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ON WEAKLY LANDSBERG FOURTH ROOT (α, β)-METRICS

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ABSTRACT. In this paper, we consider one of open problems in Finsler geometry which have been proposed by Z. Shen about the existence of non-Berwaldian Landsberg 4-th root metric. We show that every weakly Landsberg (and then Landsberg) 4-th root (α, β)- metric is a Berwald metric.

1. INTRODUCTION

In Finsler geometry, there is a non-Riemannian quantity which is determined by the Busemann-Hausdorff volume form, that is the so called distortion $\tau = \tau(x, y)$. The vertical differential of τ on each tangent space gives rise to the mean Cartan torsion $\mathbf{I} := \tau_{y^k} dx^k$. The horizontal derivative of $\mathbf{I}$ along geodesics is called the mean Landsberg curvature $\mathbf{J} := \mathbf{I}_{|k} y^k$. Finsler metrics with $\mathbf{J} = 0$ are called weakly Landsberg metrics. The mean Landsberg curvature $\mathbf{J}_y$ is the rate of change of $\mathbf{I}_y$ along geodesics for any $y \in T_x M_0$. It has been shown that on a weakly Landsberg manifold, the volume function $V = Vol(x)$ is a constant [5]. The constancy of the volume function is required to establish a Gauss-Bonnet theorem for Finsler manifolds [4]. There is an induced Riemannian metric of Sasaki type on TM_0. In [11], Shen showed that if $\mathbf{J} = 0$, then all the slit tangent spaces $T_x M_0$ are minimal in TM_0. Some rigidity problems also lead to weakly Landsberg manifolds. For example, for a closed Finsler manifold of non-positive flag curvature, if the S-curvature is a constant, then it is weakly Landsbergian [12]. Apparently, weakly Landsberg Finsler manifolds deserve further investigation.

Let (M, F) be an n-dimensional Finsler manifold, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. Let $F = F(x, y)$ be a scalar function on TM defined by $F = \sqrt[m]{A}$, where $A := a_{i_1 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}$ and $a_{i_1 \ldots i_m}$ is symmetric in all its indices. F is called an m-th root Finsler metric. The theory of m-th root metrics has been developed by Shimada [14], and applied to Biology as an ecological metric by Antonelli [1]. The fourth root metrics $F = \sqrt[4]{a_{ijkl}(x) y^i y^j y^k y^l}$ are called the quartic metric. It is remarkable that, the special 4-th root metric $F = \sqrt[4]{(y^1)^4 + \cdots + (y^4)^4}$ represents the historic primary Finsler fundamental function considered by Riemann in his "Habilitation address". The recent attempts of modeling relativity

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based on a palette of physical models relying on the 4-th root Finsler metrics. The 4-th root metric $F = \sqrt[4]{y^1y^2y^3y^4}$ is called Berwald-Moór metric which plays an important role in theory of space-time structure, gravitation and general relativity [2] [3]. For more progress, see [6], [10], [15], [17], [18], [19], [22] and [23].

In [8], Matsumoto proved that every 3-th root Finsler metric with vanishing Landsberg curvature is a Berwald metric. But he had not any progress for the class of 4-th root metrics with vanishing Landsberg curvature. Since the study of the class of 4-th root metrics becomes urgent necessity for the Finsler geometry as well as for theoretical physics, then in [13] Shen introduced the following open problem:

Is there any non-Berwaldian Landsberg 4-th root Finsler metric?

The class of weakly Landsberg metrics contains the class of Landsberg metrics as a special case. In this paper, we are going to prove the following.

Theorem 1.1. *Let $F = F(x,y)$ be a 4-th root (α,β)-metric on a manifold M of dimension $n \geq 3$. Then F is a weakly Landsberg metric if and only if it is a Berwald metric.*

2. Preliminaries

Let M be a n-dimensional C^∞ manifold and $TM = \bigcup_{x\in M} T_xM$ the tangent bundle. Let (M,F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y$ on T_xM is called fundamental tensor

$$\mathbf{g}_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y+su+tv)\Big]|_{s=t=0}, \quad u,v \in T_xM.$$

For $y \in T_xM_0$, define $\mathbf{C}_y : T_xM\otimes T_xM\otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]_{t=0} = \frac{1}{4}\frac{\partial^3}{\partial r\partial s\partial t}\Big[F^2(y+ru+sv+tw)\Big]_{r=s=t=0},$$

where $u,v,w \in T_xM$. By definition, $\mathbf{C}_y$ is a symmetric trilinear form on T_xM. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by $\mathbf{I}_y(u) = \sum_{i=1}^n g^{ij}(y)\mathbf{C}_y(u,\partial_i,\partial_j)$, where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. Thus, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Deicke's theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

For a vector $y \in T_xM$, the Landsberg and mean Landsberg curvature of F can be defined by following

$$\mathbf{L}_y(u,v,w) := \frac{d}{dt}\Big[\mathbf{C}_{\dot\sigma(t)}\Big(U(t),V(t),W(t)\Big)\Big]|_{t=0}, \quad \mathbf{J}_y(u) := \frac{d}{dt}\Big[\mathbf{I}_{\dot\sigma(t)}\Big(U(t)\Big)\Big]|_{t=0},$$

where $\sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U(t),V(t),W(t)$ are linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$. In this case, the Landsberg curvature (resp. mean Landsberg curvature) measures the rate of change of the Cartan (resp. mean Cartan) torsion along geodesics.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$, where $G^i = G^i(x, y)$ are local functions on TM_0 satisfying $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$, $\lambda > 0$, and given by

$$G^i = \frac{1}{4} g^{il} \Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \Big].$$

The vector field $\mathbf{G}$ is called the associated spray to (M, F). The projection of an integral curve of the spray $\mathbf{G}$ is called a geodesic in M.

Define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} = \frac{\partial^2 N^i_j}{\partial y^k \partial y^l}.$$

$\mathbf{B}_y(u, v, w)$ is symmetric in u, v and w. $\mathbf{B}$ is called the Berwald curvature. F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following.

Lemma 3.1. *Let $F = \sqrt[4]{A}$ be a 4-th root metric on a manifold M, where $A = a_{ijkl} y^i y^j y^k y^l$. If $dim(M) \geq 3$ and F is a function of a non-degenerate quadratic form $\alpha^2 = \alpha_{ij}(x) y^i y^j$ and a one-form $\beta = \beta_i(x) y^i$ which is homogeneous in α and β of degree one, then it is written in the form $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$, where c_1, c_2 and c_3 are real constants.*

Proof. By the same argument used in [9] for the cubic metrics admitting (α, β)-metrics, we get the proof. □

Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on a manifold M (see [16], [22] and [23]). Let $G^i = G^i(x, y)$ and $G^i_\alpha = G^i_\alpha(x, y)$ denote the coefficients of F and α, respectively, in the same coordinate system. For an (α, β)-metric, let us define $b_{i|j}$ by $b_{i|j}\theta^j := db_i - b_j \theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik} dx^k$ denote the Levi-Civita connection form of α. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r_{i0} := r_{ij} y^j, \ r_{00} := r_{ij} y^i y^j, \ r_j := b^i r_{ij},$$
$$s_{i0} := s_{ij} y^j, \quad s_j := b^i s_{ij}, \quad s^i{}_j = a^{im} s_{mj}, \quad s^i{}_0 = s^i{}_j y^j, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j.$$

where $a^{ij} = (a_{ij})^{-1}$ and $b^i := a^{ij} b_j$. Put

$$Q := \frac{\phi'}{\phi - s\phi'}, \qquad \Delta := 1 + sQ + (b^2 - s^2)Q', \qquad \Theta := \frac{Q - sQ'}{2\Delta},$$
$$\Psi := \frac{Q'}{2\Delta} = \frac{\phi''}{2\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]}.$$

By definition, we have

$$G^i = G^i_\alpha + \alpha Q {s^i}_0 + (r_{00} - 2Q\alpha s_0)(\alpha^{-1}\Theta y^i + \Psi b^i). \qquad (1)$$

where

$$P := \alpha^{-1}\Theta\Big[-2Q\alpha s_0 + r_{00}\Big], \quad Q^i := \alpha Q {s^i}_0 + \Psi\Big[-2Q\alpha s_0 + r_{00}\Big]b^i.$$

Clearly, if β is parallel with respect to α, that is $r_{ij} = 0$ and $s_{ij} = 0$, then $P = 0$ and $Q^i = 0$. In this case, $G^i = G^i_\alpha$ are quadratic in y. In this case, F is a Berwald metric.

Put

$$\Phi := -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q''.$$

By a direct computation, we can obtain a formula for the mean Cartan torsion of (α, β)- metrics as follows

$$I_i = -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i - sy_i). \qquad (2)$$

According to Deickes theorem, a Finsler metric is Riemannian if and only if $\mathbf{I} = 0$. Clearly, an (α, β)-metric $F = \alpha\phi(s)$ is Riemannian if and only if $\Phi = 0$. Then by Theorem 1.1 in [7], we get the following.

Lemma 3.2. ([7]) *Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be an almost regular (α, β)-metric on an n-dimensional manifold $M(n \geq 3)$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$. Suppose that β is not parallel with respect to α and $\phi \neq k_1\sqrt{1 + k_2s^2}$ for any constants k_1 and k_2. Let $b(x) := ||\beta_x||_\alpha \neq 0$. Then F is a weakly Landsberg metric if and only if β satisfies*

$$r_{ij} = k(b^2 a_{ij} - b_ib_j), \qquad s_{ij} = 0, \qquad (3)$$

where $k = k(x)$ is a scalar function on M and $\phi = \phi(s)$ satisfies

$$\Phi = \frac{\lambda}{\sqrt{b^2 - s^2}}\Delta^{\frac{3}{2}}, \qquad (4)$$

where λ is a constant.

Proof of Theorem 1.1: For an (α, β)-metric $F = \alpha\phi(s)$, the mean Landsberg curvature is given by

$$\begin{aligned} J_i = -\frac{1}{2\Delta\alpha^4}\Bigg[&\frac{2\alpha^2}{b^2 - s^2}\Big[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\Big](r_0 + s_0)h_i \\ &+ \frac{\alpha}{b^2 - s^2}(\Psi_1 + s\frac{\Phi}{\Delta})(r_{00} - 2\alpha Q s_0)h_i + \alpha\Big[-\alpha Q' s_0 h_i + \alpha Q(\alpha^2 s_i - y_i s_0) \\ &+ \alpha^2\Delta s_{i0} + \alpha^2(r_{i0} - 2\alpha Q s_i) - (r_{00} - 2\alpha Q s_0)y_i\Big]\frac{\Phi}{\Delta}\Bigg]. \qquad (5)\end{aligned}$$

where

$$\Psi_1 := \sqrt{b^2 - s^2}\Delta^{\frac{1}{2}}\Big[\frac{\sqrt{b^2 - s^2}\Phi}{\Delta^{\frac{3}{2}}}\Big]', \quad h_i := b_i - \alpha^{-1}sy_i.$$

By Lemmas 3.1 and 3.2, the mean Landsberg curvature of a 4-th root metric is given by following

$$J_i := \lambda(Ab_i + By_i), \tag{6}$$

where A and B are listed in Appendix, and

$$\lambda := 1/T\Big[2c_1c_2(b^2\alpha^2+2\beta^2)\alpha^2+(12c_1c_3-{c_2}^2)b^2\alpha^2\beta^2+2b^2\beta^4c_2c_3+4\alpha^4{c_1}^2-8\beta^4c_1c_3+3\beta^4{c_2}^2\Big]^3.$$

where $T := -(2c_1\alpha^2+c_2\beta^2)$. By assumption, $J_i = 0$. Contracting it with b^i implies that

$$k\Big[d_8\alpha^{14}+d_7\beta^2\alpha^{12}+d_6\beta^4\alpha^{10}+d_5\beta^6\alpha^8+d_4\beta^8\alpha^6+d_3\beta^{10}\alpha^4+d_2\beta^{12}\alpha^2+d_1\beta^{14}\Big]=0, \tag{7}$$

where d_i $(i=1,\cdots,8)$ are given by following

$$\begin{aligned}
d_8 :=& -96{c_1}^4b^6(4c_1c_3-{c_2}^2)(b^2c_2+2c_1),\\
d_7 :=& \ 48\,b^4{c_1}^3(4c_1c_3-{c_2}^2)\Big(nb^4{c_2}^2+24b^4c_1c_3-6b^4{c_2}^2+4nb^2c_1c_2+14nb^2c_1c_2+4{c_1}^2+28{c_1}^2\Big),\\
d_6 :=& \ 8b^2{c_1}^2(4c_1c_3-{c_2}^2)\Big(60n\,b^6c_1c_2c_3-4nb^6c_2^3+144b^6c_1c_2c_3-7b^6c_2^3+120nb^4c_1^2c_3-576b^4{c_1}^2c_3\\
&+318b^4c_1{c_2}^2-12nb^2{c_1}^2c_2+96b^2{c_1}^2c_2-56n{c_1}^3-272{c_1}^3\Big),\\
d_5 :=& \ 4c_1(4c_1c_3-{c_2}^2)\Big(288nb^8c_1^2{c_3}^2-20nb^8c_1{c_2}^2c_3+nb^8{c_2}^4-288b^8{c_1}^2{c_3}^2+328b^8c_1{c_2}^2c_3-8b^8{c_2}^4\\
&-104nb^6c_1^2c_2c_3+28nb^6c_1c_2^3-704b^6c_1^2c_2c_3+334b^6c_1c_2^3-704nb^4c_1^3c_3+32nb^4c_1^2{c_2}^2+2176b^4c_1^3c_3\\
&-940b^4{c_1}^2{c_2}^2-152nb^2{c_1}^3c_2-848b^2{c_1}^3c_2+64n{c_1}^4+256{c_1}^4\Big),\\
d_4 :=& \ 8c_1(4c_1c_3-{c_2}^2)\Big(36nb^8c_1c_2c_3^2-3nb^8c_2^3c_3+60b^8c_1c_2{c_3}^2+24b^8{c_2}^3c_3-504nb^6{c_1}^2{c_3}^2-5nb^6{c_2}^4\\
&+94nb^6c_1{c_2}^2c_3+696b^6{c_1}^2{c_3}^2-392b^6c_1c_2^2c_3+40b^6{c_2}^4-212nb^4c_1^2c_2c_3-nb^4c_1c_2^3+616b^4{c_1}^2c_2c_3\\
&-370b^4c_1{c_2}^3+328nb^2{c_1}^3c_3-94nb^2{c_1}^2{c_2}^2-1040b^2{c_1}^3c_3+92b^2{c_1}^2{c_2}^2+64n{c_1}^3c_2+256{c_1}^3c_2\Big),\\
d_3 :=& -4(4c_1c_3-{c_2}^2)\Big(8nb^8c_1{c_2}^2{c_3}^2-nb^8{c_2}^4c_3-64b^8c_1{c_2}^2{c_3}^2+8b^8{c_2}^4c_3+248nb^6{c_1}^2c_2{c_3}^2-nb^6c_2^5\\
&-10nb^6c_1{c_2}^3c_3+104b^6{c_1}^2c_2{c_3}^2+80b^6c_1{c_2}^3c_3-1264nb^4{c_1}^3{c_3}^2+500nb^4{c_1}^2{c_2}^2c_3-832b^4{c_1}^2{c_2}^2c_3\\
&+8b^6{c_2}^5-25nb^4c_1{c_2}^4+2192b^4{c_1}^3{c_3}^2+200b^4c_1{c_2}^4-600nb^2{c_1}^3c_2c_3+98nb^2c_1^2{c_2}^3+1584b^2c_1^3c_2c_3\\
&-768{c_1}^4c_3-484b^2{c_1}^2{c_2}^3+192n{c_1}^4c_3-144n{c_1}^3{c_2}^2-192{c_1}^3{c_2}^2\Big),
\end{aligned}$$

$$d_2 := -8(4c_1c_3 - c_2{}^2)\Big(nb^8c_2{}^3c_3{}^2 + b^8c_2{}^3c_3{}^2 - 2nb^6c_1c_2{}^2c_3{}^2 + 3nb^6c_2{}^4c_3 + 70b^6c_1c_2{}^2c_3{}^2 - 6b^6c_2{}^4c_3$$
$$- 128nb^4c_1{}^2c_2c_3{}^2 + 15nb^4c_1c_2{}^3c_3 + 2nb^4c_2{}^5 + 88b^4c_1{}^2c_2c_3{}^2 + 6b^4c_1c_2{}^3c_3 - 7b^4c_2{}^5 + 336nb^2c_1{}^3c_3{}^2$$
$$- 222nb^2c_1{}^2c_2^2c_3 + 20nb^2c_1c_2{}^4 - 672b^2c_1^3c_3{}^2 + 348b^2c_1{}^2c_2{}^2c_3 - 88b^2c_1c_2^4 + 96nc_1{}^3c_2c_3 - 40nc_1^2c_2{}^3$$
$$- 384c_1{}^3c_2c_3 + 32c_1{}^2c_2{}^3\Big),$$

$$d_1 := 4(4c_1c_3 - c_2{}^2)\Big(2nb^6c_2{}^3c_3{}^2 + 2b^6c_2{}^3c_3{}^2 + 4nb^4c_1c_2{}^2c_3{}^2 + 5nb^4c_2{}^4c_3 + 76b^4c_1c_2{}^2c_3{}^2 - 4nb^4c_2{}^4c_3$$
$$- 80b^2c_1{}^2c_2\, c_3{}^2 + 26nb^2c_1c_2{}^3c_3 + 3n\, b^2c_2{}^5 + 160b^2c_1{}^2c_2\, c_3{}^2 + 44b^2c_1c_2{}^3c_3 - 6b^2c_2{}^5 + 128nc_1{}^3c_3{}^2$$
$$- 112nc_1{}^2c_2{}^2c\_3 + 24nc_1c_2{}^4 - 256c_1{}^3c_3{}^2 + 320c_1{}^2c_2{}^2c_3 - 48c_1c_2{}^4\Big).$$

If $k \neq 0$, then by (7) we get

$$d_1\beta^{14} = f\alpha^2, \tag{8}$$

where $f = f(x,y)$ is a homogeneous scalar function of degrees 12 with respect to y. But, (8) contradict with the positive-definiteness of α. Thus $k = 0$. Putting it in (3) implies $r_{ij} = 0$. □

Remark 3.1. By a simple calculation, for the 4-th root (α, β)-metric $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$, we get the following

$$\Delta = \frac{2b^2c_2c_3s^4 + 12b^2c_1c_3s^2 - b^2c_2{}^2s^2 - 8c_1c_3s^4 + 3c_2{}^2s^4 + 2b^2c_1c_2 + 4c_1c_2s^2 + c_1{}^2}{\Big(c_2s^2 + 2c_1\Big)^2},$$

$$\Phi = \frac{-2s}{(c_2s^2 + 2c_1)^4}\Big(- 8nb^2c_1c_2c_3{}^2s^6 + 2nb^2c_2{}^3c_3s^6 - 8b^2c_1c_2c_3{}^2s^6 + 2b^2c_2{}^3c_3s^6 - 48nb^2c_1{}^2c_3{}^2s^4$$

$$+16nb^2c_1c_2{}^2c_3s^4 - nb^2c_2{}^4s^4 + 32nc_1{}^2c_3{}^2s^6 - 20nc_1c_2{}^2c_3s^6 + 3nc_2{}^4s^6 + 48b^2c_1{}^2c_3{}^2s^4 - 20b^2c_1c_2{}^2c_3s^4$$

$$+2b^2c_2{}^4s^4 - 64c_1{}^2c_3{}^2s^6 + 16c_1c_2{}^2c_3s^6 - 8nb^2c_1{}^2c_2c_3s^2 + 2nb^2c_1c_2{}^3s^2 - 16nc_1{}^2c_2c_3s^4 + 4nc_1c_2{}^3s^4$$

$$+40b^2c_1{}^2c_2c_3s^2 - 10b^2c_1c_2{}^3s^2 - 64c_1{}^2c_2c_3s^4 + 16c_1c_2{}^3s^4 - 16nc_1{}^3c_3s^2 + 4nc_1{}^2c_2{}^2s^2 + 48b^2c_1{}^3c_3\Big).$$

Thus $\sqrt{b^2 - s^2}\Phi/\sqrt{\Delta^3}$ is not a constant which shows that (4) is not hold. Then β is parallel with respect to α and F reduces to a Berwald metric.

4. APPENDIX

$$
\begin{aligned}
A :=& -4k \Big(12n\alpha^{12}b^6\beta^2c_1^3c_2^4 - 48n\alpha^{12}b^6\beta^2c_1^4c_2^2c_3 - 480n\alpha^{10}b^6\beta^4c_1^4c_2c_3^2 + 152n\alpha^{10}b^6\beta^4c_1^3c_2^3c_3 - 8n\alpha^{10}b^6\beta^4c_1^2c_2^5 - 1152n\alpha^8b^6\beta^6c_1^4c_3^3 \\
&+ 368n\alpha^8b^6\beta^6c_1^3c_2^2c_3^2 - 24n\alpha^8b^6\beta^6c_1^2c_2^4c_3 + n\alpha^8b^6\beta^6c_1c_2^6 - 288n\alpha^6b^6\beta^8c_1^3c_2c_3^3 + 96n\alpha^6b^6\beta^8c_1^2c_2^3c_3^2 - 6n\alpha^6b^6\beta^8c_1c_2^5c_3 - 24\alpha^{14}b^6c_1^4c_2^3 \\
&+ 32n\alpha^4b^6\beta^{10}c_1^2c_2^2c_3^3 - 12n\alpha^4b^6\beta^{10}c_1c_2^4c_3^2 + n\alpha^4b^6\beta^{10}c_2^6c_3 + 8n\alpha^2b^6\beta^{12}c_1c_2^3c_3^3 - 2n\alpha^2b^6\beta^{12}c_2^5c_3^2 + 96\alpha^{14}b^6c_1^5c_2c_3 - 1152\alpha^{12}b^6\beta^2c_1^5c_3^2 \\
&+ 576\alpha^{12}b^6\beta^2c_1^4c_2^2c_3 - 72\alpha^{12}b^6\beta^2c_1^3c_2^4 - 1152\alpha^{10}b^6\beta^4c_1^4c_2c_3^2 + 344\alpha^{10}b^6\beta^4c_1^3c_2^3c_3 - 14\alpha^{10}b^6\beta^4c_1^2c_2^5 + 1152\alpha^8b^6\beta^6c_1^4c_3^3 - 8\alpha^4b^6\beta^{10}c_2^6c_3 \\
&- 1600\alpha^8b^6\beta^6c_1^3c_2^2c_3^2 + 360\alpha^8b^6\beta^6c_1^2c_2^4c_3 - 8\alpha^8b^6\beta^6c_1c_2^6 - 480\alpha^6b^6\beta^8c_1^3c_2c_3^3 - 72\alpha^6b^6\beta^8c_1^2c_2^3c_3^2 + 48\alpha^6b^6\beta^8c_1c_2^5c_3 - 256\alpha^4b^6\beta^{10}c_1^2c_2^2c_3^3 \\
&+ 96\alpha^4b^6\beta^{10}c_1c_2^4c_3^2 + 8\alpha^2b^6\beta^{12}c_1c_2^3c_3^3 - 2\alpha^2b^6\beta^{12}c_2^5c_3^2 - 192n\alpha^{12}b^4\beta^2c_1^5c_2c_3 + 48n\alpha^{12}b^4\beta^2c_1^4c_2^3 - 960n\alpha^{10}b^4\beta^4c_1^5c_3^2 + 192n\alpha^{10}b^4\beta^4c_1^4c_2^2c_3 \\
&+ 12n\alpha^{10}b^4\beta^4c_1^3c_2^4 - 64n\alpha^8b^4\beta^6c_1^4c_2c_3^2 - 64n\alpha^8b^4\beta^6c_1^3c_2^3c_3 + 20n\alpha^8b^4\beta^6c_1^2c_2^5 + 2880n\alpha^6b^4\beta^8c_1^4c_3^3 - 1392n\alpha^6b^4\beta^8c_1^3c_2^2c_3^2 + n\alpha^4b^4\beta^{10}c_2^7 \\
&+ 204n\alpha^6b^4\beta^8c_1^2c_2^4c_3 - 9n\alpha^6b^4\beta^8c_1c_2^6 + 704n\alpha^4b^4\beta^{10}c_1^3c_2c_3^3 - 192n\alpha^4b^4\beta^{10}c_1^2c_2^3c_3^2 + 16n\alpha^2b^4\beta^{12}c_1^2c_2^2c_3^3 + 16n\alpha^2b^4\beta^{12}c_1c_2^4c_3^2 - 5n\alpha^2b^4\beta^{12}c_2^6c_3 \\
&+ 192\alpha^{14}b^4c_1^6c_3 - 48\alpha^{14}b^4c_1^5c_2^2 - 576\alpha^{12}b^4\beta^2c_1^5c_2c_3 + 144\alpha^{12}b^4\beta^2c_1^4c_2^3 + 3456\alpha^{10}b^4\beta^4c_1^5c_3^2 - 3120\alpha^{10}b^4\beta^4c_1^4c_2^2c_3 + 564\alpha^{10}b^4\beta^4c_1^3c_2^4 \\
&+ 1664\alpha^8b^4\beta^6c_1^4c_2c_3^2 - 1696\alpha^8b^4\beta^6c_1^3c_2^3c_3 + 320\alpha^8b^4\beta^6c_1^2c_2^5 - 4416\alpha^6b^4\beta^8c_1^4c_3^3 + 2928\alpha^6b^4\beta^8c_1^3c_2^2c_3^2 - 744\alpha^6b^4\beta^8c_1^2c_2^4c_3 + 72\alpha^6b^4\beta^8c_1c_2^6 \\
&- 64\alpha^4b^4\beta^{10}c_1^3c_2c_3^3 + 144\alpha^4b^4\beta^{10}c_1^2c_2^3c_3^2 - 8\alpha^4b^4\beta^{10}c_2^7 + 304\alpha^2b^4\beta^{12}c_1^2c_2^2c_3^3 - 92\alpha^2b^4\beta^{12}c_1c_2^4c_3^2 + 4\alpha^2b^4\beta^{12}c_2^6c_3 - 192n\alpha^{12}b^2\beta^2c_1^6c_3 \\
&+ 48n\alpha^{12}b^2\beta^2c_1^5c_2^2 - 96n\alpha^{10}b^2\beta^4c_1^5c_2c_3 + 24n\alpha^{10}b^2\beta^4c_1^4c_2^3 + 1856n\alpha^8b^2\beta^6c_1^5c_3^2 - 640n\alpha^8b^2\beta^6c_1^4c_2^2c_3 + 44n\alpha^8b^2\beta^6c_1^3c_2^4 + 1632n\alpha^6b^2\beta^8c_1^4c_2c_3^2 \\
&- 48n\alpha^6b^2\beta^8c_1^3c_2^3c_3 + 18n\alpha^6b^2\beta^8c_1^2c_2^5 - 21n\alpha^4b^2\beta^{10}c_1^4c_3^3 + 18n\alpha^4b^2\beta^{10}c_1^3c_2^2c_3^2 - 36n\alpha^4b^2\beta^{10}c_1^2c_2^4c_3 + 16n\alpha^4b^2\beta^{10}c_1c_2^6 - 30n\alpha^2b^2\beta^{12}c_1^3c_2c_3^3 \\
&+ 184n\alpha^2b^2\beta^{12}c_1^2c_2^3c_3^2 - 14n\alpha^2b^2\beta^{12}c_1c_2^5c_3 - 3n\alpha^2b^2\beta^{12}c_2^7 - 1152\alpha^{12}b^2\beta^2c_1^6c_3 + 288\alpha^{12}b^2\beta^2c_1^5c_2^2 - 1344\alpha^{10}b^2\beta^4c_1^5c_2c_3 + 336\alpha^{10}b^2\beta^4c_1^4c_2^3 \\
&- 5248\alpha^8b^2\beta^6c_1^5c_3^2 + 2816\alpha^8b^2\beta^6c_1^4c_2^2c_3 - 376\alpha^8b^2\beta^6c_1^3c_2^4 - 3264\alpha^6b^2\beta^8c_1^4c_2c_3^2 + 2496\alpha^6b^2\beta^8c_1^3c_2^3c_3 - 420\alpha^6b^2\beta^8c_1^2c_2^5 + 4352\alpha^4b^2\beta^{10}c_1^4c_3^3 \\
&- 2592\alpha^4b^2\beta^{10}c_1^3c_2^2c_3^2 + 888\alpha^4b^2\beta^{10}c_1^2c_2^4c_3 - 128\alpha^4b^2\beta^{10}c_1c_2^6 + 640\alpha^2b^2\beta^{12}c_1^3c_2c_3^3 + 16\alpha^2b^2\beta^{12}c_1^2c_2^3c_3^2 - 68\alpha^2b^2\beta^{12}c_1c_2^5c_3 + 6\alpha^2b^2\beta^{12}c_2^7 \\
&+ 256n\alpha^{10}\beta^4c_1^6c_3 - 64n\alpha^{10}\beta^4c_1^5c_2^2 + 512n\alpha^8\beta^6c_1^5c_2c_3 - 12n\alpha^8\beta^6c_1^4c_2^3 - 768n\alpha^6\beta^8c_1^5c_3^2 + 768n\alpha^6\beta^8c_1^4c_2^2c_3 - 144n\alpha^6\beta^8c_1^3c_2^4 - 192\alpha^6\beta^8c_1^3c_2^4 \\
&- 768n\alpha^4\beta^{10}c_1^4c_2c_3^2 + 512n\alpha^4\beta^{10}c_1^3c_2^3c_3 - 80n\alpha^4\beta^{10}c_1^2c_2^5 + 512n\alpha^2\beta^{12}c_1^4c_3^3 - 576n\alpha^2\beta^{12}c_1^3c_2^2c_3^2 + 208n\alpha^2\beta^{12}c_1^2c_2^4c_3 - 24n\alpha^2\beta^{12}c_1c_2^6 \\
&+ 1024\alpha^{10}\beta^4c_1^6c_3 - 256\alpha^{10}\beta^4c_1^5c_2^2 + 2048\alpha^8\beta^6c_1^5c_2c_3 - 512\alpha^8\beta^6c_1^4c_2^3 + 3072\alpha^6\beta^8c_1^5c_3^2 + 3072\alpha^4\beta^{10}c_1^4c_2c_3^2 - 1024\alpha^4\beta^{10}c_1^3c_2^3c_3 + 64\alpha^4\beta^{10}c_1^2c_2^5 \\
&- 1024\alpha^2\beta^{12}c_1^4c_3^3 + 1536\alpha^2\beta^{12}c_1^3c_2^2c_3^2 - 512\alpha^2\beta^{12}c_1^2c_2^4c_3 + 48\alpha^2\beta^{12}c_1c_2^6\Big),
\end{aligned}
$$

$$
\begin{aligned}
B :=& -4k \Big(48n\alpha^{10}b^6\beta^3c_1^4c_2^2c_3 - 12n\alpha^{10}b^6\beta^3c_1^3c_2^4 + 480n\alpha^8b^6\beta^5c_1^4c_2c_3^2 - 152n\alpha^8b^6\beta^5c_1^3c_2^3c_3 + 8n\alpha^8b^6\beta^5c_1^2c_2^5 + 1152n\alpha^6b^6\beta^7c_1^4c_3^3 - n\alpha^6b^6\beta^7c_1c_2^6 \\
&- 368n\alpha^6b^6\beta^7c_1^3c_2^2c_3^2 + 24n\alpha^6b^6\beta^7c_1^2c_2^4c_3 + 288n\alpha^4b^6\beta^9c_1^3c_2c_3^3 - 96n\alpha^4b^6\beta^9c_1^2c_2^3c_3^2 + 6n\alpha^4b^6\beta^9c_1c_2^5c_3 - 3n\alpha^2b^6\beta^{11}c_1^2c_2^2c_3^3 + 12n\alpha^2b^6\beta^{11}c_1c_2^4c_3^2 \\
&- \alpha^2b^6\beta^{11}c_2^6c_3 - 8nb^6\beta^{13}c_1c_2^3c_3^3 + 2nb^6\beta^{13}c_2^5c_3^2 - 96\alpha^{12}b^6\beta c_1^5c_2c_3 + 24\alpha^{12}b^6\beta c_1^4c_2^3 + 1152\alpha^{10}b^6\beta^3c_1^5c_3^2 - 576\alpha^{10}b^6\beta^3c_1^4c_2^2c_3 + 72\alpha^{10}b^6\beta^3c_1^3c_2^4 \\
&+ 1152\alpha^8b^6\beta^5c_1^4c_2c_3^2 - 344\alpha^8b^6\beta^5c_1^3c_2^3c_3 + 14\alpha^8b^6\beta^5c_1^2c_2^5 - 1152\alpha^6b^6\beta^7c_1^4c_3^3 + 1600\alpha^6b^6\beta^7c_1^3c_2^2c_3^2 - 360\alpha^6b^6\beta^7c_1^2c_2^4c_3 + 8\alpha^6b^6\beta^7c_1c_2^6 \\
&+ 480\alpha^4b^6\beta^9c_1^3c_2c_3^3 + 72\alpha^4b^6\beta^9c_1^2c_2^3c_3^2 - 48\alpha^4b^6\beta^9c_1c_2^5c_3 + 256\alpha^2b^6\beta^{11}c_1^2c_2^2c_3^3 - 96\alpha^2b^6\beta^{11}c_1c_2^4c_3^2 + 8\alpha^2b^6\beta^{11}c_2^6c_3 - 8b^6\beta^{13}c_1c_2^3c_3^3 + 2b^6\beta^{13}c_2^5c_3^2 \\
&+ 192n\alpha^{10}b^4\beta^3c_1^5c_2c_3 - 48n\alpha^{10}b^4\beta^3c_1^4c_2^3 + 960n\alpha^8b^4\beta^5c_1^5c_3^2 - 192n\alpha^8b^4\beta^5c_1^4c_2^2c_3 - 12n\alpha^8b^4\beta^5c_1^3c_2^4 + 64n\alpha^6b^4\beta^7c_1^4c_2c_3^2 + 64n\alpha^6b^4\beta^7c_1^3c_2^3c_3 \\
&- 20n\alpha^6b^4\beta^7c_1^2c_2^5 - 2880n\alpha^4b^4\beta^9c_1^4c_3^3 + 1392n\alpha^4b^4\beta^9c_1^3c_2^2c_3^2 - 204n\alpha^4b^4\beta^9c_1^2c_2^4c_3 + 9n\alpha^4b^4\beta^9c_1c_2^6 - 704n\alpha^2b^4\beta^{11}c_1^3c_2c_3^3 + 192n\alpha^2b^4\beta^{11}c_1^2c_2^3c_3^2 \\
&- n\alpha^2b^4\beta^{11}c_2^7 - 16nb^4\beta^{13}c_1^2c_2^2c_3^3 - 16nb^4\beta^{13}c_1c_2^4c_3^2 + 5nb^4\beta^{13}c_2^6c_3 - 192\alpha^{12}b^4\beta c_1^6c_3 + 48\alpha^{12}b^4\beta c_1^5c_2^2 + 576\alpha^{10}b^4\beta^3c_1^5c_2c_3 - 144\alpha^{10}b^4\beta^3c_1^4c_2^3 \\
&- 3456\alpha^8b^4\beta^5c_1^5c_3^2 + 3120\alpha^8b^4\beta^5c_1^4c_2^2c_3 - 564\alpha^8b^4\beta^5c_1^3c_2^4 - 1664\alpha^6b^4\beta^7c_1^4c_2c_3^2 + 1696\alpha^6b^4\beta^7c_1^3c_2^3c_3 - 320\alpha^6b^4\beta^7c_1^2c_2^5 + 4416\alpha^4b^4\beta^9c_1^4c_3^3 \\
&- 2928\alpha^4b^4\beta^9c_1^3c_2^2c_3^2 + 744\alpha^4b^4\beta^9c_1^2c_2^4c_3 - 72\alpha^4b^4\beta^9c_1c_2^6 + 64\alpha^2b^4\beta^{11}c_1^3c_2c_3^3 - 144\alpha^2b^4\beta^{11}c_1^2c_2^3c_3^2 + 8\alpha^2b^4\beta^{11}c_2^7 - 304b^4\beta^{13}c_1^2c_2^2c_3^3 \\
&+ 92b^4\beta^{13}c_1c_2^4c_3^2 - 4b^4\beta^{13}c_2^6c_3 + 192n\alpha^{10}b^2\beta^3c_1^6c_3 - 48n\alpha^{10}b^2\beta^3c_1^5c_2^2 + 96n\alpha^8b^2\beta^5c_1^5c_2c_3 - 24n\alpha^8b^2\beta^5c_1^4c_2^3 - 1856n\alpha^6b^2\beta^7c_1^5c_3^2 - 44n\alpha^6b^2\beta^7c_1^3c_2^4 \\
&+ 640n\alpha^6b^2\beta^7c_1^4c_2^2c_3 - 1632n\alpha^4b^2\beta^9c_1^4c_2c_3^2 + 480n\alpha^4b^2\beta^9c_1^3c_2^3c_3 - 18n\alpha^4b^2\beta^9c_1^2c_2^5n + 2176n\alpha^2b^2\beta^{11}c_1^4c_3^3 - 1872n\alpha^2b^2\beta^{11}c_1^3c_2^2c_3^2 \\
&+ 36n\alpha^2b^2\beta^{11}c_1^2c_2^4c_3 - 16n\alpha^2b^2\beta^{11}c_1c_2^6 + 320nb^2\beta^{13}c_1^3c_2c_3^3 - 14nb^2\beta^{13}c_1^2c_2^3c_3^2 + 14nb^2\beta^{13}c_1c_2^5c_3 + 3nb^2\beta^{13}c_2^7 + 1152\alpha^{10}b^2\beta^3c_1^6c_3 - 288\alpha^{10}b^2\beta^3c_1^5c_2^2 \\
&+ 1344\alpha^8b^2\beta^5c_1^5c_2c_3 - 336\alpha^8b^2\beta^5c_1^4c_2^3 + 5248\alpha^6b^2\beta^7c_1^5c_3^2 - 2816\alpha^6b^2\beta^7c_1^4c_2^2c_3 + 376\alpha^6b^2\beta^7c_1^3c_2^4 + 3264\alpha^4b^2\beta^9c_1^4c_2c_3^2 - 2496\alpha^4b^2\beta^9c_1^3c_2^3c_3 \\
&+ 420\alpha^4b^2\beta^9c_1^2c_2^5 - 4352\alpha^2b^2\beta^{11}c_1^4c_3^3 + 2592\alpha^2b^2\beta^{11}c_1^3c_2^2c_3^2 - 888\alpha^2b^2\beta^{11}c_1^2c_2^4c_3 + 128\alpha^2b^2\beta^{11}c_1c_2^6 - 640b^2\beta^{13}c_1^3c_2c_3^3 - 16b^2\beta^{13}c_1^2c_2^3c_3^2 \\
&+ 68b^2\beta^{13}c_1c_2^5c_3 - 6b^2\beta^{13}c_2^7 - 256n\alpha^8\beta^5c_1^6c_3 + 64n\alpha^8\beta^5c_1^5c_2^2 - 512n\alpha^6\beta^7c_1^5c_2c_3 + 128n\alpha^6\beta^7c_1^4c_2^3 + 768n\alpha^4\beta^9c_1^5c_3^2 - 768n\alpha^4\beta^9c_1^4c_2^2c_3 \\
&+ 144n\alpha^4\beta^9c_1^3c_2^4 + 768n\alpha^2\beta^{11}c_1^4c_2c_3^2 - 512n\alpha^2\beta^{11}c_1^3c_2^3c_3 + 80n\alpha^2\beta^{11}c_1^2c_2^5 - 512n\beta^{13}c_1^4c_3^3 + 576n\beta^{13}c_1^3c_2^2c_3^2 - 208n\beta^{13}c_1^2c_2^4c_3 + 24n\beta^{13}c_1c_2^6 \\
&+ 256\alpha^8\beta^5c_1^5c_2^2 - 2048\alpha^6\beta^7c_1^5c_2c_3 + 512\alpha^6\beta^7c_1^4c_2^3 - 3072\alpha^4\beta^9c_1^5c_3^2 + 192\alpha^4\beta^9c_1^3c_2^4 - 3072\alpha^2\beta^{11}c_1^4c_2c_3^2 - 1024\alpha^8\beta^5c_1^6c_3 + 1024\alpha^2\beta^{11}c_1^3c_2^3c_3 \\
&- 64\alpha^2\beta^{11}c_1^2c_2^5 + 1024\beta^{13}c_1^4c_3^3 - 1536\beta^{13}c_1^3c_2^2c_3^2 + 512\beta^{13}c_1^2c_2^4c_3 - 48\beta^{13}c_1c_2^6\Big).
\end{aligned}
$$

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