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TRANS-SASAKIAN MANIFOLDS WITH RESPECT TO A NON-SYMMETRIC NON-METRIC CONNECTION

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ABSTRACT. Trans-Sasakian manifolds with respect to a non-symmetric non-metric connection are studied. Some properties of the curvature tensor, the conformal curvature tensor, the conharmonic curvature tensor of trans-Sasakian manifolds admitting a non-symmetric non-metric connection are studied. Nijenhuis tensor studied as well.

1. INTRODUCTION

Systematic study of semi-symmetric connection in a Riemannian manifold was initiated by Yano [14]. In 1992, Agashe and Chafle [1] introduced the notion of semi-symmetric non-metric connection. In 2007, S. K. Chaubey [4] introduced another non-symmetric non-metric connection. Later he studied the same connection on Kenmotsu manifold [5].

On the other hand there is a class of almost contact metric manifold, namely trans-Sasakian manifold [9], which generalizes both α -Sasakian [13] and β-Kenmotsu [8] structure. It was also studied by several geometers [6], [7], [11] . In this paper we study the non-symmetric non-metric connection [4] on trans-Sasakian manifold.

2. PRELIMINARIES

Let M be a $(2n+1)$-dimensional almost contact metric manifold [2] equipped with almost contact metric structure (ϕ, ξ, η, g), where ϕ is $(1,1)$ tensor field, ξ is a vector field, η is 1-form and g is compatible Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta o \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y), \qquad g(X,\xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in TM$.

An almost contact metric manifold M [11] is called trans-Sasakian manifold if

$$(\nabla_X \phi)Y = \alpha \left\{ g\left(X,Y\right)\xi - \eta\left(Y\right)X \right\} + \beta \left\{ g\left(\phi X, Y\right)\xi - \eta\left(Y\right)\phi X \right\} \tag{2.4}$$

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where ∇ is Levi-Civita connection of Riemannian metric g and α and β are smooth functions on M. From equation (2.4) and equations (2.1), (2.2) and (2.3), we have

$$\nabla_X \xi = -\alpha\phi X + \beta\left[X - \eta(X)\xi\right], \tag{2.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.6}$$

Also the following relations hold in a trans-Sasakian manifold [7]:

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2-\beta^2)\left[\eta(Y)X - \eta(X)Y\right] - \beta d\eta(X,Y)\xi \\ &\quad +2\alpha\beta\left[\eta(Y)\phi X - \eta(X)\phi Y\right] + (Y\alpha)\phi X \\ &\quad -(X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \end{aligned} \tag{2.7}$$

$$\begin{aligned} R(\xi,Y)X &= \left(\alpha^2-\beta^2\right)\left[g(X,Y)\xi - \eta(X)Y\right] \\ &\quad +2\alpha\beta\left[g(\phi X,Y)\xi - \eta(X)\phi Y\right] \\ &\quad +(X\alpha)\phi Y + g(\phi X,Y)(grad\alpha) \\ &\quad +X\beta\left[Y - \eta(Y)\xi\right] - g(\phi X,\phi Y)(grad\beta), \end{aligned} \tag{2.8}$$

$$R(\xi,X)\xi = \left(\alpha^2-\beta^2-\xi\beta\right)\left[\eta(X)\xi - X\right], \tag{2.9}$$

$$\begin{aligned} S(X,\xi) &= \left(2n\left(\alpha^2-\beta^2\right) - \xi\beta\right)\eta(X) \\ &\quad -(2n-1)X\beta - (\phi X)\alpha, \end{aligned} \tag{2.10}$$

$$\begin{aligned} Q\xi &= \left(2n\left(\alpha^2-\beta^2\right) - \xi\beta\right)\xi \\ &\quad -(2n-1)grad\beta + \phi(grad\alpha), \end{aligned} \tag{2.11}$$

where R is the curvature tensor, S is the Ricci-curvature and Q is the Ricci-operator of trans-Sasakian manifold of type (α,β). S and Q are related to each other by

$$S(X,Y) = g(QX,Y) \tag{2.12}$$

and

$$2\alpha\beta + \xi\alpha = 0. \tag{2.13}$$

Equation (2.10) implies that in trans-Sasakian manifold of type (α,β), α and β are not arbitrary functions but related to each other by structural vector field ξ. Equation (2.10) also implies that α and β are not non-zero constants simultaneously. If $\xi\alpha = 0$ and $\alpha \neq 0$, we have $\beta = 0$ and we can state the following.

Now we shall give two proper examples of trans-Sasakian manifold of type (α,β) which are neither α-Sasakian nor β-Kenmotsu and both the examples satisfy equation (2.10).

Example 2.1. *Let (x,y,z) be Cartesian coordinate in R^3, then (ϕ,ξ,η,g) given by $\xi = \frac{\partial}{\partial z}$, $\eta = dz - ydx$, $\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix}$, $g = \begin{pmatrix} e^z+y^2 & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \end{pmatrix}$ is a trans-Sasakian structure of type $\left(-\frac{1}{2e^z}, \frac{1}{2}\right)$ in R^3 [11].*

Example 2.2. *In 3-dimensional K-contact manifold with structure tensors (ϕ, ξ, η, g), for a non-constant function f ,defined $g' = fg + (1-f)\,\eta \otimes \eta$ then (ϕ, ξ, η, g') is a trans-Sasakian structure of type $\left(\frac{1}{f}, \frac{1}{2}\xi\,(\ln f)\right)$ [11].*

It is easy to verify that examples 2.1 and 2.2 satisfy the condition

$$2\alpha\beta + \xi\alpha = 0.$$

The Nijenhuis tensor $N(X,Y)$ of ϕ in an (M_{2n+1}, g) is a vector valued bilinear function such that

$$N(X,Y) = \left(\nabla_{\phi X}\phi\right)(Y) - \left(\nabla_{\phi Y}\phi\right)(X) - \phi\left(\left(\nabla_X\phi\right)(Y)\right) + \phi\left(\left(\nabla_Y\phi\right)(X)\right). \tag{2.14}$$

If we define

$$'N(X,Y,Z) = g\left(N(X,Y), Z\right). \tag{2.15}$$

then

$$\begin{aligned} 'N(X,Y,Z) &= g\left(\left(\nabla_{\phi X}\phi\right)(Y), Z\right) - g\left(\left(\nabla_{\phi Y}\phi\right)(X), Z\right) \\ &\quad - g\left(\phi\left(\left(\nabla_X\phi\right)(Y)\right), Z\right) + g\left(\phi\left(\left(\nabla_Y\phi\right)(X)\right), Z\right). \end{aligned} \tag{2.16}$$

3. A NON-SYMMETRIC NON-METRIC CONNECTION

On the other hand, a linear connection $\widetilde{\nabla}$ [4], [5] defined as

$$\widetilde{\nabla}_X Y = \nabla_X Y + g\left(\phi X, Y\right)\xi \tag{3.1}$$

satisfying

$$\widetilde{T}\left(X,Y\right) = 2g\left(\phi X, Y\right)\xi \tag{3.2}$$

and

$$\left(\widetilde{\nabla}_X g\right)(Y,Z) = -\eta\left(Z\right) g\left(\phi X, Y\right) - \eta\left(Y\right) g\left(\phi X, Z\right) \tag{3.3}$$

for arbitrary vector field X, Y and Z, is called a non-symmetric non-metric connection. It is also known [4]

$$\left(\widetilde{\nabla}_X\phi\right)(Y) = \left(\nabla_X\phi\right)(Y) + g\left(\phi X, \phi Y\right)\xi, \tag{3.4}$$

$$\left(\widetilde{\nabla}_X\eta\right)(Y) = \left(\nabla_X\eta\right)(Y) + g\left(\phi X, Y\right), \tag{3.5}$$

$$\left(\widetilde{\nabla}_X g\right)(\phi Y, Z) = -\eta\left(Z\right) g\left(\phi X, \phi Y\right). \tag{3.6}$$

On changing Y by ξ in the equation (3.1), we have

$$\widetilde{\nabla}_X\xi = \nabla_X\xi. \tag{3.7}$$

Hence we can say

Proposition 3.1. *The vector field ξ is invariant with respect to Levi-Civita connection ∇ and non-symmetric non-metric connection $\widetilde{\nabla}$.*

On taking $X = \xi$ in the equation (3.3),we get

$$\left(\widetilde{\nabla}_\xi g\right)(Y,Z) = 0. \tag{3.8}$$

Hence we can say

Proposition 3.2. *Co-variant differentiation of Riemannian metric g with respect to contra-variant vector field ξ vanish identically in a contact metric manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$.*

The curvature tensor $\widetilde{R}$ of $\widetilde{\nabla}$ defined as follows

$$\widetilde{R}(X,Y,Z) = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z \tag{3.9}$$

Using equations (3.1), (3.4), (3.5), (3.6) and (3.7), we get

$$\begin{aligned}\widetilde{R}(X,Y,Z) &= R(X,Y,Z) + g((\nabla_X\phi)Y,Z)\xi - g((\nabla_Y\phi)X,Z)\xi \\ &\quad + g(\phi Y,Z)\nabla_X\xi - g(\phi X,Z)\nabla_Y\xi,\end{aligned} \tag{3.10}$$

where

$$R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{3.11}$$

is the Riemannian curvature tensor of Levi-Civita connection [2].

4. Trans-Sasakian manifolds with respect to a non-symmetric non-metric connection

If we define

$$\widetilde{d}\eta(X,Y) = \left(\widetilde{\nabla}_X\eta\right)Y - \left(\widetilde{\nabla}_Y\eta\right)X. \tag{4.1}$$

Using equations (2.6) and (3.5), we get

$$\widetilde{d}\eta(X,Y) = 2(\alpha-1)g(\phi X,Y). \tag{4.2}$$

If we take $\alpha = 1$ in the equation (2.13), we get $\beta = 0$. Hence we can say

Theorem 4.1. *1-form η is closed in a Sasakian manifold as well as in a β-Kenmotsu manifold admitting the non-symmetric non-metric connection.*

Let us define Nijenhuis tensor $\widetilde{N}(X,Y)$ of ϕ in an (M_{2n+1},g) admitting the non-symmetric non-metric connection

$$\begin{aligned}\widetilde{N}(X,Y) &= \left(\widetilde{\nabla}_{\phi X}\phi\right)(Y) - \left(\widetilde{\nabla}_{\phi Y}\phi\right)(X) \\ &\quad -\phi\left(\left(\widetilde{\nabla}_X\phi\right)(Y)\right) + \phi\left(\left(\widetilde{\nabla}_Y\phi\right)(X)\right).\end{aligned} \tag{4.3}$$

In view of equations (2.1) and (3.4), we get

$$\begin{aligned}\widetilde{N}(X,Y) &= \left(\nabla_{\phi X}\phi\right)(Y) - \left(\nabla_{\phi Y}\phi\right)(X) \\ &\quad -\phi\left(\left(\nabla_X\phi\right)(Y)\right) + \phi\left(\left(\nabla_Y\phi\right)(X)\right) + 2g(\phi X,Y)\xi.\end{aligned} \tag{4.4}$$

Using equation (2.4), we get

$$\widetilde{N}(X,Y) = (\alpha+2)g(\phi X,Y)\xi. \tag{4.5}$$

Hence we can say

Theorem 4.2. *In a (-2)-Sasakian manifold as well as in a β-Kenmotsu manifold admitting the non-symmetric non-metric connection, $\widetilde{\nabla}$ is integrable.*

In view of equation (2.15), we have

$$'\widetilde{N}(X,Y,Z) = (\alpha+2)g(\phi X,Y)\eta(Z). \tag{4.6}$$

Hence we can say

Corollary 4.1. *In a trans-Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$, we have*

$$'\widetilde{N}(\xi,Y,Z) =' \widetilde{N}(X,\xi,Z) =' \widetilde{N}(X,Y,\phi Z) = 0. \tag{4.7}$$

Theorem 4.3. *If a trans-Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$, then scalar curvature of $\widetilde{\nabla}$ coincide with that of ∇.*

Proof. Using equations (2.1), (2.3), (2.4) and (2.5) in (3.10), we have

$$\begin{aligned}\widetilde{R}(X,Y,Z) \quad = \quad & R(X,Y,Z) + \alpha[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ & + g(\phi X,Z)(\phi Y) - g(\phi Y,Z)(\phi X)] \\ & + \beta[2g(\phi X,Y)\eta(Z)\xi + g(\phi Y,Z)X - g(\phi X,Z)Y]. \end{aligned} \tag{4.8}$$

Contracting equation (4.8) with respect to X, we have

$$\widetilde{S}(Y,Z) = S(Y,Z) + 2n\beta g(\phi Y,Z). \tag{4.9}$$

By virtue of equation (2.12), equation (4.9) gives

$$\widetilde{Q}(Y) = Q(Y) + 2n\beta(\phi Y). \tag{4.10}$$

Again contracting equation (4.9), we get

$$\widetilde{r} = r, \tag{4.11}$$

where $\widetilde{S}(Y,Z)$; $S(Y,Z)$ and $\widetilde{r}$; r are the Ricci tensor and scalar curvature of the connections $\widetilde{\nabla}$ and ∇ respectably. □

Theorem 4.4. *In a trans-Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$, the necessary and sufficient condition for the Ricci-tensor $\widetilde{S}$ of $\widetilde{\nabla}$ to be skew-symmetric is that the manifold is Ricci-flat.*

Proof. In view of equations (2.3), (4.9) and $S(Y,X) = S(X,Y)$, we have

$$\widetilde{S}(X,Y) + \widetilde{S}(Y,X) = 2S(X,Y). \tag{4.12}$$

The proof of the theorem is obvious from equation (4.12). □

In view of equations (2.3), (4.9) and $S(Y,X) = S(X,Y)$, we have

$$\widetilde{S}(X,Y) - \widetilde{S}(Y,X) = 2n\beta g(\phi X,Y). \tag{4.13}$$

Hence we can state

Theorem 4.5. *In a trans-Sasakian manifold of dimension ≥ 3 admitting the non-symmetric non-metric connection $\widetilde{\nabla}$, the Ricci-tensor $\widetilde{S}$ of $\widetilde{\nabla}$ is non-symmetric.*

The equation (4.13) is independent of α. Hence we can say that in a α-Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$, the Ricci-tensor $\widetilde{S}$ of $\widetilde{\nabla}$ is symmetric.

Theorem 4.6. *If Riemannian curvature tensor of $\widetilde{\nabla}$ in a α-Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$ vanish, then the manifold is Ricci-flat.*

Proof. On taking $\widetilde{R}(X,Y,Z) = 0$ in the equation (4.8), we have

$$\begin{aligned} R(X,Y,Z) &= \alpha[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &\quad + g(\phi Y,Z)(\phi X) - g(\phi X,Z)(\phi Y)] \\ &\quad + \beta[g(\phi X,Z)Y - g(\phi Y,Z)X - 2g(\phi X,Y)\eta(Z)\xi]. \end{aligned} \tag{4.14}$$

For α-Sasakian manifold, equation (4.14) reduced to

$$\begin{aligned} R(X,Y,Z) &= \alpha[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &\quad + g(\phi Y,Z)(\phi X) - g(\phi X,Z)(\phi Y)]. \end{aligned} \tag{4.15}$$

In view of $'R(X,Y,Z,W) = g(R(X,Y,Z),W)$ and equation (4.15), we have

$$\begin{aligned} 'R(X,Y,Z,W) &= \alpha[g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) \\ &\quad + g(\phi Y,Z)g(\phi X,W) + g(X,\phi Z)g(\phi Y,W)]. \end{aligned} \tag{4.16}$$

Contracting equation (4.15) with respect to vector fields X, we get

$$S(Y,Z) = 0. \tag{4.17}$$

Which leads to the proof. □

Theorem 4.7. *If curvature tensor of a $(2n+1)$-dimensional trans-Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$ vanish, then the curvature tensor of the manifold satisfies the following relations*

(i) $S(X,Y)\xi = nR(X,Y,\xi) = -2n\beta g(\phi X,Y)\xi,$
(ii) $R(\xi,X,\xi) = 0,$
(iii) $R(\xi,X,Y) = -[\alpha g(\phi X,\phi Y) + \beta g(\phi X,Y)],$
(iv) $'R(\phi X,Y,\xi,Z) - \beta g(\phi X,\phi Y)\eta(Z) = 0.$

Proof. On taking $Z = \xi$ in the equation (4.14), we have

$$R(X,Y,\xi) = -2\beta g(\phi X,Y)\xi. \tag{4.18}$$

In view of $'R(X,Y,Z,W) = g(R(X,Y,Z),W)$ and equation (4.14), we have

$$\begin{aligned} 'R(X,Y,Z,W) &= \alpha[g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) \\ &\quad + g(\phi Y,Z)g(\phi X,W) + g(X,\phi Z)g(\phi Y,W)] \\ &\quad + \beta[g(\phi X,Z)g(Y,W) - g(\phi Y,Z)g(X,W) \\ &\quad - 2g(\phi X,Y)\eta(Z)\eta(W)]. \end{aligned} \tag{4.19}$$

Contracting equation (4.14) with respect to vector fields X, we get

$$S(Y,Z) = -2n\beta g(\phi Y,Z). \tag{4.20}$$

(i) is obvious in reference of equation (4.18) and (4.20).

Changing X by ξ and then replacing Y by X in the equation (4.18) leads to the proof of (ii).

Taking $X = \xi$ and replacing $Y; Z$ respectively by $X; Y$ in the equation (4.14) gives (iii).

On changing $Z = \xi$ and then replacing W by Z gives

$$'R(X,Y,\xi,Z) = -2\beta g(\phi X,Y)\eta(Z). \tag{4.21}$$

On replacing X by ϕX in the equation (4.21) gives (iv). □

Theorem 4.8. *In a trans-Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$, the necessary and sufficient condition for the conformal curvature tensor of $\widetilde{\nabla}$ coincides with that of ∇ is that the conharmonic curvature tensor of $\widetilde{\nabla}$ is equal to that of ∇.*

Proof. The conformal curvature tensor of $\widetilde{\nabla}$ is defined as

$$\begin{aligned}\widetilde{C}(X,Y,Z) &= \widetilde{R}(X,Y,Z) - \frac{1}{(2n-1)}[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y \\ &\quad + g(Y,Z)\widetilde{Q}X - g(X,Z)\widetilde{Q}Y] \\ &\quad + \frac{\widetilde{r}}{2n(2n-1)}\left[g(Y,Z)X - g(X,Z)Y\right].\end{aligned} \tag{4.22}$$

Using equations (4.8), (4.9), (4.10) and (4.11) in the equation (4.22), we have

$$\begin{aligned}\widetilde{C}(X,Y,Z) - C(X,Y,Z) &= \alpha[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad + g(\phi Y,Z)X - g(\phi X,Z)Y] \\ &\quad + \beta[2g(\phi X,Y)\eta(Z)\xi + g(\phi Y,Z)X - g(\phi X,Z)Y] \\ &\quad \frac{2n\beta}{(2n-1)}[g(\phi Y,Z)X - g(\phi X,Z)Y \\ &\quad + g(Y,Z)\phi X - g(X,Z)\phi Y],\end{aligned} \tag{4.23}$$

where

$$\begin{aligned}C(X,Y,Z) &= R(X,Y,Z) - \frac{1}{(2n-1)}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY] \\ &\quad + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{4.24}$$

Again we define conharmonic curvature tensor of $\widetilde{\nabla}$ as

$$\begin{aligned}\widetilde{L}(X,Y,Z) &= \widetilde{R}(X,Y,Z) - \frac{1}{(2n-1)}[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y \\ &\quad + g(Y,Z)\widetilde{Q}X - g(X,Z)\widetilde{Q}Y].\end{aligned} \tag{4.25}$$

Using equations (4.8), (4.9), (4.10) and (4.11) in the equation (4.25), we have

$$\begin{aligned}\widetilde{L}(X,Y,Z) - L(X,Y,Z) &= \alpha[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad + g(\phi Y,Z)X - g(\phi X,Z)Y] \\ &\quad + \beta[2g(\phi X,Y)\eta(Z)\xi + g(\phi Y,Z)X - g(\phi X,Z)Y] \\ &\quad \frac{2n\beta}{(2n-1)}[g(\phi Y,Z)X - g(\phi X,Z)Y \\ &\quad + g(Y,Z)\phi X - g(X,Z)\phi Y],\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}L(X,Y,Z) &= R(X,Y,Z) - \frac{1}{(2n-1)}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY].\end{aligned} \tag{4.27}$$

Proof of the theorem is obvious in view of equations (4.23) and (4.26). □

Theorem 4.9. *In a trans-Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$, the necessary and sufficient condition for the concircular curvature tensor coincides with curvature tensor is scalar curvature tensor of $\widetilde{\nabla}$ to be zero.*

Proof. The concircular curvature tensor [12] of a Riemannian manifold is defined as

$$V(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y]. \tag{4.28}$$

Proof of the theorem is obvious in view of equations (4.11), (4.28) and $\widetilde{r} = 0$. □

Example 4.1. *Consider a three dimensional manifold $M^3 = \{(x,y,z) \in R^3 : z > 0$ with the standard coordinate system (x,y,z) of R^3. Let $e_1 = z\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right), e_2 = z\frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z} = \xi$, which are linear independent vector fields at each point of M^3 and form basis of tangent space at each point.*

Let g be a Riemannian metric defined by

$$\begin{aligned} g(e_1,e_2) = g(e_2,e_3) = g(e_3,e_1) = 0 \\ g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1 \end{aligned} \tag{4.29}$$

and ϕ is an $(1,1)-$tensor fiend defined by

$$\phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3) = 0. \tag{4.30}$$

Using linearity of ϕ and g, we have

$$\eta(e_3) = 1, \quad \phi^2 X = -X + \eta(X)e_3 \tag{4.31}$$

for any $X \in TM$. Here η is a 1-form on M^3 defined by $\eta(X) = g(X,e_3)$ for any $X \in TM$. Hence for $\xi = e_3$, the structure (ϕ,ξ,η,g) defines an almost contact metric structure on M^3.

Let ∇ is a Levi-civita connection with respect to Riemannian metric g. Then we have

$$[e_1,e_2] = ye_2 - z^2 e_3, \quad [e_2,e_3] = \frac{-1}{z}e_2, \quad [e_3,e_1] = \frac{1}{z}e_1. \tag{4.32}$$

Using the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y,Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) \end{aligned} \tag{4.33}$$

and the Riemannian metric g, we can easily calculate

$$\begin{aligned} &\nabla_{e_1}e_1 = \tfrac{1}{z}e_3, && \nabla_{e_1}e_2 = \tfrac{-z^2}{2}e_3, && \nabla_{e_1}e_3 = \tfrac{-1}{z}e_1 + \tfrac{z^2}{2}e_2, \\ &\nabla_{e_2}e_1 = \tfrac{z^2}{2}e_3 - ye_2, && \nabla_{e_2}e_2 = ye_1 + \tfrac{1}{z}e_3, && \nabla_{e_2}e_3 = \tfrac{-1}{z}e_2 - \tfrac{z^2}{2}e_1, \\ &\nabla_{e_3}e_1 = \tfrac{z^2}{2}e_2, && \nabla_{e_3}e_2 = \tfrac{-z^2}{2}e_1, && \nabla_{e_3}e_3 = 0\,. \end{aligned} \tag{4.34}$$

Now for $X = X^1e_1 + X^2e_2 + X^3e_3$ and $\xi = e_3$, we have

$$\begin{aligned} \nabla_X\xi &= \nabla_{X^1e_1+X^2e_2+X^3e_3}e_3 \\ &= X^1\nabla_{e_1}e_3 + X^2\nabla_{e_2}e_3 + X^3\nabla_{e_3}e_3 \\ &= \frac{z^2}{2}(X^1e_2 - X^2e_1) + \frac{-1}{z}(X^1e_1 + X^2e_2), \end{aligned} \tag{4.35}$$

$$\begin{aligned} \phi X &= \phi(X^1e_1 + X^2e_2 + X^3e_3e_3) \\ &= X^1\phi e_1 + X^2\phi e_2 + X^3\phi e_3 \\ &= -X^1e_2 + X^2e_1 \end{aligned} \tag{4.36}$$

and

$$\begin{aligned} X - \eta(X)\xi &= (X^1e_1 + X^2e_2 + X^3e_3) - g(X^1e_1 + X^2e_2 + X^3e_3, e_3)e_3 \\ &= X^1e_1 + X^2e_2. \end{aligned} \tag{4.37}$$

In view of equations (4.35), (4.36), (4.37) and (2.5), we have $\alpha = \frac{z^2}{2}$ and $\beta = \frac{-1}{z}$. Equation (2.4) is also satisfied by all above values. Hence the structure (ϕ, ξ, η, g) is an trans-Sasakian structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is a $\left(\frac{z^2}{2}, \frac{-1}{z}\right)$-trans-Sasakian manifold.

In reference of equations (3.1), (4.29), (4.30)and (4.34), we have the following

$$\begin{array}{lll} \widetilde{\nabla}_{e_1}e_1 = \frac{1}{z}e_3, & \widetilde{\nabla}_{e_1}e_2 = \left(\frac{-z^2}{2} + 1\right)e_3, & \widetilde{\nabla}_{e_1}e_3 = \frac{-1}{z}e_1 + \frac{z^2}{2}e_2, \\ \widetilde{\nabla}_{e_2}e_1 = \left(\frac{z^2}{2} + 1\right)e_3 - ye_2, & \widetilde{\nabla}_{e_2}e_2 = ye_1 + \frac{1}{z}e_3, & \widetilde{\nabla}_{e_2}e_3 = \frac{-1}{z}e_2 - \frac{z^2}{2}e_1, \\ \widetilde{\nabla}_{e_3}e_1 = \frac{z^2}{2}e_2, & \widetilde{\nabla}_{e_3}e_2 = \frac{-z^2}{2}e_1, & \widetilde{\nabla}_{e_3}e_3 = 0\,. \end{array} \tag{4.38}$$

On changing $X = e_1$, $Y = e_2$ and $Z = e_3$ in the equations (3.2) and (3.3), we have

$$\widetilde{T}(e_1, e_2) = 2g(\phi e_1, e_2)e_3 = 2g(-e_2, e_2)e_3 = -2e_3 \neq 0$$

and

$$\begin{aligned} (\widetilde{\nabla}_{e_1}g)(e_2, e_3) &= -\eta(e_3)g(\phi e_1, e_2) - \eta(e_2)g(\phi e_1, e_3) \\ &= -1g(-e_2, e_2) = 1 \neq 0. \end{aligned}$$

Hence the connection is a non-symmetric non-metric connection. Again for $X = X^1e_1 + X^2e_2 + X^3e_3$ and $\xi = e_3$, we have

$$\begin{aligned} \widetilde{\nabla}_X\xi &= \widetilde{\nabla}_{X^1e_1+X^2e_2+X^3e_3}e_3 \\ &= X^1\widetilde{\nabla}_{e_1}e_3 + X^2\widetilde{\nabla}_{e_2}e_3 + X^3\widetilde{\nabla}_{e_3}e_3 \\ &= \frac{z^2}{2}(X^1e_2 - X^2e_1) + \frac{-1}{z}(X^1e_1 + X^2e_2), \end{aligned} \tag{4.39}$$

In view of equations (4.35)and (4.39), we can say that the example verify proposition 3.1.

From equation (4.9), we have

$$\sum_{i=1}^{3}\widetilde{S}(e_i, e_i) = \sum_{i=1}^{3}S(e_i, e_i + 2\beta\sum_{i=1}^{3}g(\phi e_i, e_i).$$

Which gives

$$\begin{aligned}\widetilde{r} &= r+2\left(\frac{-1}{z}\right)[g(\phi e_1,e_1)+g(\phi e_2,e_2)+g(\phi e_3,e_3)]\\ &= r+\left(\frac{-2}{z}\right)[g(-e_2,e_1)+g(e_1,e_2)+g(0,e_3)]\\ &= r.\end{aligned}$$

Hence we can say that the example also verify theorem 4.3.

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