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ON THE GEOMETRY OF WARPED PRODUCT PSEUDO-SLANT SUBMANIFOLDS IN A NEARLY COSYMPLECTIC MANIFOLD

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ABSTRACT. In this paper, we study the non-trivial warped product pseudo-slant submanifold of a nearly cosymplectic manifold such that the extrinsic sphere is slant immersion. outset of the considering, we also show some integrability and totally geodesic foliations results for the distributions of pseudo-slant submanifold. Next, we demonstrate the characterization theorem for the existence of warped product isometrically immersed into nearly cosymplectic manifolds.

1. INTRODUCTION

The geometry of warped product submanifolds have been studied dynamically since B. Y. Chen [12] has investigated the idea of CR-warped product submanifold in a Kaehler manifold. Further in the different geometric aspect, he has studied the warping fuction in the form of some partial differential equations. In fact, distinct forms of warped product submanifolds of different class of structures were studied by the many authors (see [1, 2, 3, 4, 5, 6, 8, 7, 12, 13, 20, 22, 23, 26, 27]). Recently, in [5], the author established general inequalities for warped product pseudo-slant isometrically immersed in nearly Kenmotsu manifolds for mixed totally geodesic submanifold. Moreover, S. Uddin et al. has obtained some existence results for warped product pseudo-slant submanifolds in terms of endomorphism in a nearly cosymplectic manifold (see [25]). In the present paper, we define clearly a characterization theorem of the non-trivial warped product pseudo-slant submanifolds of the form $M_\perp \times_f M_\theta$ which are the natural extension of CR-warped product submanifolds. Since every CR-warped product submanifold is a non-trivial warped product pseudo-slant submanifold of the forms $M_\perp \times_f M_\theta$ and $M_\theta \times_f M_\perp$ with slant angle $\theta = 0$. But the warped product pseudo-slant submanifold never generalize the CR-warped product submanifold. First, we consider the non-trivial warped product pseudo-slant submanifold $M = M_\perp \times_f M_\theta$ such that M_θ and $M_\perp$ are proper slant and anti-invariant submanifolds, respectively. Further, we establish a necessary and sufficent condition involving the slant angle and warping functions, to reduce a pseudo-slant submanifolds into warped products. However, similar results has been obtained by many prestigious geometers (see [5, 6, 4, 2, 20, 22, 23, 26]) for distinct warped product submanifolds in different type of ambient manifolds.

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The paper is arranged as follows: Section 2, we notice some preliminary formulas and definitions. Section 3, is come up with full of attention to the study of pseudo-slant submanifolds of nearly cosymplectic manifolds and some theorems are given on total manifolds. Section 4, we study the warped product pseudo-slant submanifolds of a nearly cosymplectic manifold and obtain some results on its characterization.

2. Preliminaries

An odd $(2n+1)-$dimensional Riemannian manifolds $(\widetilde{M}, g)$ is called *nearly cosymplectic* manifold, if it is consisting off an endomorphism φ of its tangent bundle $T\widetilde{M}$, a structure vector fields ξ and a 1-form η which satisfies the following:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \ \ \eta o \varphi = 0, \tag{2.1}$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \tag{2.2}$$

$$(\widetilde{\nabla}_U \varphi)V + (\widetilde{\nabla}_V \varphi)U = 0, \tag{2.3}$$

for any vector fields U, V on $\widetilde{M}$ such that $\widetilde{\nabla}$ denotes the Riemannian connection with respect to Riemannian metric g (see [25]). Furthermore, the fundamental $2-$form denoted by Φ, i.e., $\Phi(U, V) = g(\varphi U, V)$. The pair (Φ, η) defines a locally conformal cosymplectic structure was proved, i.e.,

$$d\Phi = 2\Phi \wedge \eta, \quad d\eta = 0.$$

It is fascinating to see that the structure vector field ξ is killing in nearly cosymplectic manifolds. For this killing vector field we have the following an important theorem which was proved in [16]:

Theorem 2.1. *On a nearly cosymplectic manifold ξ is killing.*

From the above theorem, we have

$$g(\widetilde{\nabla}_V \xi, U) + g(\widetilde{\nabla}_U \xi, V) = 0, \tag{2.4}$$

for any vector fields U, V tangent to $\widetilde{M}$, where $\widetilde{M}$ is a *nearly cosymplectic manifold*. If we consider $\mathcal{P}_U V$ and $\mathcal{Q}_U V$ be the tangential and normal components of $(\widetilde{\nabla}_U \varphi)V$, i.e.,

$$(\widetilde{\nabla}_U \varphi)V = \mathcal{P}_U V + \mathcal{Q}_U V \tag{2.5}$$

Similarly, we have

$$(\widetilde{\nabla}_U \varphi)N = \mathcal{P}_U N + \mathcal{Q}_U N \tag{2.6}$$

for any $N \in \Gamma(T^{\perp}\widetilde{M})$. From the above relation it can be concluded that for nearly cosymplectic manifolds the following conditions are satisfied:

$$\mathcal{P}_U V + \mathcal{P}_V U = 0, \ \ \mathcal{Q}_U V + \mathcal{Q}_V U = 0. \tag{2.7}$$

The following properties of $\mathcal{P}$ and$\mathcal{Q}$ can be directly verified which are developed here of later usage (see [25, 27]):

$$\left.\begin{array}{l} (i)\ \mathcal{P}_{U+V} W = \mathcal{P}_U W + \mathcal{P}_V W,\ (ii)\ \mathcal{Q}_{U+V} W = \mathcal{Q}_U W + \mathcal{Q}_V W, \\ (iii)\ \mathcal{P}_U (W+Z) = \mathcal{P}_U W + \mathcal{P}_U Z, \\ (iv)\ \mathcal{Q}_U (W+Z) = \mathcal{Q}_U W + \mathcal{Q}_U Z, \\ (v)\ g(\mathcal{P}_U V, W) = -g(V, \mathcal{P}_U W),\ (vi)\ g(\mathcal{Q}_U V, N) = -g(V, \mathcal{P}_U N), \\ (vii)\ \mathcal{P}_U \varphi V + \mathcal{Q}_U \varphi V = -\varphi(\mathcal{P}_U V + \mathcal{Q}_U V). \end{array}\right\} \tag{2.8}$$

A Riemannian manifold M is isometrically immersed into almost contact metric manifold $\widetilde{M}$ and let g denote the Riemannian metric induced on M. Suppose that $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$ be the Lie algebra of vector fields tangent to M and normal to M, respectively and $\nabla^{\perp}$ the induced connection on $T^{\perp}M$. It is represented by $\mathcal{F}(M)$ the algebra of smooth functions on M and $\Gamma(TM)$, the $\mathcal{F}(M)$-module of smooth sections of TM over M which are also denoted by ∇ the Levi-Civita connection of M then the *Gauss* and *Weingarten* formulas are given by

$$\widetilde{\nabla}_U V = \nabla_U V + h(U,V) \tag{2.9}$$

$$\widetilde{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.10}$$

for each $U,\ V \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively for the immersion of M into $\widetilde{M}$. They are defined by

$$g(h(U,V),N) = g(A_N U,V). \tag{2.11}$$

Now, for any $U \in \Gamma(TM)$, we have

$$\varphi U = PU + FU, \tag{2.12}$$

where PU and FU are tangential and normal components of φU, respectively. Similarly for any $N \in \Gamma(T^{\perp}M)$, we have

$$\varphi N = tN + fN, \tag{2.13}$$

where tN (resp. fN) are tangential (resp. normal) components of φN. A submanifold M is said to be totally geodesic and totally umbilical, if $h(U,V) = 0 \ and \ h(U,V) = g(U,V)H$, respectively.

Now, we define a class of submanifolds which are called the slant submanifold. For each non-zero vector U tangent to M at p, such that U is not proportional to ξ_p, we denote by $0 \leq \theta(U) \leq \pi/2$, the angle between φU and T_pM is called the Wirtinger angle. If the angle $\theta(U)$ is constant for all $U \in T_pM- <\xi>$ and $p \in M$, then M is said to be a slant submanifold (see [19]) and the angle θ is called slant angle of M. Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is anti-invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant.

In [11], the following proposition was given by J. L Cabrerizo for almost contact manifolds.

Proposition 2.1. *Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.14}$$

Furthermore, in such a case, if θ is slant angle, then it satisfies that $\lambda = \cos^2\theta$.

Hence, for a slant submanifold M of an almost contact metric manifold $\widetilde{M}$, we have the following relations which are consequences of the above Proposition 2.1

$$g(PU,PV) = \cos^2\theta\{g(U,V) - \eta(U)\eta(V)\}. \tag{2.15}$$

$$g(FU,FV) = \sin^2\theta\{g(U,V) - \eta(U)\eta(V)\}. \tag{2.16}$$

for any $U, V \in \Gamma(TM)$. Also, we proceed to give an another characterization which is directly related the consequence of the Proposition 2.1:

Proposition 2.2. *Let M be a slant submanifold of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in TM$. Then*

$$(i)\ tFX = -\sin^2\theta(X - \eta(X)\xi) \ \text{ and } \ (ii)\ fFX = -FPX, \tag{2.17}$$

for any $X \in \Gamma(TM)$.

3. Pseudo-slant submanifolds of nearly cosymplectic manifolds

In this section, we define pseudo-slant submanifolds of almost contact manifolds by using the slant distribution given in [10]. We investigate the geometry of leaves of distributions containing in the definition of pseudo-slant submanifolds. We also construct some necessary and sufficient conditions for such sub-immersions to be totally geodesic foliations for which later usage in characterization theorem. Further the study of pseudo-slant we refer [15]. First, we give the definition of pseudo-slant submanifold:

Definition 3.1. *A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be pseudo-slant submanifold, if there exist two orthogonal distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ such that*

(i) *$TM = \mathcal{D}^\theta \oplus \mathcal{D}^\perp \oplus \langle\xi\rangle$, where $< \xi >$ is 1-dimensional distribution spanned by ξ.*
(ii) *$\mathcal{D}^\perp$ is an anti invariant distribution under φ i.e., $\varphi\mathcal{D}^\perp \subseteq T^\perp M$.*
(iii) *$\mathcal{D}^\theta$ is slant distribution with slant angle $\theta \neq 0, \frac{\pi}{2}$.*

Let m_1 and m_2 are dimensions of distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively. If m_2=0, then M is anti invariant submanifold. If m_1=0 and $\theta = 0$, then M is invariant submanifold. If m_1=0 and $\theta \neq 0, \frac{\pi}{2}$, then M is proper-slant submanifold, or if $\theta = \frac{\pi}{2}$, then M is anti invariant submanifold and if $\theta = 0$, then M is semi-invariant submanifold. If μ is an invariant subspace of normal bundle $T^\perp M$, then for pseudo-slant case, the normal bundle $T^\perp M$ can be decomposed as follows:

$$T^\perp M = \varphi\mathcal{D}^\perp \oplus F\mathcal{D}^\theta \oplus \mu, \tag{3.1}$$

where μ is even dimensional invariant sub bundle of $T^\perp M$. Now, we obtain the following productive theorems for later usage:

Theorem 3.1. *Let M be a pseudo-slant submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then the distribution $\mathcal{D}^\perp \oplus \xi$ defines as totally geodesic foliation in M if and only if*

$$g(h(Z,W), FPZ) = \frac{1}{2}\Big\{g(A_{\varphi Z}W, PX) + g(A_{\varphi W}Z, PX)\Big\}$$

for any $Z, W \in \Gamma(\mathcal{D}^\perp \oplus \xi)$ and $X \in \Gamma(\mathcal{D}^\theta)$.

Proof. From the property of Riemannian metric and using fact that ξ is orthogonal to $\mathcal{D}^\theta$, i.e., $\eta(X) = 0$, for every $X \in \Gamma(\mathcal{D}^\theta)$, we have

$$g(\nabla_Z W, X) = g(\varphi\widetilde{\nabla}_Z W, \varphi X),$$

for $Z, W \in \Gamma(\mathcal{D}^{\perp} \oplus \xi)$. Taking account of (2.12), ones derives

$$g(\nabla_Z W, X) = g(\varphi\widetilde{\nabla}_Z W, PX) + g(\varphi\widetilde{\nabla}_Z W, FX).$$

It follow from the covariant derivative φ and the property of Riemannian metric, we obtain

$$g(\nabla_Z W, X) = g(\widetilde{\nabla}_Z \varphi W, PX) - g((\widetilde{\nabla}_Z \varphi)W, PX) - g(\widetilde{\nabla}_Z W, \varphi FX).$$

From (2.3), (2.10), we have

$$g(\nabla_Z W, X) = g((\widetilde{\nabla}_W \varphi)Z, PX) - g(\widetilde{\nabla}_Z W, tFX) - g(\widetilde{\nabla}_Z W, fFX)$$
$$-g(A_{\varphi W} Z, PX).$$

Again from the covariant derivative of endomorphism φ and Proposition 2.2, it is easily seen that

$$g(\nabla_Z W, X) = g(\widetilde{\nabla}_W \varphi Z, PX) - g(\widetilde{\nabla}_W Z, \varphi PX) + \sin^2\theta g(\widetilde{\nabla}_Z W, X)$$
$$+g(\widetilde{\nabla}_Z W, FPX) - g(A_{\varphi W} Z, PX).$$

Then, from (2.10) and (2.12), the above equation becomes

$$\cos^2\theta g(\nabla_Z W, X) = 2g(h(Z, W), FPX) - g(A_{\varphi W} Z, PX) - g(A_{\varphi Z} W, PX)$$
$$+g(\widetilde{\nabla}_W Z, P^2 X).$$

The above equation can be written in the new form by using (2.14)

$$\cos^2\theta g(\nabla_Z W, X) = 2g(h(Z, W), FPZ) - g(A_{\varphi Z} W, PX) - g(A_{\varphi W} Z, PX)$$
$$-\cos^2\theta g(\widetilde{\nabla}_W Z, X).$$

The assumption is followed from the last equation. It makes the complete proof of the theorem.■

Theorem 3.2. *On a pseudo-slant submanifold M of a nearly cosymplectic manifold $\widetilde{M}$. The distribution $\mathcal{D}^{\theta}$ is integrable if and only if*

$$2g(\nabla_X Y, Z) = \sec^2\theta \Big\{ g(h(X, PY) + h(Y, PX), \varphi Z) - g(h(X, Z), FPY)$$
$$-g(h(Y, Z), FPX) - \eta(Z)g(\widetilde{\nabla}_X \xi, Y) \Big\},$$

for any $Z \in \Gamma(\mathcal{D}^{\perp} \oplus \xi)$ and $X, Y \in \Gamma(\mathcal{D}^{\theta})$.

Proof. By using the properties of symmetric torsion and Riemannian metric g, we have

$$g([X, Y], Z) = g(\varphi\widetilde{\nabla}_X Y, \varphi Z) + \eta(Z)g(\widetilde{\nabla}_X Y, \xi) - g(\widetilde{\nabla}_Y X, Z).$$

From the covariant derivative of endomorphism φ in the first term, we get

$$g([X, Y], Z) = g(\widetilde{\nabla}_X \phi Y, \varphi Z) - g((\widetilde{\nabla}_X \varphi)Y, \varphi Z)$$
$$-g(\widetilde{\nabla}_Y X, Z) - \eta(Z)g(\widetilde{\nabla}_X \xi, Y).$$

Taking account of (2.3) in the second term and (2.12), we derive

$$g([X, Y], Z) = g(\widetilde{\nabla}_X PY, \varphi Z) + g(\widetilde{\nabla}_X FY, \varphi Z) + g((\widetilde{\nabla}_Y \varphi)X, \varphi Z)$$
$$-g(\widetilde{\nabla}_Y X, Z) - \eta(Z)g(\widetilde{\nabla}_X \xi, Y).$$

Since, FY and φZ are orthogonal then considering the property of Riemannian connection and (2.2), we have

$$g([X,Y],Z) = g(h(X,PY),\varphi Z) - g(FY,\widetilde{\nabla}_X\varphi Z) + g(\widetilde{\nabla}_Y\phi X,\varphi Z)$$
$$-2g(\widetilde{\nabla}_Y X,Z) - \eta(Z)g(\widetilde{\nabla}_X\xi,Y).$$

From (2.5) and the property of Riemannian metric, we can modify as

$$g([X,Y],Z) = g(h(X,PY),\varphi Z) - g(FY,(\widetilde{\nabla}_X\varphi)Z)$$
$$+g(\varphi FY,\widetilde{\nabla}_X Z) + g(\widetilde{\nabla}_Y PX,\varphi Z) + g(\widetilde{\nabla}_Y FX,\varphi Z)$$
$$-2g(\widetilde{\nabla}_Y X,Z) - \eta(Z)g(\widetilde{\nabla}_X\xi,Y).$$

Using (2.5) in the second term of the above equation from (2.9), (2.13), it follows that

$$g([X,Y],Z) = g(h(X,PY),\varphi Z) + g(h(PX,Y),\varphi Z) + g(FY,\mathcal{Q}_X Z)$$
$$+g(tFY,\widetilde{\nabla}_X Z) + g(fFY,\widetilde{\nabla}_X Z) - g(FX,\widetilde{\nabla}_Y\phi Z)$$
$$-2g(\widetilde{\nabla}_Y X,Z) - \eta(Z)g(\widetilde{\nabla}_X\xi,Y).$$

From (2.10) and Proposition 2.2, then the above equation is going on to be taken new form

$$g([X,Y],Z) = g(h(X,PY),\varphi Z) + g(h(PX,Y),\varphi Z) + g(\varphi Y,\mathcal{Q}_X Z)$$
$$-\sin^2\theta g(Y,\widetilde{\nabla}_X Z) - g(h(X,Z),FPY) - g(FX,(\widetilde{\nabla}_Y\varphi)Z)$$
$$+g(\varphi FX,\widetilde{\nabla}_Y Z) - 2g(\widetilde{\nabla}_Y X,Z) - \eta(Z)g(\widetilde{\nabla}_X\xi,Y).$$

Moreover, it is also known as $\mathcal{Q}$ is a normal part of structure equation, then $g(\varphi Y,\mathcal{Q}_X Z) = -g(Y,\varphi\mathcal{Q}_X Z) = 0$ and (2.13), we derive

$$g([X,Y],Z) = g(h(X,PY),\varphi Z) + g(h(PX,Y),\varphi Z) - \sin^2\theta g(Y,\widetilde{\nabla}_X Z)$$
$$-g(h(X,Z),FPY) + g(tFX,\widetilde{\nabla}_Y Z) + g(fFX,\widetilde{\nabla}_Y Z)$$
$$-2g(\widetilde{\nabla}_Y X,Z) - \eta(Z)g(\widetilde{\nabla}_X\xi,Y).$$

Taking into an account that X,Y are orthogonal to Z and Proposition 2.2 which gives us

$$\sin^2\theta g([X,Y],Z) = g(h(X,PY) + h(PX,Y),\varphi Z) - 2\cos^2\theta g(Z,\widetilde{\nabla}_X Y)$$
$$-g(h(X,Z),FPY) - g(h(Y,Z),FPX) - \eta(Z)g(\widetilde{\nabla}_X\xi,Y).$$

Let $\mathcal{D}^\theta$ is integrable, we modified by

$$2\cos^2\theta g(\nabla_X Y,Z) = g(h(X,PY) + h(PX,Y),\varphi Z) - g(h(X,Z),FPY)$$
$$-g(FPX,h(Y,Z)) - \eta(Z)g(\widetilde{\nabla}_X\xi,Y),$$

which proves our assumption. Converse part of the theorem follows from the above using by directly. Thus the proof of the theorem is going to be completed.■

4. WARPED PRODUCT SUBMANIFOLDS OF THE FROM $M_\perp \times_f M_\theta$

The most constructive generalizations of Riemannian product manifolds are warped products with warping function f. It was initiated by Bishop and Neil in (see [9]). They have described these manifolds as follows: Suppose that f be a positive differentiable function which always be defined on leaves and (M_1, g_1) and (M_2, g_2) are two Riemannian manifolds. Then the warped product of M_1 and M_2 are the Riemannian manifolds $M_1 \times_f M_2 = (M_1 \times_f M_2, g)$, where $g = g_1 + f^2 g_2$. For a warped product, we have

$$\nabla_X Z = \nabla_Z X = X \ln f Z, \tag{4.1}$$

for any vector fields X, Z and tangents M_1 and M_2, respectively, where ∇ denotes the Levi-Civita connection on M (see [9]). On the other hand, $\nabla \ln f$ is the gradient of $\ln f$ which is defined as $g(\nabla \ln f, U) = U \ln f$. If the warping function f is constant then a warped product manifold $M = M_1 \times_f M_2$ is called simply Riemannian product or *trivial* warped product manifold. For a warped product manifold $M = M_1 \times_f M_2$, M_1 is called totally geodesic and M_2 is called totally umbilical submanifolds of M, respectively. Now, we obtain some preparatory propositions.

Proposition 4.1. *Let $M = M_\perp \times_f M_\theta$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\widetilde{M}$ such that structure vector field ξ is tangent to $M_\perp$. Then*

$$2g(h(X,Z), FPX) = (Z \ln f)\cos^2\theta ||X||^2 + g(h(X, PX), \varphi Z)$$
$$+ g(h(Z, PX), FX),$$

for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. Suppose that $M = M_\perp \times_f M_\theta$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\widetilde{M}$, then from (2.9), we find that

$$g(h(X,Z), FPX) = g(\widetilde{\nabla}_Z X, FPX).$$

It is well-known that ξ is tangent to $M_\perp$, then from (2.12) and (2.14), we obtain

$$g(h(X,Z), FPX) = -g(\varphi\widetilde{\nabla}_Z X, PX) + \cos^2\theta g(\nabla_Z X, X).$$

Then from the definition of covariant derivative of tensor field φ and (4.1), we can derive

$$g(h(X,Z), FPX) = g((\widetilde{\nabla}_Z\varphi)X, PX) - g(\widetilde{\nabla}_Z \phi X, PX)$$
$$+ \cos^2\theta (Z \ln f)||X||^2.$$

Using (2.3) and the (2.12), then the above equation becomes

$$g(h(X,Z), FPX) = -g((\widetilde{\nabla}_X\varphi)Z, PX) - g(\nabla_Z PX, PX)$$
$$-g(\widetilde{\nabla}_Z FX, PX) + \cos^2\theta (X \ln f)||Z||^2.$$

We derive the following by using (4.1), (2.15) and (2.10)

$$g(h(X,Z), FPX) = -g(\widetilde{\nabla}_X \varphi Z, PX) - g(\widetilde{\nabla}_X Z, \varphi PX) - \cos^2\theta (Z \ln f)||X||^2$$
$$+ g(h(Z, PX), FX) + \cos^2\theta (Z \ln f)||X||^2.$$

From (2.10), (2.12) and (2.14) for slant submanifold, it is easily seen that

$$g(h(X,Z), FPX) = g(A_{\varphi Z} X, PX) + \cos^2\theta g(\nabla_X Z, X)$$

$$-g(\widetilde{\nabla}_X Z, FPX) + g(h(Z, PX), FX).$$

Finally,(4.1) and (2.9) imply that

$$2g(h(X, Z), FPX) = (Z \ln f) \cos^2 \theta ||X||^2 + g(h(X, PX), \varphi Z)$$
$$+g(h(Z, PX), FX),$$

which gives our assertion. So it completes the proof of proposition.■

Proposition 4.2. *Assume that $M = M_\perp \times_f M_\theta$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\widetilde{M}$. Then*

$$g(h(X, Z), FPX) = 2g(h(X, PX), \varphi Z) - g(h(Z, PX), FX),$$

for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. From (2.9), (2.12) and the fact that ξ is tangent to $M_\perp$, we have

$$g(h(Z, PX), FX) = -g(\varphi \widetilde{\nabla}_Z PX, X) - g(\widetilde{\nabla}_Z PX, PX),$$

for $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$. Then the covariant derivative of φ and (4.1), (2.15), we modified as

$$g(h(Z, PX), FX) = g((\widetilde{\nabla}_Z \varphi)PX, X) - g(\widetilde{\nabla}_Z \varphi PX, X)$$
$$- \cos^2 \theta (Z \ln f)||X||^2.$$

Taking into account of (2.3) for nearly cosymplectic manifold and the virtues of (2.9), (2.12) and (2.14), it follows that

$$g(h(Z, PX), FX) = g(h(X, Z), FPX) - \cos^2 \theta (Z \ln f)||X||^2$$
$$-g((\widetilde{\nabla}_{PX} \varphi)Z, X) + \cos^2 \theta g(\nabla_Z X, X).$$

Using (4.1) and the covariant derivative of φ, one has

$$g(h(Z, PX), FX) = g(h(X, Z), FPX) - g(\widetilde{\nabla}_{PX} \varphi Z, X)$$
$$-g(\widetilde{\nabla}_{PX} Z, \varphi X).$$

Then from (2.12) and (2.10), it is obvious that

$$2g(h(Z, PX), FX) = g(h(X, Z), FPX) + g(h(X, PX), \varphi Z)$$
$$-g(\widetilde{\nabla}_{PX} Z, PX).$$

Again from (4.1) and (2.15), then last equation reduces in the new form, i.e.,

$$2g(h(Z, PX), FX) = g(h(X, Z), FPX) - (Z \ln f) \cos^2 \theta ||X||^2$$
$$+g(h(X, PX), \varphi Z). \tag{4.2}$$

Now, interchanging X by PX in (4.2) and using (2.14), we get

$$-2\cos^2 \theta g(h(X, Z), FPX) = -(Z \ln f) \cos^4 ||X||^2 - \cos^2 \theta g(h(Z, PX), FX)$$
$$- \cos^2 \theta g(h(PX, X), \varphi Z).$$

which implies

$$2g(h(X, Z), FPX) = (Z \ln f) \cos^2 ||X||^2 + g(h(Z, PX), FX)$$
$$+g(h(PX, X), \varphi Z). \tag{4.3}$$

From (4.2) and (4.3), it follows that

$$g(h(Z,PX),FX) = 2g(h(X,PX),\varphi Z) - g(h(X,Z),FPX),$$

which completes the proof of proposition.■

Proposition 4.3. *On a warped product pseudo-slant submanifold $M = M_\perp \times_f M_\theta$ of a nearly cosymplectic manifold $\widetilde{M}$. Then*

$$g(h(PX,X),\varphi Z) = g(h(X,Z),FPX) - \frac{1}{3}\cos^2\theta(Z\lambda)||X||^2,$$

for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. From Proposition 4.1 and Proposition 4.2, we can derive the proof of Proposition 4.3.

Now, we give the proof of first main characterization theorem of this note:

Theorem 4.1. *Let $\widetilde{M}$ be nearly cosymplectic manifold and M be a proper pseudo-slant submanifold of $\widetilde{M}$ such that the slant distribution is integrable. Then $M = M_\perp \times_f M_\theta$ is locally a warped product of proper slant and anti-invariant submanifolds if and only if*

$$A_{FPX}Z - A_{\varphi Z}PX = \frac{1}{3}\cos^2\theta(Z\lambda)X, \tag{4.4}$$

for any $Z \in \Gamma(\mathcal{D}^\perp \oplus \xi)$ and any $X \in \Gamma(\mathcal{D}^\theta)$. Moreover, for a differentiable function λ on M such that $Y\lambda = 0$, for every $Y \in \Gamma(\mathcal{D}^\theta)$.

Proof. If we consider that $M = M_\perp \times_f M_\theta$ be a non-trivial warped product proper pseudo-slant submanifold of a nearly cosymplectic manifold $\widetilde{M}$ such that M_θ and $M_\perp$ are proper slant and anti-invariant submanifolds, then direct part follows from the Proposition 4.3 and we take $\ln f = \lambda$.

Conversely, suppose that M be a proper pseudo-slant submanifold in a nearly cosymplectic manifold $\widetilde{M}$ with (4.4) holds. Let us take inner product in (4.4) with $W \in \Gamma(\mathcal{D}^\perp \oplus \xi)$ and using fact that X and W are orthogonal, then we derive

$$g(h(Z,W),FPX) = g(h(PX,W),\varphi Z). \tag{4.5}$$

It is easily obtained the following by interchanging Z and W in (4.5)

$$g(h(Z,W),FPX) = g(h(PX,Z),\varphi W). \tag{4.6}$$

From (4.5) and (4.6), its follows that

$$2g(h(Z,W),FPX) = g(h(PX,W),\varphi Z) + g(h(PX,Z),\varphi W). \tag{4.7}$$

The virtue of (4.7) and Theorem 3.1 indicates that $(\mathcal{D}^\perp \oplus \xi)$ is totally geodesic foliation in M, i.e., its leaves are totally geodesic into M of $\widetilde{M}$. So far as the slant distribution $\mathcal{D}^\theta$ is concerned and it is integrable by hypothesis. Thus Theorem 3.2 signifies that the distribution $\mathcal{D}^\theta$ is integrable if and only if

$$2g(\nabla_X Y, Z) = \sec^2\theta\Big\{g(h(X,PY) + h(Y,PX), \varphi Z)$$
$$-g(h(X,Z), FPY) - g(h(Y,Z), FPX) - \eta(Z)g(\widetilde{\nabla}_X\xi, Y)\Big\},$$

for $X, Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp \oplus \xi)$. Therefore, the above equation can be expressed in the new form

$$g(\nabla_X Y, Z) = -\frac{1}{2}\sec^2\theta\Big\{g(A_{FPY}Z - A_{\varphi Z}PY, X) + g(A_{FPX}Z - A_{\varphi Z}PX, Y)$$
$$+\eta(Z)g(\widetilde{\nabla}_X\xi, Y)\Big\}.$$

Moreover, by hypothesis of the theorem we have considered that the distribution $\mathcal{D}^\theta$ is integrable. Then it is obvious we assume that M_θ is a leaf of $\mathcal{D}^\theta$ or M_θ be a integral manifold of $\mathcal{D}^\theta$ and h^θ be the second fundamental form of M_θ into M of the immersion $\widetilde{M}$. From (4.4), we obtain that

$$g(h^\theta(X,Y), Z) = -\frac{1}{3}(Z\lambda)g(X,Y) - \eta(Z)g(\widetilde{\nabla}_X\xi, Y). \tag{4.8}$$

It is directly obtained the following by interchanging X and Y in the above equation, i.e.,

$$g(h^\theta(X,Y), Z) = -\frac{1}{3}(Z\lambda)g(X,Y) - \eta(Z)g(\widetilde{\nabla}_Y\xi, X). \tag{4.9}$$

Then from (4.8) and (4.9), it follows that

$$2g(h^\theta(X,Y), Z) = -\frac{2}{3}(Z\lambda)g(X,Y) - \eta(Z)\{g(\widetilde{\nabla}_X\xi, Y) + g(\widetilde{\nabla}_X\xi, Y)\}$$

Thus, from (2.4), for killing vector field ξ, the above relation is reduced as

$$g(h^\theta(X,Y), Z) = -\frac{1}{3}g(X,Y)g(\nabla\lambda, Z).$$

Finally, the above equation implies that

$$h^\theta(X,Y) = -\frac{1}{3}g(X,Y)\nabla\lambda. \tag{4.10}$$

The equation (4.10) indicates that the leaves of $\mathcal{D}^\theta$ are totally umbilical in M such that $H^\theta = -\frac{1}{3}\nabla\lambda$ is a mean curvature vector of M. Moreover, the argument $X\lambda = 0$, for every $X \in \Gamma(\mathcal{D}^\theta)$ suggest that the leaves of $\mathcal{D}^\theta$ are extrinsic spheres in M. In distinct way, the integral manifold M_θ of $\mathcal{D}^\theta$ is totally umbilical and it's mean curvature vector field is non-zero parallel along M_θ and so from the result of Hiepko (see [17]), M is a warped product submanifold of integral manifolds M_θ and $M_\perp$ of $\mathcal{D}^\theta$ and $\mathcal{D}^\perp \oplus \xi$, respectively. This is complete the proof of the theorem.■

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