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SOME NEW EQUILATERAL TRIANGLES IN A PLANE GEOMETRY

DAO THANH OAI

ABSTRACT. In this paper, we give some equilateral triangles in plane geometry. Some equilateral triangles constructed from its reference triangles. In these ones, some equilateral triangles are homothetic to the Morley triangles, some equilateral triangles are homothetic to the Napoleon triangles, and some equilateral triangles are not homothetic to any kown triangle. Furthermore we give an equilateral triangle is the converse of L. Bankoff, P. Erds and M. Klamkins theorem. We give a closed chain of six equilateral triangle. We also introduce to the Yius equilateral triangle and Yius triple points.

1. INTRODUCTION

In plane geometry, there are several famous equilateral triangles which are constructed from its reference triangle.

An equilateral triangle constructed from a reference triangles is a topic which is intersested by plane geometry lovers. There are many results from this topic. For example, the Napoleon theorem is famous classic theorem in the plane geometry. There are several generalizations of the Napoleon theorem. For example, the Kiepert's theorem [1], the Jacobi's theorem [2], PetrDouglasNeuman theorem [3], Napoleon-Barlotti theorem [4]. There are many articles around these results, you can see in [5], [6], [7]. These theorems are also continuing creative inspiration for some results [8], [10]. The Morley equilateral triangle is also the famous nice theorem in plane geometry. There are over 100 references around the Morley theorem [11]....Viviani's theorem, Pompeiu's theorem are also theorems around equilateral triangle.

There are some classical equilateral triangles with respect to some triangle centers as the Napoleon equilateral triangles and the Napoleon points, the Morley equilateral triangles and the Morley points, equilateral pedal triangle and the isodynamic points, and Fermat points. There are some recent equilateral triangles with respect to some triangle centers as Stammler equilateral triangles, equilateral cevian triangle and equilateral cevian triangle point. Recently, I discover about 40 new equilateral triangles which are perspective to its reference triangle, you can see in [12], [13].

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Key words and phrases. Kimberling center, Morley equilateral triangle, Napoleon equilateral triangle, New equilateral triangle, Fermat point, Isodynamic point, Yiu's equilateral triangle.

We give some equilateral triangles in plane geometry. Some equilateral triangles constructed from its reference triangles. In these ones, some equilateral triangles are homothetic to the Morley triangles, some equilateral triangles are homothetic to the Napoleon triangles, and some equilateral triangles are not homothetic to any kown triangle. Furthermore we give an equilateral triangle is the converse of L. Bankoff, P. Erds and M. Klamkins theorem. We give a closed chain of six equilateral triangle. We also introduce to the Yius equilateral triangle and Yius triple points.

2. A GENERALIZATION OF THE NAPOLEON THEOREM ASSOCIATED WITH THE KIEPERT HYPERBOLA AND THE KIEPERT TRIANGLE

In this part we introduce a family equilateral triangle associated with a Kiepert triangle and give a proof of the family equilatarl triangle associated with the Kiepert hyperbola.

Theorem 2.1. *Let ABC be a triangle with H is the orthocenter. Let A_1, B_1, C_1 be chosen on rays AH, BH, CH (or rays HA, HB, HC) respectively so that $AA_1 = \frac{BC\sqrt{3}}{3}$, $BB_1 = \frac{CA\sqrt{3}}{3}$, $CC_1 = \frac{AB\sqrt{3}}{3}$, then $\triangle A_1B_1C_1$ is an equilateral triangle.*

The Theorem 2.1 was found by me since June 2013, you can see in [14], this theorem was independently discovered by Dimitris Vartziotis [15]. Theorem 2.1 is also a special case of Theorem 2.2 as follows:

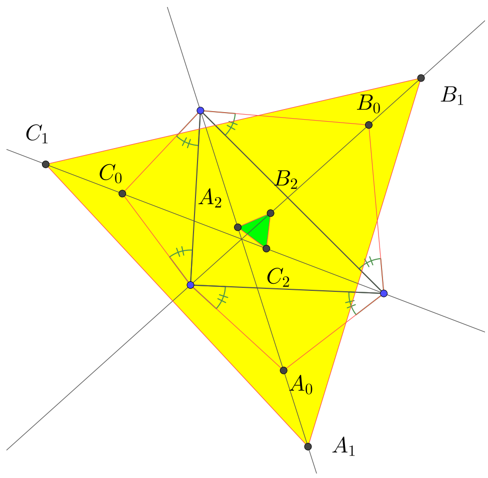

FIGURE 1. Theorem 2.2

Theorem 2.2 ([10]). *Let $\triangle ABC$ be a triangle, constructed three isosceles similar triangle BA_0C, CB_0A, AC_0B with base angles are α, either all outward or all inward. Let points A_1, B_1, C_1 be chosen on rays AA_0, BB_0, CC_0 so that:*

$$\frac{AA_1}{AA_0} = \frac{BB_1}{BB_0} = \frac{CC_1}{CC_0} = \frac{2}{3 - \sqrt{3}\tan\alpha}$$

Or

$$\frac{AA_1}{AA_0} = \frac{BB_1}{BB_0} = \frac{CC_1}{CC_0} = \frac{2}{3 + \sqrt{3}\tan\alpha}$$

Then $A_1B_1C_1$ is an equilateral triangle (Figure 1).

Let $\alpha = 30^0$, Theorem 2.2 is the Napoleon theorem.

Theorem 2.3 ([16]). *Let ABC be a triangle with F is the first (or second) Fermat point, let K be arbitrary point on the Kiepert hyperbola. Let P be arbitrary point on line FK. The line through P and perpendicular to BC meet AK at A_0. Define B_0, C_0 cyclically, then $A_0B_0C_0$ is an equilateral triangle.*

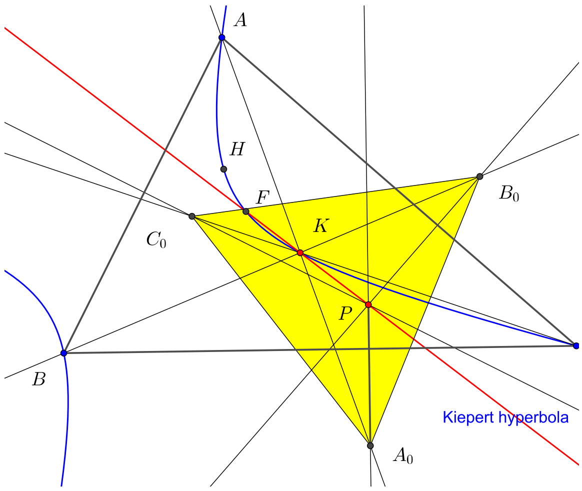

FIGURE 2. Theorem 2.3

Noted that the first Fermat point, the circumcenter, the fisrt Napoleon point are collinear; the second Fermat point, the circumcenter, the second Napoleon point are collinear [17] so if we let K is the first (or second) Napoleon point and P is the circumcenter, then equilateral triangle $A_0B_0C_0$ in Theorem 2.3 is the outer (or inner) Napoleon triangle.

We used Lemma 2.4 and Lemma 2.5 as follows to prove Theorem 3.

Lemma 2.4. *Let A, B, C, D, E be on a rectangular hyperbola, three lines through D and perpendicular to EA, EB, EC meet BC, AC, AB at A', B', C' respectively. Then A', B', C' are collinear, and line $\overline{A'B'C'}$ perpendicular to DE.*

You can see a proof of Lemma 2.4 using Cartesian coordinates system in [18], You can see many synthetic proofs of Lemma 2.4 in [19].

We omit the proof of the following easy Lemma:

Lemma 2.5. *Let ABC be a triangle, P be a point in the plane, A' be a point on AP. Let two lines through A' and parallel to AB, AC meet PB, PC at B', C' respectively, then $B'C'$ parallel to BC.*

Let three lines through F perpendicular to AB, AK, AC that meet KC, CB, BK at three points B', C', K'_a respectively. Since B, K, C, F, A lie on Kiepert hyperbola, so by Lemma 2.4 we have AF perpendicular to $B'C'$.

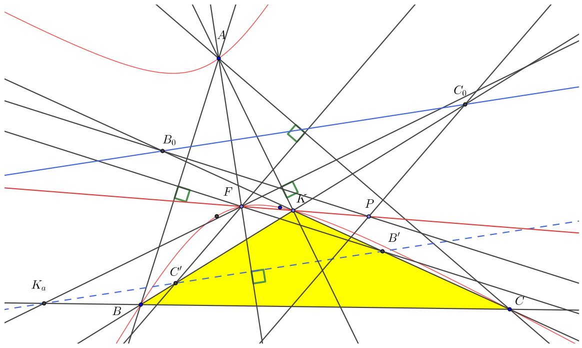

FIGURE 3. Applying Lemma 2.4 and Lemma 2.5 to prove Theorem 2.3

In triangle $FC'B'$ and point K, P lie on FK and $PC_0 \parallel FC', PB_0 \parallel FB'$, by Lemma 2.5 we get $B_0C_0 \parallel B'C'$, so $B_0C_0 \perp AF$. Similarly we have $C_0A_0 \perp BF$ and $A_0B_0 \perp CF$. On the other hand, we known that $\angle AFB = \angle BFC = \angle CFA = \frac{2\pi}{3}$. Therefore $A_0B_0C_0$ is the equilateral triangle.

You can see a synthetic proof of Theorem 4 by Telv Cohl in [20].

We give some special case of Theorem 2.3 as follows:

Corollary 2.1 ([21]). *Let ABC be a triangle with F be the first (or the second) Fermat point, I be the first (or the second) Isodynamic point. Let P be a point on the line FI. Three line through P and perpendicular to BC meets the line AF at A_0, define B_0, C_0 cyclically. Then $A_0B_0C_0$ is an equilateral triangle*

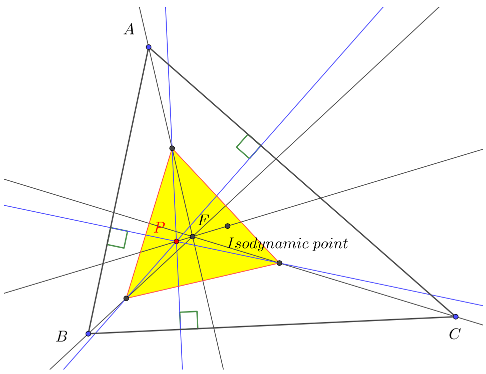

FIGURE 4. Corollary 2.1

Proof. We have FI parallel to the Euler line [22]. The circle through the two Fermat points and the centroid tangent to the Euler line at the centroid (Proposition 4, [23]). Two Fermat points and the centroid lie on the Kiepert hyperbola [24]. Then the line FI tangent to the Kiepert hyperbola at F (Lemma 1, [25]). Applying Theorem 2.3 with $K \equiv F$ we have $A_0B_0C_0$ is the equilateral triangle. □

We omit the proof of the following special case of Theorem 2.3:

Corollary 2.2. *Let ABC be a triangle, F be the the first (or the second) Fermat point, K be the point on the Kiepert hyperbola. Three lines through F perpendicular to BC, CA, AB meet AK, BK, CK at three points A_0, B_0, C_0 respectively, then $A_0B_0C_0$ is an equilateral triangle*

Corollary 2.3. *Let ABC be a triangle, F_+, F_- be the First and the second Fermat points, P be arbitrary point lies on F_+F_-. The line through P perpendicular to BC meets AF_+, AF_- at A_+, A_- respectively; define B_+, B_-, C_+, C_- cyclically, then $A_+B_+C_+$ and $A_-B_-C_-$ are two equilateral triangles.*

3. A CHAIN OF SIX EQUILATERAL TRIANGLES

In this part, around a triangle ABC, we construct six equilateral triangles and use this configuration and complex numbers, we generalize and investigate famous theorems such as Fermat-Torricelli, Napoleon.

Consider ABC be a triangle, we take an arbitrary point D on the plane of ABC. If the triangle DAC is not equilateral then we construct five equilateral triangles ADE, $\triangle BEF$, $\triangle CFG$, $\triangle AGH$, $\triangle BHJ$ with the same orientation.

Without loss of generality, in this paper we assume that the triangle ABC with negative orientation and fives equilateral triangles ADE, $\triangle BEF$, $\triangle CFG$, $\triangle AGH$, $\triangle BHJ$ with the positive orientation

We will prove that the remaining sixth triangle CJD is also equilateral with positive orientation. Let ϵ be the complex cube root of unity that rotates by an angle of $\frac{2\pi}{3}$ then $\epsilon^3 = 1$ and $1 + \epsilon + \epsilon^2 = 0$. We use the following useful lemma to prove all properties in this article.

Lemma 3.1. *If a, b, c are the complex affixes of the vertices of triangle ABC then ABC is equilateral with positive orientation if and only if $a + b\epsilon + c\epsilon^2 = 0$ and ABC is equilateral with negative orientation if and only if $a + b\epsilon^2 + c\epsilon = 0$.*

You can see the proof of Lemma 3.1 in [8].

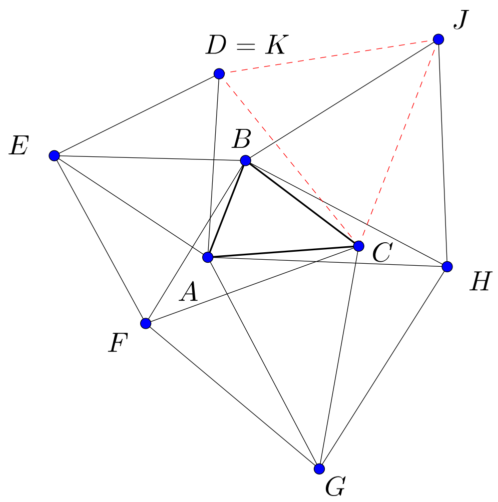

FIGURE 5. Theorem 3.2

• In this configuration, if $D \equiv B$ or $D \equiv C$ we get the FermatTorricelli point theorem and Napoleon theorem configuration.

We give some properties of this configuration and its proof by complex number from Theorem 3.2 to Theorem 3.3.

Theorem 3.2.
1. The triangle $\triangle CJD$ is an equilateral with the same orientation as triangle ADE.
2. Three triangles $\triangle ABC$, $\triangle DHF$, $\triangle EJG$ have the same centroids

Proof. We prove this theorem by using complex number coordinates of the points. Suppose the Figure 5 is in the complex plane. Each of the vertices $A, B, C, D, E, F, G, H, J$ has a complex affix $a, b, c, d, e, f, g, h, j$.

• Proof. Let K be a point such that the triangle CJK be an equilateral triangle with the same orientation ADE. We have:

$e = -d\epsilon - a\epsilon^2$
$f = -e\epsilon - b\epsilon^2 = d\epsilon^2 + a - b\epsilon^2$
$g = -f\epsilon - c\epsilon^2 = -d\epsilon^3 - a\epsilon + b\epsilon^3 - c\epsilon^3 = -d - a\epsilon + b - c\epsilon^2$
$h = -g\epsilon - a\epsilon^2 = d\epsilon + a\epsilon^2 - b\epsilon + c\epsilon^3 - a\epsilon^2 = d\epsilon - b\epsilon + c$
$j = -h\epsilon - b\epsilon^2 = -d\epsilon^2 + b\epsilon^2 - c\epsilon - b\epsilon^2 = -d\epsilon^2 - c\epsilon$
$k = -j\epsilon - c\epsilon^2 = d\epsilon^3 + c\epsilon^2 - c\epsilon^2 = d\epsilon^3 = d$

Therefore $K \equiv D$

• We have $d + h + f = d + d\epsilon - b\epsilon + c + d\epsilon^2 + a - b\epsilon^2$ $= d(1 + \epsilon + \epsilon^2) + a + (-b\epsilon - b\epsilon^2) + c = a + b + c$ therefore two triangle $\triangle ABC$, $\triangle DFH$ have the same centroids. Similarly two triangle $\triangle ABC$, $\triangle EJG$ have the same centroids □

Theorem 3.3. *(A generalization of the Napoleon theorem) The centroids of three triangles as follows form an equilateral triangle.*

- $\triangle BJC$, $\triangle CGA$, $\triangle AEB$
- $\triangle BCF$, $\triangle CAD$, $\triangle ABH$
- $\triangle DJH$, $\triangle HGF$, $\triangle FED$
- $\triangle DHA$, $\triangle HFB$, $\triangle FDC$
- $\triangle GFE$, $\triangle EDJ$, $\triangle JHG$
- $\triangle GEA$, $\triangle EJB$, $\triangle JGC$

Proof. The proof of the centroids of three triangles $\triangle BCF$, $\triangle CAD$, $\triangle ABH$ form an equilateral triangle.

$(b + c + f) + (c + a + d)\epsilon + (a + b + h)\epsilon^2$
$= b + b\epsilon^2 + c + c\epsilon + a\epsilon + a\epsilon^2 + f + d\epsilon + h\epsilon^2$
$= -b\epsilon - c\epsilon^2 - a + f + d\epsilon + h\epsilon^2$
$= -b\epsilon + d\epsilon - c\epsilon^2 + h\epsilon^2 - a + f$
$= e + f\epsilon^2 - e - a\epsilon^2 + g + f\epsilon - g - a\epsilon - a + f$
$= f(1 + \epsilon + \epsilon^2) - a(1 + \epsilon + \epsilon^2) = 0$

Therefore the centroids of three triangles $\triangle BCF$, $\triangle CAD$, $\triangle ABH$ form an equilateral triangle.

By the symmetry of $\triangle ABC$, $\triangle JGE$, $\triangle FDH$ in this configuration so the centroids of three triangles as follows form an equilateral triangle:

- $\triangle BJC$, $\triangle CGA$, $\triangle AEB$
- $\triangle DJH$, $\triangle HGF$, $\triangle FED$

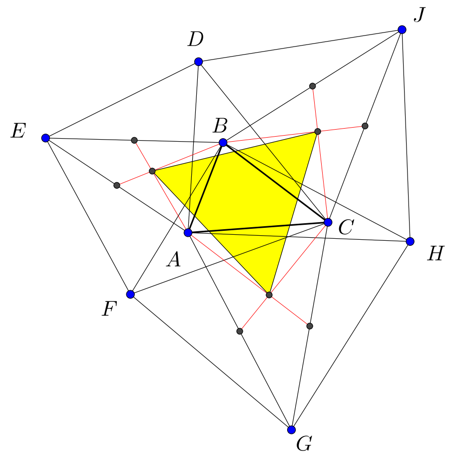

FIGURE 6. Theorem 3.3

- $\triangle DHA, \triangle HFB, \triangle FDC$
- $\triangle GFE, \triangle EDJ, \triangle JHG$
- $\triangle GEA, \triangle EJB, \triangle JGC$

□

Theorem 3.4. *(A generalization of the Napoleon theorem) The centroids of three equilateral triangles as follows form an equilateral triangle.*
- $\triangle AED, \triangle BJH, \triangle CGF$
- $\triangle AHG, \triangle BFE, \triangle CDJ$

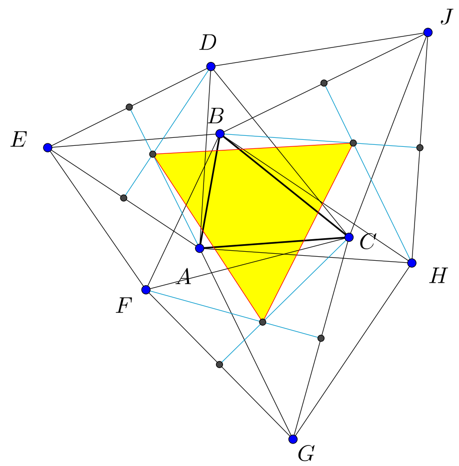

FIGURE 7. Theorem 3.4

Proof. The proof of the centroids of three triangles $\triangle AED, \triangle BJH, \triangle CGF$ form an equilateral triangle.

$(a+e+d)+(b+j+h)\epsilon+(c+g+f)\epsilon^2$
$= a+b\epsilon+c\epsilon^2+e+j\epsilon+g\epsilon^2+d+h\epsilon+f\epsilon^2$
$= a-(e+f\epsilon^2)+c\epsilon^2+e-(d+c\epsilon^2)+g\epsilon^2+d-(a+g\epsilon^2)+f\epsilon^2 = 0$

Therefore the centroids of three triangles $\triangle AED$, $\triangle BJH$, $\triangle CGF$ form an equilateral triangle.

By the symmetry of $\triangle ABC$, $\triangle JGE$, $\triangle FDH$ in this configuration so the centroids of three triangles $\triangle AHG$, $\triangle BFE$, $\triangle CDJ$ form an equilateral triangle. □

Theorem 3.5. *(see Figure 8).*

1. The first Fermat points of three triangle $\triangle ADH$, $\triangle BHF$, $\triangle CFD$ form an equilateral triangle.

2. The first Fermat points of three triangle $\triangle ADC$, $\triangle HAB$, $\triangle FBC$ form an equilateral triangle.

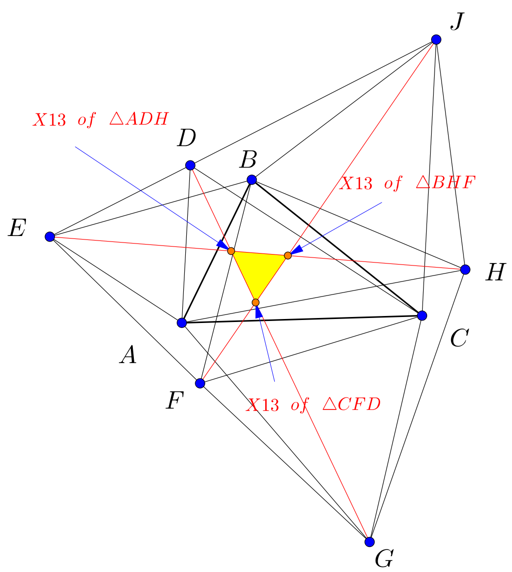

FIGURE 8. Theorem 3.5

Proof. First we prove that AJ, BG, CF form an equilateral triangle. Two equilateral triangles $\triangle BJH$ and HGA construct on the sidelines of the triangle $\triangle BHA$ so by Fermat point theorem we have angle of AJ and BG equal to 60^0. Similarly, two equilateral triangles $\triangle AED$ and DCJ construct on the sidelines of the triangle $\triangle ADC$ so by Fermat point theorem we have that angle of AJ and CF equals to 60^0. Therefore three lines AJ, BG, CF form an equilateral triangle. But we easily can deduce that in triangle BHF then X_{13} of $\triangle BHF$ lie on FJ, and in triangle $\triangle CFD$ then X_{13} of $\triangle BHF$ lie on FJ. □

4. AN EQUILATERAL TRIANGLE ASSOCIATED WITH ISOSCELES TRAPEZOID ATTACHED TO SIDES OF A TRIANGLE

Theorem 4.1. *Let ABC be a triangle, construct three arbitrary isosceles trapezoids $ABGJ$, $BCLT$, $CAHK$ with angle of their diagonals is 60^0. Then the centroids of $\triangle HAJ$, $\triangle GBT$, $\triangle LCK$ form an equilateral triangle.*

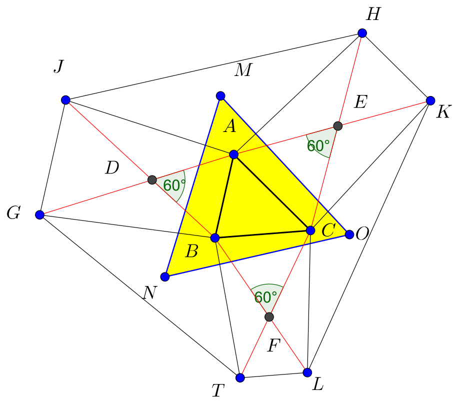

FIGURE 9. Theorem 4.1

Proof. Let $D = AG \cap BJ$, $F = BL \cap CT$, $E = CH \cap AK$. We prove this theorem by using complex number coordinates of the points. Suppose that the Figure 9 is in the complex plane. Each of the vertices A, B, C, D, E, F, M, O, N, H, K, L, T, G, J has a complex affix a, b, c, d, e, f, m, o, n, h, k, l, t, g, j.

$3(m + n\epsilon + o\epsilon^2) = (a + h + j) + (b + g + t)\epsilon + (c + l + k)\epsilon^2$
$= a + b\epsilon + c\epsilon^2 + (j + g\epsilon) + (t\epsilon + l\epsilon^2) + (h + k\epsilon^2)$
$= a + b\epsilon + c\epsilon^2 - d\epsilon^2 - f - e\epsilon$
$= a + b\epsilon + c\epsilon^2 + (b + a\epsilon) + (c\epsilon + b\epsilon^2) + (c + a\epsilon^2)$
$= a + a\epsilon + a\epsilon^2 + b + b\epsilon + b\epsilon^2 + c\epsilon + c\epsilon^2$
$= a(1 + \epsilon + \epsilon^2) + b(1 + \epsilon + \epsilon^2) + c(1 + \epsilon + \epsilon^2) = 0$

Therefore $\triangle MNP$ is the equilateral triangle □

If we let $ABGJ$, $BCLT$, $CAHK$ be three rectangles with the angle of their diagonals is 60^0, either all outward or all inward, the Theorem 4.1 is the Theorem 2.1.

5. TWO EQUILATERAL TRIANGLE ASSOCIATED WITH THE FERMAT POINTS

Theorem 5.1. *Let ABC be a triangle, Let F be the first (or the second) Fermat points. The line through F and parallel to BC meets circles (AFB), (AFC) again at A_c, A_b respectively. Define B_c, B_a, C_a, C_b cyclically. Let $A' = C_bA_b \cap A_cB_c$, $B' = B_aC_a \cap A_cB_c$, $C' = B_aC_a \cap C_bA_b$ then*

1. *$A'B'C'$ is an equilateral triangle.*
2. *Two triangle $A'B'C'$ and ABC are perspective.*

Proof. By properties of inscribed angle and two parallel line, We have $\angle C_aPC = \angle C_aBC$, $\angle A_cPA = \angle ABA_c$ and $\angle A_cPC_a = \angle CBA$ so $\angle C_aBA_c = \angle C_aBC + \angle CBA + \angle ABA_c = \angle C_aBC + \angle A_cPC_a + \angle A_cPA = \angle APC$.

By the Miquel's theorem we have three circles $(B'C_aA_c)$, (C_aB_aP), (PB_cA_c) have common point B so $\angle C'B'A' = \angle C_aB'A_c = 180^0 - \angle C_aBA_c = 180^0 - \angle APC = 60^0$.

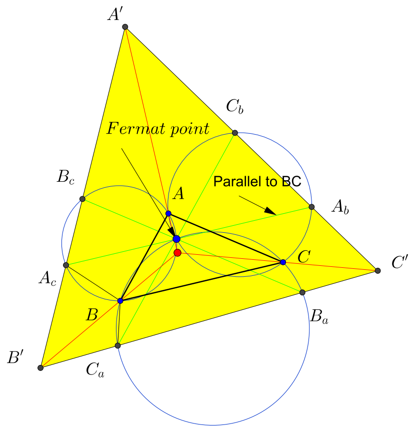

FIGURE 10. Theorem 5.1

By the Miquel theorem we have $(A'B_cPC_b)$ so applying Law of sines we have

$$\frac{\sin\angle B_cA'A}{\sin\angle AA'C_b}=\frac{AB_c}{AC_b}=\frac{\sin\angle B'A'A}{\sin\angle AA'C'}$$

By Law of sine, we have

$$\frac{AB_c}{\sin\angle APB_c}=\frac{AP}{\sin\angle PB_cA}$$

so

$$AB_c=AP\frac{\sin\angle APB_c}{\sin\angle PB_cA};\quad AC_b=AP\frac{\sin\angle C_bPA}{\sin\angle AC_bP}$$

Thus

$$\frac{AB_c}{AC_b}=\frac{\sin\angle APB_c}{\sin\angle PB_cA}\frac{\sin\angle AC_bP}{\sin\angle C_bPA};\quad \frac{BC_a}{BA_c}=\frac{\sin\angle BPC_a}{\sin\angle PC_aB}\frac{\sin\angle BA_cP}{\sin\angle A_cPB}$$

and

$$\frac{CA_b}{CB_a}=\frac{\sin\angle CPA_b}{\sin\angle PA_bC}\frac{\sin\angle CB_aP}{\sin\angle B_aPC}$$

On the other hand we have $\angle APB_c=\angle PAC=\angle PA_bC$; similarly $\angle AC_bP=\angle B_aPC$, $\angle BPC_a=\angle PB_cA$, $\angle BA_cP=\angle C_bPA$, $\angle CPA_b=\angle PC_aB$, $\angle CB_aP=\angle A_cPB$ therefore

$$\frac{\sin\angle B'A'A}{\sin\angle AA'C'}\frac{\sin\angle A'C'A}{\sin\angle ACB'}\frac{\sin\angle C'B'A}{\sin\angle AB'A'}=\frac{AB_c}{AC_b}\frac{CA_b}{CB_a}\frac{BC_a}{BA_c}$$

$$=\frac{\sin\angle APB_c}{\sin\angle PB_cA}\frac{\sin\angle AC_bP}{\sin\angle C_bPA}\frac{\sin\angle BPC_a}{\sin\angle PC_aB}\frac{\sin\angle BA_cP}{\sin\angle A_cPB}\frac{\sin\angle CPA_b}{\sin\angle PA_bC}\frac{\sin\angle CB_aP}{\sin\angle B_aPC}=1$$

By the converse of Cevas theorem, we conclude that the lines AA', BB', CC' are concurrent. The perspector of $A'B'C'$ and ABC is $X(16247)$ in [9]. □

Remarks: *Two triangles $A'B'C'$ and ABC are perspective with arbitrary point P in the plane of $\triangle ABC$.*

6. YIU'S EQUILATERAL TRIANGLES-YIU'S TRIPLE POINTS

In May, 2014, Professor Paul Yiu sent to me a very nice equilateral triangle via email as follows. I have no proof for the result and I am looking for a proof from reader.

Theorem 6.1. *Let ABC be a triangle with two Fermat points F_1, F_2. Circles with center F_1 radius F_1F_2 meets the Kiepert hyperbola again at three points D, E, F. Then triangle DEF is equilateral and perspective to ABC at triplet points:*
- DA, EB, FC are concurrent, let the point of concurrence be Y_1.
- DB, EC, FA are concurrent, let the point of concurrence be Y_2.
- DC, EA, FB are concurrent, let the point of concurrence be Y_3.
Three points Y_1, Y_2, Y_3 are collinear.

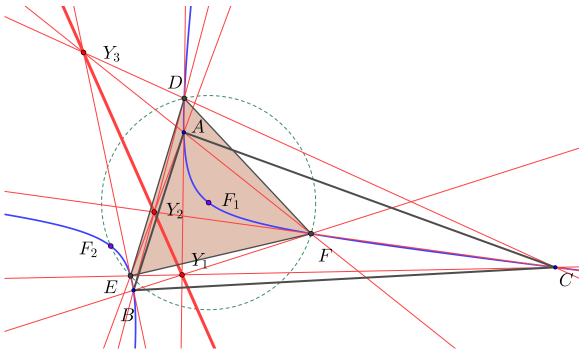

FIGURE 11. Yiu's equilateral triangle-Yiu's triple points

Theorem 6.2. *Let ABC be a triangle with two Fermat points F_1, F_2. Circles with center F_2 radius F_2F_1 meets the Kiepert hyperbola again at three points D, E, F. Then the triangle DEF is equilateral and perspective to ABC at triplet points*
- DA, EB, FC are concurrent, let the point of concurrence be Y_1.
- DB, EC, FA are concurrent, let the point of concurrence be Y_2.
- DC, EA, FB are concurrent, let the point of concurrence be Y_3.
Three points Y_1, Y_2, Y_3 are collinear.

7. EQUILATERAL TRIANGLE ASSOCIATED WITH THE INCENTER

Theorem 7.1 ([31]). *Let ABC be a triangle with the incenter I, Let points D and E be chosen on side BC, points F and G on side CA, and points H and K chosen on side AB so that IDE, IFG, IHK are three equilateral triangles. Let (IKD) meets (IEF) again at A'; (IEF) meets (IGH) again at B', (IGH) meets (IKD) again at C', then $A'B'C'$ is an equilateral triangle*

The centroid of of $\triangle A'B'C'$ is $X(16038)$ in the Encyclopedia of Triangle Centers. Two triangle ABC and $A'B'C'$ are perspective, the perspector is $X(3639)$ [27].

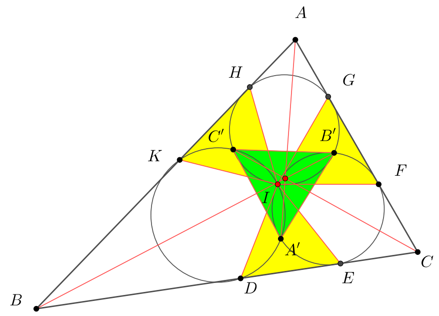

FIGURE 12. Theorem 7.2

8. AN EQUILATERAL TRIANGLE ASSOCIATED WITH L.BANKOFF P.ERDS AND M.KLAMKIN'S CONFIGURATION

In this part, we give an equilateral triangle from the converse of L. Bankoff, P. Erds and M. Klamkin's theorem.

Theorem 8.1. *Let $ABCDEF$ be the hexagon with midpoints of AD, BE, CF form an equilateral triangle. Construct equilateral triangles $\triangle ABM$, $\triangle CDP$, $\triangle EFN$ either all outward (Figure 13). Then $\triangle MNP$ is an equilateral triangle.*

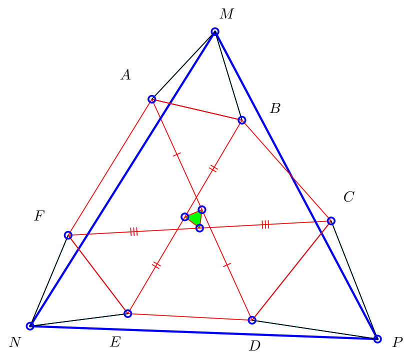

FIGURE 13. Theorem 8.1

This is the converse of L. Bankoff, P. Erds and M. Klamkin's theorem [28]. When AF, BE, CF have the same midpoint, Theorem 8.1 is a generalization of Problem 4167 in the Crux Mathematicorum [29]. Note that the problem 4167 is also a generalization of the famous Thebault's problem II and the Napoleon theorem.

Proof. We prove this theorem by using complex number coordinates of the points. Suppose the Figure 16 is in the complex plane. Each of the vertices $A, B, C, D, E, F, M, N, P$ has a complex affix $a, b, c, d, e, f, m, n, p$.

By Lemma 9, midpoints of AD, BE, CF form an equilateral triangle, so:
$\frac{c+f}{2} + \frac{a+d}{2}\epsilon + \frac{b+e}{2}\epsilon^2 = 0$

$m + n\epsilon + p\epsilon^2 = -a\epsilon - b\epsilon^2 - (e\epsilon + f\epsilon^2)\epsilon - (c\epsilon + d\epsilon^2)\epsilon^2$
$= -a\epsilon - b\epsilon^2 - e\epsilon^2 - f\epsilon^3 - c\epsilon^3 - d\epsilon^4$
$= -f - c - a\epsilon - d\epsilon - b\epsilon^2 - e\epsilon^2 = -2\left(\frac{c+f}{2} + \frac{a+d}{2}\epsilon + \frac{b+e}{2}\epsilon^2\right) = 0$ therefore $\triangle MNP$ is the equilateral triangle.

□

9. SOME EQUILATERAL TRIANGLES ASSOCIATED WITH SOME GROUP CIRCLES

In this part, we first given two construction of a group cyclic hexagon which the vertices belong to three sidelines of a triangle. A special case, we give two family equilateral triangles homothetic to the Morley triangle.

Theorem 9.1. *Let ABC be a triangle, let points D, G be chosen on side AB, points I, F be chosen on side BC, points E, H be chosen on side CA such that:*

$$\begin{cases} \angle EDA = kA + lB + (1-k-l)C \\ \angle FEC = (1-l)A + (k+l)B - kC \\ \angle GFB = (1-k-l)A + kB + lC \\ \angle HGA = -kA + (1-l)B + (k+l)C \\ \angle IHC = lA + (1-k-l)B + kC \end{cases}$$

Then six points D, E, F, G, H, I lie on a circle and $\angle DIB = (k+l)A - kB + (1-l)C$

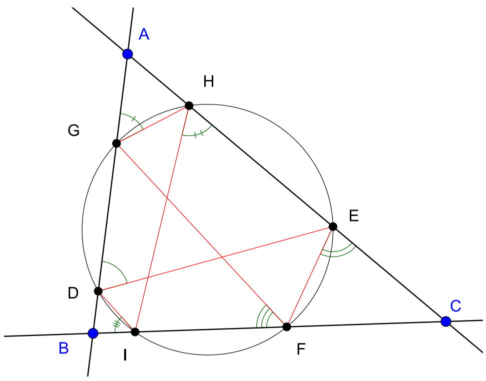

FIGURE 14. Theorem 9.1

Let $k = 0, l = 1$ the circle above is the Tucker circle of ABC.

Proof. We have $\angle EFG = \pi - \angle CFE - \angle GFB = \pi - (\pi - \angle FEC - C) - \angle GFB = C + \angle FEC - \angle GFB = C + (1-l)A + (k+l)B - kC - (1-k-l)A - kB - lC = C + A - lA + kB + lB - kC - A + kA + lA - kB - lC = kA + lB + (1-k-l)C = \angle EDG$, so $GEFD$ is a cyclic quadrilateral.

We have $\angle FGH = \pi - \angle BGF - \angle HGA = \pi - (\pi - \angle GFB - B) - \angle HGA = B + \angle GFB - \angle HGA = B + (1-k-l)A + kB + lC + kA - (1-l)B - (k+l)C = B + A - kA - lA + kB + lC + kA - B + lB - kC - lC = (1-l)A + (k+l)B - kC = \angle FEC$, so $\angle FGH + \angle HEF = \angle FEC + \angle HEF = \pi$, so $GHEF$ is a cyclic quadrilateral.

We have $\angle GHI = \pi - \angle AHG - \angle IHC = \pi - (\pi - A - \angle HGA) - \angle IHC = A + \angle HGA - \angle IHC = A - kA + (1-l)B + (k+l)C - lA - (1-k-l)B - kC = (1-k-l)A + kB + lC = \angle GFB = \angle GFI$, so $GHFI$ is a cyclic quadrilateral.

There exists one and only one circle passing through three no-collinear points. So the circle $(GEFD)$ (call it by (ω)) is the only circle that passes through G, E and F. But $GHFE$ is cyclic quadrilateral. So H is also a point on (ω). (ω) is the only circle that passes through G, H and F. But $GHFI$ is a cyclic quadrilateral. So I also lies on (ω). Then six points D, E, F, G, H and I lies on the circle (ω). We easily can show that $\angle DIB = (k+l)A - kB + (1-l)C$ □

The construction of a group circle in Theorem 9.1 equivalent to the construction in Theorem 9.2 as follows:

Theorem 9.2 ([31]). *Let ABC be a triangle and P be a point in the plane. Three lines PA, PB, PC meet three circles (BCP), (CAP), (ABP) again at A', B', C' respectively. Let points F, I be chosen on side BC, points H, E be chosen on side CA, points D, G be chosen on side AB so that DE, EF, FG, GH, HI parallel to $C'A$, CA', $A'B$, AB', $B'C$ respectively, then six points D, E, F, G, H, I lie on a circle and ID parallel to BC'.*

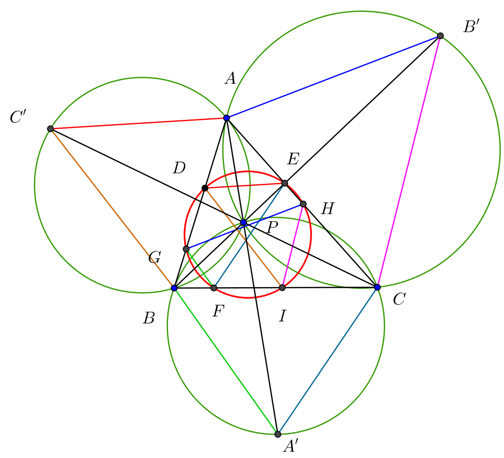

FIGURE 15. Theorem 9.2

We omit the proof of Theorem 9.2.

Theorem 9.3. *Let ABC be a triangle with angles A, B, C. Let points D and G be chosen on side AB, points I and F be chosen on side BC, points E and H be chosen on side CA so that:*

$$\begin{cases} \angle EDA = \frac{2B}{3} + \frac{C}{3} \\ \angle FEC = \frac{A}{3} + \frac{2B}{3} \\ \angle GFB = \frac{A}{3} + \frac{2C}{3} \\ \angle HGA = \frac{B}{3} + \frac{2C}{3} \\ \angle IHC = \frac{2A}{3} + \frac{B}{3} \end{cases}$$

1. Then six points D, E, F, G, H, I lie on a circle and $\angle DIB = \frac{2A}{3} + \frac{C}{3}$
2. Let $HI \cap FG \equiv A_1$, $DE \cap HI \equiv B_1$, $FG \cap DE \equiv C_1$ then $A_1B_1C_1$ be an equilateral triangle. Two triangles $A_1B_1C_1$ and ABC are perspective.
3. The triangle $A_1B_1C_1$ and the Morley triangle are homothetic.

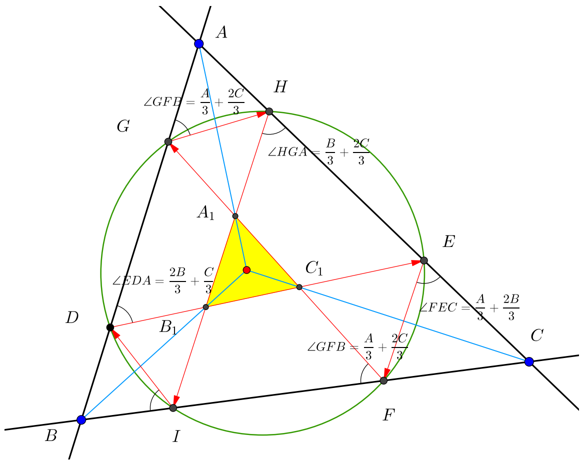

FIGURE 16. Theorem 9.3

Proof. 1. We have:

$$\begin{cases} \angle EDA = \frac{2B}{3} + \frac{C}{3} \\ \angle FEC = \frac{A}{3} + \frac{2B}{3} \\ \angle GFB = \frac{A}{3} + \frac{2C}{3} \\ \angle HGA = \frac{B}{3} + \frac{2C}{3} \\ \angle IHC = \frac{2A}{3} + \frac{B}{3} \end{cases} \Leftrightarrow \begin{cases} \angle EDA = 0.A + \frac{2}{3}.B + (1 - 0 - \frac{2}{3})C \\ \angle FEC = (1 - \frac{2}{3})A + (0 + \frac{2}{3})B - 0.C \\ \angle GFB = (1 - 0 - \frac{2}{3})A + 0.B + \frac{2}{3}.C \\ \angle HGA = -0.A + (1 - \frac{2}{3})B + (0 + \frac{2}{3})C \\ \angle IHC = \frac{2}{3}A + (1 - 0 - \frac{2}{3})B + 0.C \end{cases}$$

This is one case of theorem 1 with $k = 0, l = \frac{2}{3}$ so six points D, E, F, G, H, I lie on a circle and $\angle DIB = \frac{2A}{3} + \frac{C}{3}$.

2. We have $\angle GHA_1 = \angle HGI + A - \angle IHC$, $\angle A_1GH = \angle GFB + B - \angle HGI$ so $\angle B_1A_1C_1 = \angle HA_1G = \pi - \angle GHA_1 - \angle A_1GH = \pi - \angle HGA - A + \angle IHC - \angle GFB - B + \angle HGA = \pi - A - B + \angle IHC - \angle GFB = C + \frac{2A}{3} + \frac{B}{3} - \frac{A}{3} - \frac{2C}{3} = \frac{A+B+C}{3} = \frac{\pi}{3}$. Similarly we have $\angle C_1B_1A_1 = \angle A_1C_1B_1 = \frac{\pi}{3}$ therefore $A_1B_1C_1$ be the equilateral triangle.

Applying the Pascal theorem for hexagon $DEHIFG$ then we have DE, I, FG meet IF, GD, EH at three collinear points. Applying the Desargues's theorem we have that $A_1B_1C_1$ and ABC are perspective.

3. Considering the Morley triangle $M_aM_bM_c$ of $\triangle ABC$ we have $\angle(M_bMc, AB) = -\frac{1}{3}(A - B) + \frac{\pi}{3} = -\frac{A}{3} + \frac{B}{3} + \frac{A+B+C}{3} = \frac{2B}{3} + \frac{C}{3} = \angle(DE, AB)$ (see proposition 5, [30]), so $B_1C_1 \parallel M_bM_c$. Similarly we have $C_1A_1 \parallel M_cM_a$ and $A_1B_1 \parallel M_aM_b$. So the triangle $A_1B_1C_1$ is homothetic to the Morley triangle. □

Notation as Theorem 9.3, let the circle through D, E, F, G, H, I is (Ω) we have some nice properties in Theorem 9.4.

Theorem 9.4. *Let A', B', C' be the midpoints of arcs $\widehat{DE}$, $\widehat{FG}$, $\widehat{HI}$ of (Ω) so that A', B', C' on the half plane specified by DE, FG, HI not containing (or containing) A_1, B_1, C_1 respectively, then*
1. *$A'B'C'$ is the equilateral triangle.*
2. *Two triangle ABC and $A'B'C'$ are perspective (see Figure 17).*

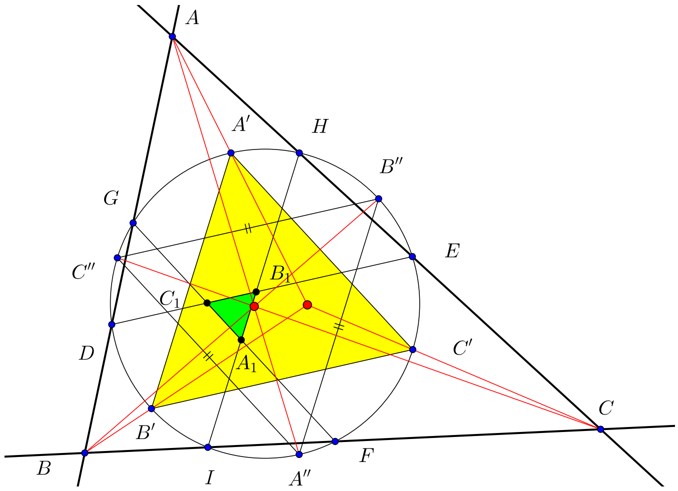

FIGURE 17. Theorem 9.4

We omit the proof of the following easy Lemma:

Lemma 9.5. *Let A, B, C, D be four points lie on a circle. AD meets BC at T. M be the midpoint of arc AB then $\frac{\sin \angle DTM}{\sin \angle MTC} = \frac{DM}{MC}$.*

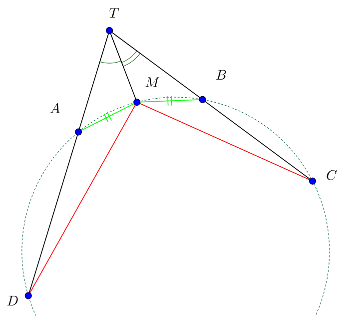

FIGURE 18. Lemma 9.5

Proof. Since $A_1B_1C_1$ be the equilateral triangle. Let O be the center of circle $(DEFGHI)$, we have $\angle A'OB' = \angle B'OC = \angle C'OA' = \frac{2\pi}{3}$ and $OA' = OB' = OC'$ then $A'B'C'$ be the equilateral triangle.

Applying Lemma 9.5 we have: $\frac{\sin \angle CAA'}{\sin \angle A'AB} = \frac{HA'}{A'G}$, $\frac{\sin \angle ABB'}{\sin \angle B'BC} = \frac{DB'}{B'I}$ and $\frac{\sin \angle BCC'}{\sin \angle C'CA} = \frac{FC'}{C'E}$.

Since A', B', C' are the midpoints of arc $\widehat{DE}$, $\widehat{HI}$, $\widehat{FG}$ respectively, so $A'D = A'E$, $B'G = B'F$ and $C'I = C'H$. On the other hand $A'B'C'$ is the equilateral triangle then $B'D = C'E$, $A'G = C'F$ and $A'H = B'I$ therefore:

$$\frac{\sin\angle CAA'}{\sin\angle A'AB}.\frac{\sin\angle ABB'}{\sin\angle B'BC}.\frac{\sin\angle BCC'}{\sin\angle C'CA}=\frac{HA'}{A'G}.\frac{DB'}{B'I}.\frac{FC'}{C'E}=1$$

.

By the converse of trigonometric version of Cevas theorem the lines AA', BB', CC' are concurrent. Similarly AA'', BB'', CC'' are concurrent.

□

Remark: $\triangle A'B'C'$, $\triangle A''B''C''$, $\triangle A_1B_1C_1$ *are homothetic to the Morley triangle.*

Theorem 9.6. *Let ABC be a triangle, let two points D, G be chosen on AB, points I, F be chosen on BC, points E, H be chosen on CA such that: $\angle EDA = \angle CFE = \angle GFB = \angle AHG = \angle IHC = \angle IDB = \frac{\pi}{3}$ then six points D, E, F, G, H, I lie on a circle.*

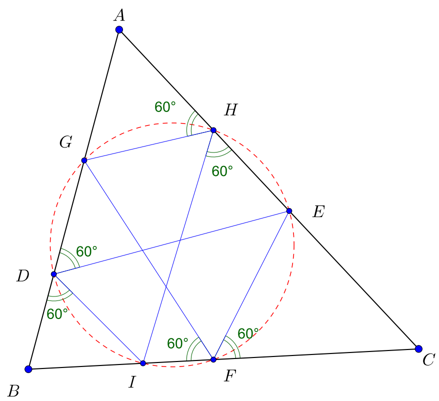

FIGURE 19. Theorem 9.6

Proof. Since $\angle CFE = \frac{\pi}{3}$ we have $\angle FEC = \pi - C - \frac{\pi}{3} = A + B + C - \frac{A+B+C}{3} - C = \frac{2A}{3} + \frac{2B}{3} - \frac{C}{3}$. Similarly $\angle HGA = -\frac{A}{3} + \frac{2B}{3} + \frac{2C}{3}$. So we have:

$$\begin{cases}\angle EDA = \frac{\pi}{3}\\ \angle FEC = \frac{2A}{3} + \frac{2B}{3} - \frac{C}{3}\\ \angle GFB = \frac{\pi}{3}\\ \angle HGA = -\frac{A}{3} + \frac{2B}{3} + \frac{2C}{3}\\ \angle IHC = \frac{\pi}{3}\end{cases} \Leftrightarrow \begin{cases}\angle EDA = \frac{1}{3}A + \frac{1}{3}B + (1 - \frac{1}{3} - \frac{1}{3})C\\ \angle FEC = (1 - \frac{1}{3})A + (\frac{1}{3} + \frac{1}{3})B - \frac{1}{3}C\\ \angle GFB = (1 - \frac{1}{3} - \frac{1}{3})A + \frac{1}{3}B + \frac{1}{3}C\\ \angle HGA = -\frac{1}{3}A + (1 - \frac{1}{3})B + (\frac{1}{3} + \frac{1}{3})C\\ \angle IHC = \frac{1}{3}A + (1 - \frac{1}{3} - \frac{1}{3})B + \frac{1}{3}C\end{cases}$$

This is a case of theorem 1 with $k = l = \frac{1}{3}$ therefore six points D, E, F, G, H, I lie on a circle. □

We also omit the proof of the following Theorem:

Theorem 9.7. *Let ABC be a triangle with the first (or the second) Fermat point F, let points D and G be chosen on side AB, points I and K be chosen on side BC, points E and H be chosen on side CA so that $\angle EDA = \angle GKB = \angle IHC = \frac{\pi}{3}$ and DE, KG, HI through the first (or second) Fermat point (Figure 20).*

1. Then six points D, E, K, G, H, I lie on a circle.

2. Let A', B', C' be the centers of three circles (FHG), (FDI), (FKE) respectively, then $A'B'C'$ is an equilateral triangle.

3. Two triangle ABC and $A'B'C'$ are perspective, the perspector is the first (or the second) Napoleon point.

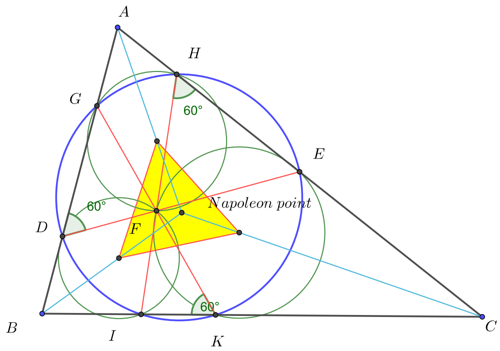

FIGURE 20. Theorem 9.7

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KIEN XUONG, THAI BINH, VIET NAM
E-mail address: daothanhoai@hotmail.com