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APPLICATIONS OF THE DIVISION BY ZERO CALCULUS TO WASAN GEOMETRY

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Abstract. From the viewpoint of the division by zero $(0/0 = 1/0 = z/0 = 0)$ and the division by zero calculus, we will show interesting applications to Wasan geometry that show unexpected new discovery for some extreme cases.

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1. Introduction

We recall the following result of the old Japanese geometry [14] (see Figure 1):

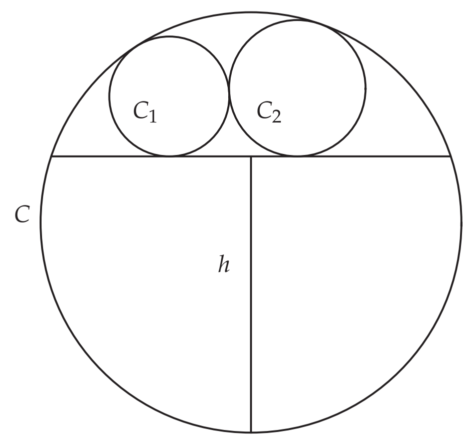

Figure 1.

Lemma. *Assume that the circle C with radius r is divided by a chord t into two arcs and let h be the distance from the midpoint of one of the arcs to t. If two externally touching circles C_1 and C_2 with radii r_1 and r_2 also touch the chord t and the other arc of the circle C internally, then h, r, r_1 and r_2 are related by*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{r_1 r_2 h}}.$$

We are interesting in the limit case $r_1 = 0$ or $r_2 = 0$. Here note the following new idea. As stated already in [6], in general, for a circle with radius r, its curvature is given by $1/r$ and by the division by zero, for the point circle, its curvature is zero. Meanwhile, for a

line corresponding the case $r = \infty$, its curvature is also zero, however, then we should consider the case as $r = 0$, not ∞. For this reality and reasonable situation, look the paper. By this interpretation, we will consider the lemma for the case $r_1 = 0$ or $r_2 = 0$. For the sake of symmetry, we need consider only, for example, for r_1. The beautiful identity is valid for all $r_1 > 0$. How will be the case $r_1 = 0$? However, following the usual sense, we can not consider such problems, but we will be able to consider the case by the concept of the division by zero calculus. We will examine this problem.
In order to see the background of the lemma, we will see its simple proof [9].
The centers of C_1 and C_2 can be on the opposite sides of the normal dropped on t from the center of C or on the same side of this normal. From the right triangles formed by the centers of C and C_i $(i = 1, 2)$, the line parallel to t through the center of C, and the normal dropped on t from the center of C_i, we have

$$\left| \sqrt{(r-r_1)^2 - (h+r_1-r)^2} \pm \sqrt{(r-r_2)^2 - (h+r_2-r)^2} \right| = 2\sqrt{r_1 r_2},$$

where we used the fact that the segment length of the common external tangent of C_1 and C_2 between the points of tangency is equal to $2\sqrt{r_1 r_2}$. The formula of the lemma follows from this equation.

2. The division by zero calculus

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-a)^n, \tag{1}$$

we obtain the identity, by the division by zero

$$f(a) = C_0. \tag{2}$$

For the correspondence (2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering the derivatives in (1), we can define any order derivatives of the function f at the singular point a.
We have considered our mathematics around an isolated singular point for analytic functions, however, we do not consider mathematics at the singular point itself. At the isolated singular point, we consider our mathematics with the limiting concept, however, the limiting values to the singular point and the value at the singular point of the function are different. By the division by zero calculus, we can consider the values and differential coefficients at the singular point.
The division by zero $(0/0 = 1/0 = z/0 = 0)$ is trivial and clear in the natural sense of the generalized division (fraction) against its mysterious and long history (see for example, [11]), since we know the Moore-Penrose generalized inverse for the elementary equation $az = b$. Therefore, the division by zero calculus above and its applications are important. See the references [1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13] for the details and the related topics. We regret that our common sense for the division by zero are still wrong; one typical comment for our division by zero results is given by some physician:
Here is how I see the problem with prohibition on division by zero, which is the biggest scandal in modern mathematics as you rightly pointed out (2017.10.14.8:55).

However, in this paper we do not need any information and results in the division by zero, we need only the definition (2) of the division by zero calculus. – Interestingly, the definition of the division by zero calculus (2) is even wrong, we will be able to obtain an interesting new result from the definition.

3. Results

We introduce the coordinates in the following way: the bottom of the circle C is the origin and tangential line at the origin of the circle C is the x axis and the y axis is given as in the center of the circle C is $(0, r)$. We denote the centers of the circles $C_j; j = 1, 2$ by (x_j, y_j), then we have

$$y_1 = h + r_1, \quad y_2 = h + r_2.$$

Then, from the touching conditions, we obtain the three equations:

$$(x_2 - x_1)^2 + (r_1 - r_2)^2 = (r_1 + r_2)^2,$$
$$x_1^2 + (h - r + r_1)^2 = (r - r_1)^2$$

and

$$x_2^2 + (h - r + r_2)^2 = (r - r_2)^2.$$

Solving the equations for x_1, x_2 and r_2, we get four sets of the solutions. Let $h = 2r_3$, $v = r - r_1 - r_3$. Then two sets are:

$$\begin{aligned} x_1 &= \pm 2\sqrt{r_3 v}, \\ x_2 &= \pm 2\frac{r_1\sqrt{rr_3} + r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} - \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

The other two sets are

$$\begin{aligned} x_1 &= \pm 2\sqrt{r_3 v}, \\ x_2 &= \mp 2\frac{r_1\sqrt{rr_3} - r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} + \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

We now consider the solution

$$\begin{aligned} x_1 &= 2\sqrt{r_3 v}, \\ x_2 &= 2\frac{r_1\sqrt{rr_3} + r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} - \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

Then

$$(x - x_2)^2 + (y - y_2)^2 - r_2^2 = \frac{g_0 + g_1 r_1 + g_2 r_1^2 + g_3}{(r_1 + r_3)^2},$$

where

$$g_0 = r_3^2(x^2 + y(y - 4r_3) + 4rr_3),$$
$$g_1 = 2r_3((x - \sqrt{rr_3})^2 + y^2 - (2r + 3r_3)y + 3rr_3),$$

$$g_2 = (x - 2\sqrt{rr_3})^2 + y^2 - 2r_3y,$$

and

$$g_3 = 4r_3\sqrt{v}(r_1(\sqrt{r}y - \sqrt{r_3}x) - r_3\sqrt{r_3}x).$$

We now consider another solution

$$\begin{aligned} x_1 &= 2\sqrt{r_3v}, \\ x_2 &= -2\frac{r_1\sqrt{rr_3} - r_3\sqrt{r_3v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1r_3(2\sqrt{r}(\sqrt{r} + \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

Then

$$(x - x_2)^2 + (y - y_2)^2 - r_2^2 = \frac{k_0 + k_1r_1 + k_2r_1^2 + k_3}{(r_1 + r_3)^2},$$

where

$$k_0 = r_3^2(x^2 + y(y - 4r_3) + 4rr_3),$$
$$k_1 = 2r_3((x + \sqrt{rr_3})^2 + y^2 - (2r + 3r_3)y + 3rr_3),$$
$$k_2 = (x + 2\sqrt{rr_3})^2 + y^2 - 2r_3y,$$

and

$$k_3 = -4r_3\sqrt{v}(r_1(\sqrt{r}y + \sqrt{r_3}x) + r_3\sqrt{r_3}x).$$

We thus see that the circle C_2 is represented by

$$(g_0 + g_3) + g_1r_1 + g_2r_1^2 = 0$$

and

$$(k_0 + k_3) + k_1r_1 + k_2r_1^2 = 0.$$

For the symmetry, we consider only the above case. We obtain the division by zero calculus, first by setting $r_1 = 0$, the next by setting $r_1 = 0$ after dividing by r_1 and the last by setting $r_1 = 0$ after dividing by r_1^2,

$$g_0 + g_3 = 0,$$
$$g_1 = 0,$$

and

$$g_2 = 0.$$

That is,

$$\left(x - \sqrt{2rh - h^2}\right)^2 + (y - h)^2 = 0,$$
$$\left(x - \sqrt{\frac{rh}{2}}\right)^2 + \left(y - \left(r + \frac{3h}{4}\right)\right)^2 = r^2 + \frac{9}{16}h^2,$$

and

$$\left(x - \sqrt{2rh}\right)^2 + \left(y - \frac{h}{2}\right)^2 = \left(\frac{h}{2}\right)^2.$$

The first equation represents one of the points of intersection of the circle C and the chord t (see Figure 2). The second equation expresses the red circle in the figure. The third equation expresses the circle touching C externally, the x-axis and the extended chord t denoted by the green circle in the figure. The last two circles are orthogonal to the circle with center origin passing through the points of intersection of C and t.

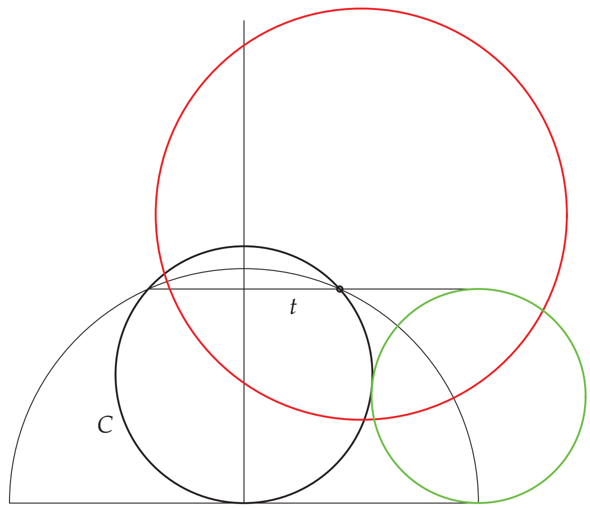

Figure 2.

Now for the beautiful identity in the lemma, for $r_1 = 0$, we have, by the division by zero,

$$\frac{1}{0} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{0 \cdot r_2 h}}$$

and

$$r_2 = -\frac{h}{2}.$$

Here, the minus sigh will mean that the green circle touches the circle C from the outside of the circle C; that is, we can consider that when the circle C_1 is reduced to the point $(\sqrt{2rh - h^2}, h)$, then the circle C_2 is suddenly changed to the green circle and the beautiful identity is still valid. Note, in particular, the green circle touches the circle C and the chord t.

Meanwhile, for the curious red circle, we do not know its property except the orthogonality mentioned before.

4. Conclusion

When a circle degenerates to a point, some identity seems to be not valid in a usual way, for such a case, by the division by zero calculus, the identity is still valid with new surprising phenomena, in discontinuously. The division by zero calculus will open a new world as a general principle.

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