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A NEW PROOF OF THE GRAS THEOREM

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ABSTRACT. By using the Paskalev-Tchobanov distance formula, we give a short and simple proof of a theorem, published by Marie-Nicole Gras in 2014.

1. INTRODUCTION

In 2014 Marie-Nicole Gras ([1], Theorem 1) has calculated the distances of segments ΩO^2, ΩI^2, ΩH^2 where Ω is the the circumcenter of the extouch triangle and O, I, H are respectively the circumcenter, the incenter, and the orthocenter of a triangle ABC.

We present here a short and simple proof of the Gras theorem, based on the Paskalev-Tchobanov distance formula, published in 1985 in [5].

Recall that the Extouch triangle is the triangle formed by the points of tangency of a triangle ABC with its excircles. See Extouch triangle in [6], [3]. In [4], the Circumcenter of the Extouch triangle is point X(1158).

Given triangle ABC, we denote the sidelengths as follows: $a = BC, b = CA, c = AB$.

We denote by

- $s = \frac{a+b+c}{2}$ the semiperimeter.
- Δ the area.
- R the radius of the Circumcircle.
- r the radius of the Incircle.

The Gras theorem states:

Theorem 1.

$$\Omega O^2 = R^2 - \frac{4R^3(R-r)}{r^2}\cos A\cos B\cos C. \quad (1)$$

$$\Omega I^2 = 2R^2 - 4Rr - \frac{4R^3(R-2r)}{r^2}\cos A\cos B\cos C. \quad (2)$$

$$\Omega H^2 = 2R^2 - 4Rr - 2r^2 - \frac{4R^2(R-r)(R-3r)}{r^2}\cos A\cos B\cos C. \quad (3)$$

The Paskalev-Tchobanov distance theorem states:

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Theorem 2. *Given two points $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$ in normalized barycentric coordinate. Denote $x = u_1 - u_2, y = v_1 - v_2$ and $z = w_1 - w_2$. Then the square of the distance between P and Q is*

$$PQ^2 = -a^2yz - b^2zx - c^2xy.$$

2. Proof of the Gras theorem

We will prove theorem 1 in the following equivalent form:

Theorem 3.

$$\Omega O^2 = \frac{abcE_9}{(a+b+c)(a+b-c)^3(b+c-a)^3(c+a-b)^3}. \quad (4)$$

$$\Omega I^2 = \frac{4abcE_6E_3}{(a+b+c)(b+c-a)^3(c+a-b)^3(a+b-c)^3}, \quad (5)$$

$$\Omega H^2 = \frac{E_{12}}{(a+b+c)(b+c-a)^3(c+a-b)^3(a+b-c)^3}. \quad (6)$$

where

$$\begin{aligned}
E_3 &= a^3+b^3+c^3+3abc-ba^2-b^2a-ca^2-c^2a-c^2b-b^2c,\\
E_6 &= a^6+b^6+c^6+6b^2a^2c^2+3abc^4-2a^3c^2b-2b^3a^2c-2ba^2c^3\\
&\quad -2ab^3c^2-2a^3b^2c-2ab^2c^3+3a^4bc+3ab^4c-a^4b^2\\
&\quad -a^4c^2-a^5b-a^5c-b^4a^2-c^4a^2-c^5a-c^2b^4-c^4b^2\\
&\quad -c^5b-b^5a-b^5c+2a^3b^3+2a^3c^3+2c^3b^3,\\
E_9 &= a^9+b^9+c^9-c^8a-c^8b-a^8b-a^8c-b^8a-b^8c+2c^6a^3+2c^6b^3\\
&\quad -2c^7a^2-2c^7b^2+2c^3a^6+2c^3b^6-2c^2a^7-2c^2b^7-2a^7b^2\\
&\quad +2a^3b^6+2a^6b^3-2a^2b^7+10c^2a^2b^5-6c^2a^3b^4\\
&\quad -2c^2ab^6-5a^5bc^3-5a^5b^3c+10a^5b^2c^2-2a^6b^2c\\
&\quad -2a^6bc^2+10c^5a^2b^2-6c^4a^2b^3+6c^4ab^4-2c^6a^2b\\
&\quad -6c^2a^4b^3-6c^3a^4b^2+6c^4a^4b-6c^3a^2b^4\\
&\quad -5c^5a^3b-6c^4a^3b^2+18c^3a^3b^3-2c^6ab^2\\
&\quad +5c^7ab+5a^7bc-5c^3ab^5-5c^5ab^3+6a^4b^4c-2a^2cb^6+5b^7ac-5a^3b^5c,\\
E_{12} &= a^{12}+b^{12}+c^{12}-b^{11}c-bc^{11}-a^{11}c-a^{11}b-ab^{11}-ac^{11}\\
&\quad -8a^6b^6-8c^6a^6-4b^{10}c^2+3b^9c^3+7b^8c^4-2b^7c^5-8b^6c^6-2b^5c^7\\
&\quad +7b^4c^8+3b^3c^9-4b^2c^{10}-2a^7b^5-2a^7c^5+7a^8c^4-4a^{10}b^2-2a^5c^7\\
&\quad +3a^9b^3-2a^5b^7+7a^4b^8+3a^9c^3-4a^{10}c^2+7a^8b^4+7a^4c^8+3a^3b^9\\
&\quad +3a^3c^9-4a^2b^{10}-4a^2c^{10}+5abc^{10}-13ab^3c^8-2ab^4c^7+10ab^5c^6\\
&\quad +ab^2c^9-13ab^8c^3+5ab^{10}c+10ab^6c^5-2ab^7c^4-2a^2b^5c^5-8a^2b^4c^6\\
&\quad +12a^2b^2c^8+a^9b^2c+5a^{10}bc-6a^5b^3c^4-6a^4b^5c^3-2a^5b^5c^2-2a^7b^4c\\
&\quad -2a^7bc^4-8a^6b^4c^2-8a^6b^2c^4-2a^4b^7c+10a^6bc^5-8a^4b^6c^2+10a^5bc^6\\
&\quad -13a^8b^3c-2a^5b^2c^5+12a^8b^2c^2-6a^5b^4c^3+16a^6b^3c^3+10a^5b^6c\\
&\quad -13a^8bc^3+10a^6b^5c-8a^4b^2c^6-2a^4bc^7+a^9bc^2+18a^4b^4c^4\\
&\quad -6a^4b^3c^5+16a^3b^6c^3-13a^3b^8c-6a^3b^4c^5-6a^3b^5c^4+16a^3b^3c^6\\
&\quad -13a^3bc^8+12a^2b^8c^2-8a^2b^6c^4+a^2b^9c+a^2bc^9+ab^9c^2.
\end{aligned}$$

Proof. By using the formulas

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},\ R = \frac{abc}{4\Delta},\ r = \frac{\Delta}{s},$$

$$cosA = \frac{b^2+c^2-a^2}{2bc},\ cosB = \frac{c^2+a^2-b^2}{2ca},\ cosC = \frac{a^2+b^2-c^2}{2ab},$$

we transform (1) to (4). We see that (1) and (4) are equivalent. Similarly, (2) and (5) are equivalent, and (3) and (6) are equivalent.

The first barycentric coordinate of point Ω is as follows:

$$\begin{aligned}\Omega \ =\ & a(a^6+2a^3b^2c+2a^3bc^2-2ab^4c+2ab^3c^2+2ab^2c^3-2abc^4\\ & -3a^4b^2+2a^4bc-3a^4c^2-b^6+b^4c^2+b^2c^4-c^6\\ & +3a^2b^4-2a^2b^3c-2a^2b^2c^2-2a^2bc^3+3a^2c^4),\end{aligned}$$

and the first barycentric coordinates of points O, I, H are as follows:

$$I = a,\ O = a^2(b^2+c^2-a^2),\ H = (c^2+a^2-b^2)(a^2+b^2-c^2).$$

We use the Paskalev-Tchobanov distanve formula and we obtain (4), (5) and (6). This completes the proof. □

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