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HOMOGENEOUS FINSLER SPACES WITH SPECIAL NON-RIEMANNIAN CURVATURES

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ABSTRACT. We prove that relatively isotropic J-curvature homogeneous Finsler space with negative scalar function is Riemannian. Secondly, we show that a generalized symmetric Finsler space has almost vanishing Ξ-curvature if and only if it has vanishing Ξ-curvature.

1. INTRODUCTION

We know that in Riemannian geometry, inner product spaces in any dimension up to isomorphism are unique, but such is not the case for Minkowski's norms. Therefore, various Finsler metrics on tangent space of a Finsler manifold are not isomorphic, in the words of Shen, a Finsler manifold is usually colorful. Thus, it is attractive to study Finsler manifolds with single color. Y. Ichijyo in [7] has studied these manifolds as the title *Finsler metrics modeled on a Minkowski space*. Homogeneous Finsler spaces are examples of these spaces which by S. Deng, D. Latifi and some others have studied [3], [8]. Non-Riemannian curvatures such as Cartan torsion, Landsberg curvature, mean Landsberg curvature, Berwald curvature, S-curvature, stretch curvature, weakly stretch curvature and Ξ-curvature have been introduced and examined [3], [11], [12]. All these quantities vanishes for Riemannian case. It is important to understand the geometric meanings of these quantities. Some theorems in the Finsler geometry can be proved for a Finslerian homgeneous space with less assumptions.

Theorem 1.1. *Let (M,F) be a homogeneous connected Finsler manifold satisfying $J+c(x)FI=0$. Suppose that $c(x)\le c_0<0$. Then F is Riemannian.*

The study of the role and effect of non-Riemannian quantities in the Finslerian homogeneous space can help us to understand these spaces profoundly. We examine some of these quantities on special homogeneous Finsler spaces.

Theorem 1.2. *Let (M,F) be a generalized symmetric Finsler space. Then (M,F) has almost vanishing Ξ-curvature if and only if F has vanishing Ξ-curvature.*

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2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on TM_0 (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F\,|_{T_xM}$. Cartan introduced a quantity to measure the non-Euclidean feature of F_x as follows $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \longrightarrow \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}[\mathbf{g}_{y+tw}(u, v)]\,|_{t=0}, \qquad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. Deicke showed that $\mathbf{C} = 0$ if and only if F is Riemannian. For $y \in T_xM_0$, define the mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$ [9].

There is a notion of distortion $\tau = \tau(x, y)$ on TM associated with the Busemann-Hausdorff volume form $dV = \sigma(x)dx$ of the Finsler metric F, which is defined by $\tau(x, y) = ln\frac{\sqrt{det(g_{ij}(x,y))}}{\sigma(x)}$. We have $I_i = \frac{\partial \tau}{\partial y^i}$ [8].
The rate of change of the distortion along geodesics is called S-curvature and defined by $\mathbf{S}(x, y) = \tau_{|i}y^i$, where $|$ is horizontal covariant derivative with respect to the Chern connection. A Finsler metric F on an n-dimensional manifold M is said to have almost isotropic S-curvature if there is a scalar function $c = c(x)$ on M such that

$$S = (n + 1)\{cF + \eta\}$$

where $\eta = \eta_i(x)y^i$ is a closed 1-form. F is said to have isotropic S-curvature if $\eta = 0$ [2]. Another non-Riemannian quantity defined as,

$$\Xi = \Xi_i dx^i$$

where $\Xi_i := S_{.i|m}y^m - S_{|i}$, "." and " $|$ " denote the vertical and horizontal covariant derivatives, respectively, with respect to the Chern connection.
We say a Finsler metric have almost vanishing Ξ-curvature if $\Xi_i = -(n + 1)F^2(\frac{\theta}{F})_{y^i}$ where $\theta = \theta_i(x)y^i$ is a 1-form on M [11].

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics gives rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \longrightarrow \mathbb{R}$ defined by

$$\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k,$$

where $L_{ijk} := C_{ijk|s}y^s$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i(x, y)$ are local functions on TM given by

$$G^i(x, y) := \frac{1}{4} g^{il} \left\{ \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right\}.$$

The vector field $\mathbf{G}$ is called the associated spray to (M, F) [9]. In local coordinates, a curve $c = c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$. A Finsler manifold is said to be forward geodesically complete if every geodesic $c(t), a \leq t \leq b$, parametrized to have constant Finslerian speed, can be extended to a geodesic defined on $a \leq t < \infty$ [8].

The horizontal covariant derivatives of $\mathbf{I}$ along geodesic gives rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where $J_i := g^{jk} L_{ijk} = I_{i|s} y^s$. A Finsler metric is said to be weakly Landsbergian if $\mathbf{J} = 0$. The Finsler metric F is said to be of relatively isotropic $\mathbf{L}$-curvature (resp. relatively isotropic $\mathbf{J}$-curvature) if there is a scalar function $c(x)$ on M such that $L + c(x)FC = 0$ (resp. $J + c(x)FI = 0$) [9].

Define the stretch curvature $\mathbf{\Sigma}_y : T_xM \otimes T_xM \otimes T_xM \otimes T_xM \longrightarrow \mathbb{R}$ by $\mathbf{\Sigma}_y(u, v, w, z) := \Sigma_{ijkl}(y) u^i v^j w^k z^l$, where

$$\Sigma_{ijkl} := 2(L_{ijk|l} - L_{ijl|k}).$$

A Finsler metric is said to be stretch metric if $\mathbf{\Sigma} = 0$. For $y \in T_xM_0$, define $\bar{\mathbf{\Sigma}}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\bar{\mathbf{\Sigma}}_y(u, v) := \bar{\Sigma}_{ij}(y) u^i v^j$, where $\bar{\Sigma}_{ij} := g^{kl} \sum_{klij}$. A Finsler metric is said to be a weakly stretch metric if $\bar{\mathbf{\Sigma}} = 0$ [12].

For $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \longrightarrow T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \longrightarrow \mathbb{R}$ by

$$\mathbf{B}_y(u, v, w) := B^i_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i} |_x \quad and \quad \mathbf{E}_y(u, v) := E_{jk}(y) u^j v^k$$

where $B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, E_{jk} := \frac{1}{2} B^m_{jkm}$. The tensors $\mathbf{B}$ and $\mathbf{E}$ are called Berwald curvature and mean Berwald curvature, respectively. Then F is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = 0$ and $\mathbf{E} = 0$, respectively [9].

2.1. **Homogeneous Finsler space.** Many works by some authors like Akbarzadeh, Shen had done on the geometric properties of Finsler geometry but group aspects of this space had been neglected. It was Deng who proved that group of isometries $I(M, F)$ of M is a Lie transformation group of M and since then, many works have been done about group aspect in Finsler geometry. Let $\phi : (M, F) \longrightarrow (M, F)$ be a diffeomorphism. Then ϕ is called an isometry of (M, F) if

$$F(\phi(x), d\phi_x(X)) = F(x, X), \forall x \in M, X \in T_xM.$$

A Finsler space (M,F) is called Finslerian homogeneous space if the group of isometries, i.e., $I(M,F)$ acts transitively on M. Hence, in homogeneous Finsler space the tangent Minkowski spaces (T_xM, F_x) are all linearly isometric to each other. Every Finslerian homogeneous space is forward complete [8].

An isometry of (M,F) with x as an isolated fixed point is called a *symmetry* at x, and will usually be denoted as s_x. A family $\{s_x \mid x \in M\}$ of symmetries on a connected Finsler manifold (M,F) is called an *s-structure* on (M,F). An s-structure $\{s_x \mid x \in M\}$ is called of order k $(k \geq 2)$ if $(s_x)^k = id$, for all $x \in M$ and k is the least integer satisfying the above property.

An s-structure $\{s_x\}$ on (M,F) is called regular if for every pair of points $x, y \in M$,

$$s_x \circ s_y = s_z \circ s_x,$$

where $z = s_x(y)$.

A *generalized symmetric* Finsler space is a connected Finsler manifold (M,F) admitting a regular s-structure. A Finsler space (M,F) is said to be k-symmetric $(k \geq 2)$ if it admits a regular s-structure of order k [2]. A connected Finsler space (M,F) is said to be *symmetric* if it is a regular s-structure of order 2. Symmetric Finsler spaces are examples of reversible homogeneous Finsler spaces [3].

3. Proof of Theorem 1.1

Symmetric Finsler spaces and locally symmetric Finsler spaces are Berwaldian and so have vanishing S-curvatures [3] [13]. Weakly symmetric and restrictively CW-homogeneous Finsler spaces must be a Finsler g.o space [3] [4]. A Finsler g.o has vanishing S-curvature [3], therefore weakly symmetric Finsler spaces and restrictively CW-homogeneous Finsler spaces have vanishing S-curvatures. A generalized symmetric Finsler space has almost isotropic S-curvature if and only if it has vanishing S-curvature [5]. It seems that distorsion along geodesics in homogeneous Finsler spaces does not have any changes and is constant. Therefore, two curvetures of Berwald and Landsberg in the Finslerian homogeneous space play a decisive role. Specially, Cartan and Landsberg tensors in homogeneous Finsler spaces are bounded. We try to examine the behavior of these two curvetures and their particular situations.

Lemma 3.1. *Cartan tensor and its vertical derivative of homogeneous Finsler spaces are bounded.*

Proof. Let (M,F) be an n-dimensional Finsler manifold and φ a local isometry of F, i.e., in a standard local coordinates, we have

$$F(x^i, y^i) = F(\phi^i(x), y^j \frac{\partial \phi^i}{\partial x^j}) \tag{1}$$

Putting $\tilde{x}^i = \varphi^i(x)$ and $\tilde{y}^i = \tilde{y}^j \frac{\partial \phi^i}{\partial x^j}$ yields $F(x^i, y^i) = F(\tilde{x}^i, \tilde{y}^i)$. Thus, we have

$$\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k} = \frac{\partial \varphi^l}{\partial x^i} \frac{\partial \varphi^p}{\partial x^j} \frac{\partial \varphi^s}{\partial x^k} \frac{\partial^3 F^2}{\partial \tilde{y}^l \partial \tilde{y}^p \partial \tilde{y}^s} \tag{2}$$

which means that the Cartan tensor is invariant under isometries. Each two points of a homogeneous Finsler space map to each other by an isometry. Therefore, the norm of Cartan tensor of a homogeneous Finsler space, which is defined by $\|C\| = sup_{x\in M}\|C_x\|$ is a constant function on the underlying manifold. Thus the Cartan tensor of a homogeneous Finsler space is bounded.
Taking a vertical derivative of (2), we conclude that the vertical derivative of Cartan tensor is also invariant under isometries and a similar argument shows that it is bounded for a homogeneous Finsler space. □

Corollary 3.1. *Using Akbarzadeh theorem [1] we conclude that a complete homogeneous Finsler manifold (M, F) with constant zero sectional curvature, is a locally Minkowskian. Moreover if F is of constant negative sectional curvature, then it is a Riemannian.*

Proposition 3.1. *[8] Let (M, F) be a homogeneous Finsler manifold. Then F is forward geodesically complete.*

Proof of Theorem 1.1. Consider the arbitrary point p in M and vector $y \in T_pM$. Let $c : [0, +\infty) \longrightarrow M$ be the geodesic parametrized by arc length on M with the initiating point $c(0) = p$ and tangent vector $c'(0) = y$. Suppose that $U(t)$ is a parallel vector field along geodesic $c(t)$ with $U(0) = u$. We define $I(t) = I(U(t))$ and $J(t) = J(U(t))$. According to $J_i = I_{i|m}y^m$ we have $J(t) = I'(t)$. Restricting $J + c(x)FI = 0$ to the geodesic c, we get the following ODE

$$I'(t) + c(t)I(t) = 0,$$

which its general solution is $I(t) = I(0)e^{-\int_0^t c(s)ds}$. By Lemma 3.1, the mean Cartan tensor is bounded. Let $I(0) \neq 0$. Proposition 3.1 implies that M is forward geodesically complete and the parameter t takes all the values in $[0, +\infty)$. Letting $t \to +\infty$ we have $I(t)$ is unbounded which is a contradiction. Therefore $I(0) = 0$ and consequently $I(t) = 0$. Actually $I(U(t)) = 0$ along any geodesic, so $I = 0$. It follows from Deicke's theorem F is a Riemannian metric. □

Corollary 3.2. *Every symmetric (also weakly symmetric) Finsler space with non-zero negative constant isotropic J-cutrvature is Riemannian.*

A Finsler metric $\Theta = \Theta(x, y)$ on an open subset in $\mathbb{R}^n$ is called a Funk metric if it satisfies $\Theta_{x^k} = \Theta\Theta_{y^k}$. The Funk metric on a strongly convex domain in $\mathbb{R}^n$ is forward complete but not complete [8]. It is well-known that Θ is of J-isotropic with constant flag curvature $-\frac{1}{4}$ and is not Riemannian. Therefore, Θ is not a homogeneous Finsler metric.

Theorem 3.2. *Every weakly stretch homogeneous Finsler manifold is weakly Landsbergian.*

Proof. Take an arbitrary unit vector $y \in T_xM$ and an arbitrary vector $v \in T_xM$. Suppose that $c = c(t)$ is a geodesic with $c(0) = x$ and $\dot{c}(0) = y$ and $V(t)$ be the parallel vector field along c with $V(0) = v$. According to ([12],Lemma 3.1) we have $I(t) = tJ(0) + I(0)$, as Proposition 3.1, $I(t)$ is a bounded function on $[0, \infty)$. Letting $t \to \infty$ in the equation, implies that $J_y(v) = J(0) = 0$. Thus F is a weakly Landsberg metric. □

4. Proof of Theorem 1.2

Since (M,F) is a generalized symmetric, for any $x \in M$, there is an symmetry s_x with x as an isolated fixed point. Suppose (M,F) has almost vanishing Ξ-curvature then

$$\Xi_i(x,y) = -(n+1)F^2(x,y)(\frac{\theta(x,y)}{F(x,y)})_{y^i}, \quad x \in M, y \in T_xM \tag{3}$$

where $\theta(x,y) := \theta_i(x)y^i$. Since s_x is a diffeomorphism, ds_x is a linear isometry. We have $\Xi_i(x,y) = \Xi_i(s_x(x), ds_x(y))$. Thus,

$$-(n+1)F^2(x,y)(\frac{\theta(x,y)}{F(x,y)})_{y^i} = -(n+1)F^2(s_x(x),ds_x(y))(\frac{\theta(s_x(x),ds_x(y))}{F(s_x(x),ds_x(y))})_{y^i} \tag{4}$$

Since $F(x,y) = F(s_x(x), ds_x(y))$ so $(\frac{\theta(x,y)}{F(x,y)})_{y^i} = (\frac{\theta(s_x(x),ds_x(y))}{F(s_x(x),ds_x(y))})_{y^i}$, and

$$\theta_i(x)F(x,y) - \theta(x,y)F(x,y)_{y^i} = \theta_i(x,ds_x(y))F(x,y) - \theta(x,ds_x(y))F(x,y)_{y^i},$$

We have $\theta((ds_x - id)(y)) = 0$, where id is the identity transformation on T_xM. Choose a basis $y_1, y_2, ..., y_n$ of T_xM so we have $\theta((ds_x - id)(y_i)) = 0$, for all i.

Base on Theorem 3.2 in [6] thus $(ds_x - id)$ is also a non-singular transformation on T_xM and $(ds_x - id)(y_1), (ds_x - id)(y_2), ..., (ds_x - id)(y_n)$ is a basis of T_xM. This implies that $\theta = 0$.

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