\gamma}(0),\end{aligned}$$

which means that $X = Ad(g^{-1})Y$ is a geodesic vector.
Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\tilde{a}$ be a left-invariant Riemannian metric on G. In [8], it is proved that a vector $X \in \mathfrak{g}$ is a geodesic vector if and only if

$$\tilde{a}(X, [X, Y]) = 0 \quad , \quad \forall Y \in \mathfrak{g}. \tag{3.1}$$

For results on homogeneous geodesics in homogeneous Finsler manifolds we refer to [4]. The basic formula characterizing geodesic vector in the Finslerian case was derived in [4], Theorem 3.1.

Theorem 3.3. *A vector $y \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if*

$$g_y(y, [y, z]) = 0 \quad , \quad \forall z \in \mathfrak{g}. \tag{3.2}$$

In the following we shall give a necessary and sufficient condition for an invariant Randers metric to be a Douglas metric on a homogeneous manifold ([1]).

Theorem 3.4. *Let $\tilde{a}$ be an invariant Riemannian metric on the Lie group G, where $\mathfrak{g}$ be the Lie algebra of G. Then there exists a bijection between the set of all invariant Randers metrics on G with the underlying Riemannian metric $\tilde{a}$ and the set*

$$V_1 = \{X \in \mathfrak{g} | \tilde{a}(X, X) < 1\} \tag{3.3}$$

The Randers metric is of the Douglas type if and only if X is orthogonal to $[\mathfrak{g}, \mathfrak{g}]$ with respect to $\tilde{a}$.

In this section we study simply connected two-step nilpotent Lie groups of dimension five equipped with left-invariant Randers metric and has $1-$dimensional center. In [9] S. Homolya and O. Kowalski showed that there exist an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{n}$ such that

$$[e_1, e_2] = \lambda e_5 \quad , \quad [e_3, e_4] = \mu e_5, \tag{3.4}$$

where $\{e_5\}$ is a basis for the center of $\mathfrak{n}$, and $\lambda \geq \mu > 0$. Also it is considered that the other commutators are zero.

Example 3.5. For example $O(2) \times SO(2)$ be a Lie group for $\lambda \neq \mu$ and $U(2) \times \mathbb{Z}_2$ be a Lie group for $\lambda = \mu$.

Let F be a left invariant Randers metric on simply connected two-step nilpotent Lie groups of dimension five defined by the Riemannian metric $\tilde{a}$ and the vector field $X = \sum_{i=1}^{5} x_i e_i$. We want to describe all geodesic vectors of (N, F). We note, that (N, F) is not of the Berwald type (see [7].).
By using the formula $g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial t \partial s} F^2(y + su + tv)|_{s=t=0}$ and some computations, for the Randers metric F we have:

$$\begin{aligned}g_y(u, v) &= \tilde{a}(u, v) + \tilde{a}(X, u)\tilde{a}(X, v) - \frac{\tilde{a}(X, y)\tilde{a}(y, u)\tilde{a}(y, v)}{\tilde{a}(y, y)^{\frac{3}{2}}} \\ &+ \frac{1}{\sqrt{\tilde{a}(y, y)}}\Big\{\tilde{a}(X, u)\tilde{a}(y, v) + \tilde{a}(X, y)\tilde{a}(u, v) + \tilde{a}(X, v)\tilde{a}(y, u)\Big\}.\end{aligned} \tag{3.5}$$

So for all $z \in \mathfrak{n}$ we have

$$g_y(y,[y,z]) = \tilde{a}\Big(X + \frac{y}{\sqrt{\tilde{a}(y,y)}},[y,z]\Big)F(y). \tag{3.6}$$

By using Theorem 3.3 and equation (3.6) a vector $y = \sum_{i=1}^{5} y_i e_i$ of $\mathfrak{n}$ is a geodesic vector if and only if

$$\tilde{a}\Big(\sum_{i=1}^{5} x_i e_i + \frac{\sum_{i=1}^{5} y_i e_i}{\sqrt{\sum_{i=1}^{5} y_i^2}},[\sum_{i=1}^{5} y_i e_i, e_j]\Big) = 0, \tag{3.7}$$

for each $j = 1,2,3,4,5$.
So we get :

$$\lambda y_1\Big(x_5 + \frac{y_5}{\sqrt{\sum_{i=1}^{5} y_i^2}}\Big) = 0,$$
$$\lambda y_2\Big(x_5 + \frac{y_5}{\sqrt{\sum_{i=1}^{5} y_i^2}}\Big) = 0,$$
$$\mu y_3\Big(x_5 + \frac{y_5}{\sqrt{\sum_{i=1}^{5} y_i^2}}\Big) = 0,$$
$$\mu y_4\Big(x_5 + \frac{y_5}{\sqrt{\sum_{i=1}^{5} y_i^2}}\Big) = 0. \tag{3.8}$$

As a special case, if $X = \sum_{i=1}^{4} x_i e_i$, Then a vector y of $\mathfrak{n}$ is a geodesic vector if and only if:
-$y \in \mathrm{Span}\{e_1, e_2, e_3, e_4\}$,
or
-y=βe_5 for $\beta \neq 0$.

Corollary 3.6. *Let F be the Randers metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie groups of dimension five with one dimensional center(see (3.4)). Then geodesic vectors depending only on $\tilde{a}(X, e_5)$.*

Theorem 3.7. *Let (N,F) be the Randers metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X = \sum_{i=1}^{4} x_i e_i$ on simply connected two-step nilpotent Lie groups of dimension five with one dimensional center(see (3.4)). Then $y \in \mathfrak{n}$ is a geodesic vector of (N,F) if and only if y is a geodesic vector of $(N,\tilde{a})$.*

Proof. Let $y = \sum_{i=1}^{5} y_i e_i \in \mathfrak{n}$ is a geodesic vector of $(N,\tilde{a})$ by using (3.1) we have $\tilde{a}(y,[y,e_i]) = 0$ for each $i = 1,2,3,4,5$.
Therefore by using equations (3.7), y is a geodesic vector of (N,F).
Conversely. Let $y = \sum_{i=1}^{5} y_i e_i \in \mathfrak{n}$ is a geodesic vector of (N,F), becouse $\tilde{a}(X,[y,e_i]) = 0$ for each $i = 1,2,3,4,5$ by using (3.7) we have

$$\tilde{a}(y,[y,e_i]) = 0 \qquad , \qquad i = 1,2,3,4,5. \tag{3.9}$$

□

In the following we shall give a necessary and sufficient condition for an invariant Randers metric to be a Douglas metric on a simply connected two-step nilpotent Lie groups of dimension five with one dimensional center(see (3.4)).

Theorem 3.8. *Let (N, F) be the Randers metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie groups of dimension five with one dimensional center(see (3.4)). Then (N, F) is of Douglas type if and only if $\tilde{a}(X, e_5) = 0$.*

Proof. In [1], it is proved that the corresponding Randers metric is of Douglas type if and only if X satisfies

$$\tilde{a}([Z, Y], X) = 0 \qquad , \qquad \forall Y, Z \in \mathfrak{n}. \tag{3.10}$$

Therefore by using formula 3.4 the proof is completed. □

Corollary 3.9. *Let (N, F) be the Randers metric of Douglas type induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie groups of dimension five with one dimensional center(see (3.4)). Then we have*
i) *$y \in \mathfrak{n}$ is a geodesic vector of (N, F) if and only if y is a geodesic vector of $(N, \tilde{a})$.*
ii) *A vector y of $\mathfrak{n}$ is a geodesic vector if and only if:*
-$y \in Span\{e_1, e_2, e_3, e_4\}$,
or
-$y=\beta e_5$ for $\beta \neq 0$

4. Lie algebras with 2-dimensional center

In this section we consider the Lie algebra $\mathfrak{n}$ has $2-$dimensional center. In [9] S. Homolya and O. Kowalski showed that there exist an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{n}$ such that

$$[e_1, e_2] = \lambda e_4 \qquad , \qquad [e_1, e_3] = \mu e_5, \tag{4.1}$$

where $\{e_4, e_5\}$ is a basis for the center of $\mathfrak{n}$, the other commutators are zero and $\lambda \geq \mu > 0$.

Example 4.1. For example $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ be a Lie group for $\lambda \neq \mu$ and $O(2) \times \mathbb{Z}_2$ be a Lie group for $\lambda = \mu$.

Let F be a left invariant Randers metric on simply connected two-step nilpotent Lie groups of dimension five with two dimensional center defined by the Riemannian metric $\tilde{a}$ and the vector field $X = \sum_{i=1}^{5} x_i e_i$. We note, that (N, F) is not of the Berwald type (see [7].).
By using Theorem 3.3 and equation(3.6) a vector $\sum_{i=1}^{5} y_i e_i$ of F is a geodesic vector if and only if

$$\tilde{a}\Big(\sum_{i=1}^{5} x_i e_i + \frac{\sum_{i=1}^{5} y_i e_i}{\sqrt{\sum_{i=1}^{5} y_i^2}}, [\sum_{i=1}^{5} y_i e_i, e_j]\Big) = 0, \tag{4.2}$$

for each $j = 1, 2, 3, 4, 5$.
So we have

$$\lambda y_2\Big(x_4 + \frac{y_4}{\sqrt{\sum_{i=1}^5 y_i^2}}\Big) + \mu y_3\Big(x_5 + \frac{y_5}{\sqrt{\sum_{i=1}^5 y_i^2}}\Big) = 0,$$
$$\lambda y_1\Big(x_4 + \frac{y_4}{\sqrt{\sum_{i=1}^5 y_i^2}}\Big) = 0,$$
$$\mu y_1\Big(x_5 + \frac{y_5}{\sqrt{\sum_{i=1}^5 y_i^2}}\Big) = 0. \tag{4.3}$$

Corollary 4.2. *Let F be the Randers metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie groups of dimension five with two dimensional center(see (4.1)). Then geodesic vectors dependig only on $\tilde{a}(X, e_4)$, $\tilde{a}(X, e_5)$, λ and μ.*

Theorem 4.3. *Let (N, F) be the Randers metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X = \sum_{i=1}^3 x_i e_i$ on simply connected two-step nilpotent Lie groups of dimension five with two dimensional center(see (4.1)). Then $y \in \mathfrak{n}$ is a geodesic vector of (N, F) if and only if y is a geodesic vector of $(N, \tilde{a})$.*

Proof. Let $y = \sum_{i=1}^5 y_i e_i \in \mathfrak{n}$ is a geodesic vector of $(N, \tilde{a})$ by using (3.1) we have $\tilde{a}(y, [y, e_i]) = 0$ for each $i = 1, 2, 3, 4, 5$.
Therefore by using equations (4.2), y is a geodesic vector of (N, F).
Conversely. Let $y = \sum_{i=1}^5 y_i e_i \in \mathfrak{n}$ is a geodesic vector of (N, F), becouse $\tilde{a}(X, [y, e_i]) = 0$ for each $i = 1, 2, 3, 4, 5$ by using (4.2) we have

$$\tilde{a}(y, [y, e_i]) = 0 \quad , \quad i = 1, 2, 3, 4, 5. \tag{4.4}$$

□

In the following we shall give a necessary and sufficient condition for an invariant Randers metric to be a Douglas metric on a simply connected two-step nilpotent Lie groups of dimension five with two dimensional center(see (4.1)).

Theorem 4.4. *Let (N, F) be the Randers metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie groups of dimension five with two dimensional center(see (4.1)). Then (N, F) is of Douglas type if and only if $\tilde{a}(X, e_4) = 0$ and $\tilde{a}(X, e_5) = 0$.*

Proof. In [1], it is proved that the corresponding Randers metric is of Douglas type if and only if X satisfies

$$\tilde{a}([Z, Y], X) = 0 \quad , \quad \forall Y, Z \in \mathfrak{n}. \tag{4.5}$$

Therefore by using formula 4.1 the proof is completed. □

Corollary 4.5. *Let (N, F) be the Randers metric of Douglas type induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie groups of dimension five with two dimensional center(see (4.1)). Then $y \in \mathfrak{n}$ is a geodesic vector of (N, F) if and only if y is a geodesic vector of $(N, \tilde{a})$.*

5. Lie algebras with 3-dimensional center

Now we study simply connected two-step nilpotent Lie groups of dimension five equipped with left-invariant Randers metric with three dimensional center.
In this section we consider the Lie algebra $\mathfrak{n}$ has $3-$dimensional center. In [9] S. Homolya and O. Kowalski showed that there exist an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{n}$ such that

$$[e_1, e_2] = \lambda e_3, \tag{5.1}$$

where $\{e_3, e_4, e_5\}$ is a basis for the center of $\mathfrak{n}$, the other commutators are zero and $\lambda > 0$.

Example 5.1. For example $H_3 \times \mathbb{R}^2$ or $O(2) \times O(2)$ be a Lie group with The metric Heisenberg Lie algebra $\mathfrak{h}_3(\lambda) \oplus \mathbb{R}^2$.

Let F be a left invariant Randers metric on simply connected two-step nilpotent Lie groups of dimension five defined by the Riemannian metric $\tilde{a}$ and the vector field $X = \sum_{i=1}^{5} x_i e_i$. We want to describe all geodesic vectors of (N, F).
By using Theorem 3.3 and equation(3.6) a vector $\sum_{i=1}^{5} y_i e_i$ of $\mathfrak{n}$ is a geodesic vector if and only if

$$\tilde{a}\Big(\sum_{i=1}^{5} x_i e_i + \frac{\sum_{i=1}^{5} y_i e_i}{\sqrt{\sum_{i=1}^{5} y_i^2}}, [\sum_{i=1}^{5} y_i e_i, e_j]\Big) = 0, \tag{5.2}$$

for each $j = 1, 2, 3, 4, 5$.
This condition leads to the system of equations

$$\begin{aligned} \lambda y_1\Big(x_3 + \frac{y_3}{\sqrt{\sum_{i=1}^{5} y_i^2}}\Big) &= 0, \\ \lambda y_2\Big(x_3 + \frac{y_3}{\sqrt{\sum_{i=1}^{5} y_i^2}}\Big) &= 0. \end{aligned} \tag{5.3}$$

As a special case, if $X = x_1 e_1 + x_2 e_2 + x_4 e_4 + x_5 e_5$, Then a vector y of $\mathfrak{n}$ is a geodesic vector if and only if:
- $y \in Span\{e_3, e_4, e_5\}$,
or
- $y \in Span\{e_1, e_2, e_4, e_5\}$.

Corollary 5.2. *Let F be the Randers metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie groups of dimension five with three dimensional center(see (5.1)). Then geodesic vectors dependig only on $\tilde{a}(X, e_3)$.*

Theorem 5.3. *Let (N, F) be the Randers metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X = x_1 e_1 + x_2 e_2 + x_4 e_4 + x_5 e_5$ on simply connected two-step nilpotent Lie groups of dimension five with $3-$dimensional center. Then $y \in \mathfrak{n}$ is a geodesic vector of (N, F) if and only if y is a geodesic vector of $(N, \tilde{a})$.*

Proof. By using equations (3.1) and (5.2) completes the proof. □

Corollary 5.4. *Let F be the Randers metric of Berwald type on simply connected two-step nilpotent Lie groups of dimension five N with three dimensional center(see (5.1)) induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X. Then its geodesic vectors are forms of*

- $y \in Span\{e_3, e_4, e_5\}$,
or
- $y \in Span\{e_1, e_2, e_4, e_5\}$.

Proof. If (N, F) be the Randers metric of Berwald type then $X = ae_4 + be_5$.(see [7].). So completed the proof. □

In the following we shall give a necessary and sufficient condition for an invariant Randers metric to be a Douglas metric on a simply connected two-step nilpotent Lie groups of dimension five with three dimensional center(see (5.1)).

Theorem 5.5. *Let (N, F) be the Randers metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie groups of dimension five with three dimensional center(see (5.1)). Then (N, F) is of Douglas type if and only if $\tilde{a}(X, e_3) = 0$.*

Proof. In [1], it is proved that the corresponding Randers metric is of Douglas type if and only if X satisfies

$$\tilde{a}([Z, Y], X) = 0 \qquad , \qquad \forall Y, Z \in \mathfrak{n}. \tag{5.4}$$

Therefore by using formula 5.1 the proof is completed. □

Corollary 5.6. *Let (N, F) be the Randers metric of Douglas type induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie groups of dimension five with three dimensional center(see (5.1)). Then we have*
i) *$y \in \mathfrak{n}$ is a geodesic vector of (N, F) if and only if y is a geodesic vector of $(N, \tilde{a})$.*
ii) *Its Geodesics are forms of*
- $y \in Span\{e_3, e_4, e_5\}$,
or
- $y \in Span\{e_1, e_2, e_4, e_5\}$.

Corollary 5.7. *Let (N, F) be the Randers metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie groups. The following are equivalent:*
i) *(N, F) is of Douglas type.*
ii) *X is orthogonal to the center of $\mathfrak{n}$ with respect to $\tilde{a}$.*

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