Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.6, (2017), Issue 2, pp.119-125

THE REPRESENTATION OF THE REMAINDER IN CLASSICAL BERNSTEIN APPROXIMATION FORMULA

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ABSTRACT. In the present paper we establish for the first time the representation of the remainder in classical Bernstein approximation formula, using divided differences. As a consequence, we get an upper bound estimation for the remainder term, when approximated function fulfills some given properties.

1. INTRODUCTION AND AUXILIARY RESULTS

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A recent study concerning the generalization of Bernstein operator associated to any real-valued function $F : [a,b] \to \mathbb{R}$, where a, b are two real and finite numbers, such that $a < b$, is done in the papers [2], [3], [5], [4], [8]. This classical Bernstein operator is given by

$$B_n^*(F;x) = \sum_{k=0}^{n} p_{n,k}^*(x) F\left(a + \tfrac{k(b-a)}{n}\right) = \tag{1.1}$$
$$= \tfrac{1}{(b-a)^n} \sum_{k=0}^{n} \tbinom{n}{k} (x-a)^k (b-x)^{n-k} F\left(a + \tfrac{k(b-a)}{n}\right),$$

for any $x \in [a,b]$ and any $n \in \mathbb{N}$. If $a := 0$, $b := 1$, from (1.1) for any $x \in [0,1]$ and any $n \in \mathbb{N}$, then it follows the Bernstein operator [1] associated to any real-valued function $f : [0,1] \to \mathbb{R}$, given by

$$B_n(f;x) = \sum_{k=0}^{n} \tbinom{n}{k} x^k (1-x)^{n-k} f\left(\tfrac{k}{n}\right). \tag{1.2}$$

Thanks to some important properties as approximation, shape preservation and variation diminishing, Bernstein operator is an indispensable tool in computer aided geometric design as well as in other areas of mathematics.

Now, let $a \le x_0 < x_1 < \cdots < x_n \le b$ be some nodes, for any $n \in \mathbb{N}$.

Definition 1.1. [7] The divided difference of the function F with respect to the points $x_0, x_1, \cdots, x_n$, is defined to be the coefficient at x^n in the Lagrange interpolating polynomial $L_n(F;x)$ and is

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2010 *Mathematics Subject Classification.* 26A51, 41A36, 41A80.

Key words and phrases. Bernstein operator, divided difference, convex function, Popoviciu theorem, remainder term.

denoted by $[x_0, x_1, \cdots, x_n; F]$, where $L_n(F; x) = \sum_{k=0}^{n} F(x_k) \frac{l(x)}{(x-x_k)l'(x_k)}$ and $l(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$.

In most books on Numerical Analysis, divided difference for distinct nodes is defined recursively

$$[x_0, x_1, \cdots, x_n; F] = \frac{1}{x_n - x_0}([x_1, \cdots, x_n; F] - [x_0, \cdots, x_{n-1}; F]). \tag{1.3}$$

The first order divided difference of the function F with respect to the distinct nodes x_0, x_1 is defined by $[x_0, x_1; F] = \frac{1}{x_1 - x_0}(F(x_1) - F(x_0))$, respectively the second order divided difference of the function F with respect to the distinct nodes x_0, x_1, x_2 is defined by

$$[x_0, x_1, x_2; F] = \frac{F(x_0)}{(x_0-x_1)(x_0-x_2)} + \frac{F(x_1)}{(x_1-x_0)(x_1-x_2)} + \frac{F(x_2)}{(x_2-x_0)(x_2-x_1)}. \tag{1.4}$$

A function F is called convex, non-concave, polynomial, non-convex, respectively, concave of nth order, if the following hold

$$[x_0, x_1, \cdots, x_{n+1}; F] > 0,\ \geq 0,\ = 0,\ \leq 0,\ \text{respectively}\ < 0,$$

for any $n + 2$ distinct nodes in $[a, b]$ and $n \in \mathbb{N}_0$.

Remark 1.1.

i) For $n = 0$ we get monotone functions: increasing, non-decreasing, constant, non-increasing, respectively, decreasing.

ii) For $n = 1$ we get: convex, non-concave, linear, non-convex, respectively, concave functions in the classical sense.

The convex functions of higher order represented a theme of study for Popoviciu. In 1944 his contributions in this area were collected in [10]. The next result is a well-known extension of Lagrange's mean value theorem to the case of divided differences.

Theorem 1. *[9] If $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and has a derivative of order n on (a, b), then there exists a $\xi \in (a, b)$ such that*

$$[x_0, x_1, \cdots, x_n; F] = \frac{1}{n!} F^{(n)}(\xi).$$

In [12], Stancu established a representation of the remainder in Bernstein approximation formula

$$f(x) = B_n(f; x) + R_n(f; x), \tag{1.5}$$

for any real-valued function $f : [0, 1] \to \mathbb{R}$.

Theorem 2. *[12] For any $x \in [0, 1] \setminus \left\{ \frac{k}{n} \,\middle|\, k = 0, \cdots, n \right\}$ and any $n \in \mathbb{N}$, the representation of the remainder associated to the Bernstein operator* (1.2) *is given by*

$$R_n(f; x) = -\frac{x(1-x)}{n} \sum_{k=0}^{n-1} p_{n-1,k}(x) \left[x, \frac{k}{n}, \frac{k+1}{n}; f \right]. \tag{1.6}$$

Further research concerning the representation of the remainder (1.6) was done on quadrature formulas based on linear positive operators (Bernstein type operators), which represented a theme of intensive study for Stancu. His contributions in this area were collected in the second volume of the monograph "*Numerical Analysis and Approximation Theory*" [13].

The aim of this paper is to establish for the first time the representation of the remainder in generalized Bernstein approximation formula for arbitrary functions, as a convex combination of divided differences of second order on known nodes. As a consequence, we get an upper bound estimation for the remainder term when approximated function possesses bounded divided differences of second order.

2. Main results

Bernstein polynomials (1.2) have opened a new era in approximation theory starting with the year 1912, when Bernstein presented his famous proof of the Weierstrass approximation theorem and continuing with thousands of papers until today. The approximation by Bernstein polynomials is made on the interval $[0,1]$. Thinking at practice utility of Bernstein operators, it is easy to remark that approximation of functions defined on the interval $[a,b]$, where a, b being real and finite numbers, excepting $a := 0$, $b := 1$ appears more often than approximation of functions defined on the interval $[0,1]$. It is also true the fact that any interval $[a,b]$ can be transformed into $[0,1]$ using the application $l(x) = \frac{x-a}{b-a}$, for all $x \in [a,b]$ and then apply the Bernstein operators. In this case occurs the problem: *Is it worth to transform the interval $[a,b]$ into $[0,1]$ in any situation?*
Now, having an arbitrary real-valued function $F : [a,b] \to \mathbb{R}$ arise the question: *Can we approximate functions defined on the interval $[a,b]$ by Bernstein operators, without transforming interval $[a,b]$ into $[0,1]$?* The answer is affirmative and is given by classical Bernstein operators (1.1).

Lemma 2.1. [8] For any $x \in [a,b]$ and any $n \in \mathbb{N}$ we present the computation of the test functions for generalized Bernstein operator (1.1).

$$B_n^*(e_0;x) = 1, \quad B_n^*(e_1;x) = x, \quad B_n^*(e_2;x) = x^2 + \frac{(x-a)(b-x)}{n}.$$

Remark 2.1. The above lemma show us that presented operator (1.1) is a Bernstein type operator, namely classical Bernstein operator.

For any $F \in C[a,b]$, any $x \in [a,b]$ and any $n \in \mathbb{N}$, the following

$$F(x) = B_n^*(F;x) + R_n^*(F;x) \tag{2.1}$$

is called classical Bernstein approximation formula, where R_n^* is the remainder operator associated to the classical Bernstein operator B_n^*. The formula (2.1) is an indispensable tool in approximation of any arbitrary function defined on the interval $[a,b]$, with the condition to know the form of remainder term. In [4], an upper bound estimation for the remainder term R_n^* associated to the Bernstein operator (1.1) was used.

Lemma 2.2. [4] Suppose that $F \in C^2[a,b]$. For any $x \in [a,b]$, the remainder term of the relation (2.1) verifies

$$|R_n^*(F;x)| \leq \frac{(x-a)(b-x)}{2n(b-a)^2} M_2[F], \tag{2.2}$$

where $M_2[F] = \max\limits_{\xi \in [a,b]} |F''(\xi)|$.

Remark 2.2. A first form of the upper bound estimation (2.2) appeared in [2], and then used also in the paper [4]. It is essential to mention the fact that a rigorous proof of this upper bound estimation it was not done in neither of the recalled papers. Moreover, the result concerning upper bound estimation seems to be wrong.

Being motivated by the Remark 2.2, in the following we establish the representation of the remainder in classical Bernstein approximation formula for arbitrary functions, as well as an upper bound estimation when approximated function possesses bounded divided differences of second order.

Theorem 3. *The representation of the remainder term associated to the classical Bernstein operator* (1.1), *is given by*

$$R_n^*(F;x) = -\tfrac{(x-a)(b-x)}{n} \sum_{k=0}^{n-1} p_{n-1,k}^*(x) \left[x, a+\tfrac{k(b-a)}{n}, a+\tfrac{(k+1)(b-a)}{n}; F\right], \tag{2.3}$$

for any $n \in \mathbb{N}$ *and* $x \in [a,b] \setminus \left\{a+\frac{k(b-a)}{n} \mid k=0,\cdots,n\right\}$.

Proof. In order to establish the representation of the remainder term, we notice that the classical Bernstein operator (1.1) preserves constants, such that

$$R_n^*(F;x) = F(x) - B_n^*(F;x) = F(x) - \sum_{k=0}^{n} p_{n,k}^*(x) F\left(a+\tfrac{k(b-a)}{n}\right) =$$

$$= \sum_{k=0}^{n} p_{n,k}^*(x) \left(F(x) - F\left(a+\tfrac{k(b-a)}{n}\right)\right).$$

Using the definition of divided difference of first order, it follows

$$R_n^*(F;x) = \sum_{k=0}^{n} p_{n,k}^*(x) \left(x - \left(a+\tfrac{k(b-a)}{n}\right)\right) \left[x, a+\tfrac{k(b-a)}{n}; F\right] =$$

$$= \sum_{k=0}^{n} p_{n,k}^*(x) \left(\tfrac{(n-k)(x-a)-k(b-x)}{n}\right) \left[x, a+\tfrac{k(b-a)}{n}; F\right] =$$

$$= \sum_{k=0}^{n-1} \tbinom{n-1}{k} \tfrac{(x-a)^{k+1}(b-x)^{n-k}}{(b-a)^n} \left[x, a+\tfrac{k(b-a)}{n}; F\right] - \sum_{k=1}^{n} \tbinom{n-1}{k-1} \tfrac{(x-a)^{k}(b-x)^{n+1-k}}{(b-a)^n} \left[x, a+\tfrac{k(b-a)}{n}; F\right] =$$

$$= \tfrac{(x-a)(b-x)}{b-a} \sum_{k=0}^{n-1} p_{n-1,k}^*(x) \left[x, a+\tfrac{k(b-a)}{n}; F\right] - \sum_{k=0}^{n-1} \tbinom{n-1}{k} \tfrac{(x-a)^{k+1}(b-x)^{n-k}}{(b-a)^n} \left[x, a+\tfrac{(k+1)(b-a)}{n}; F\right] =$$

$$= \tfrac{(x-a)(b-x)}{b-a} \sum_{k=0}^{n-1} p_{n-1,k}^*(x) \left(\left[x, a+\tfrac{k(b-a)}{n}; F\right] - \left[x, a+\tfrac{(k+1)(b-a)}{n}; F\right]\right) =$$

$$= -\tfrac{(x-a)(b-x)}{n} \sum_{k=0}^{n-1} p_{n-1,k}^*(x) \left[x, a+\tfrac{k(b-a)}{n}, a+\tfrac{(k+1)(b-a)}{n}; F\right].$$

□

In what follows, we want to present an alternative way for proving the Theorem 3, which implies the transformation of the interval $[a,b]$ into $[0,1]$ and vice versa.

An alternative way for proving Theorem 3.

Proof. The classical Bernstein approximation formula (2.1) provides

$$R_n^*(F;x) = F(x) - B_n^*(F;x)$$

and then, using the equality $F(x) =: f\left(\frac{x-a}{b-a}\right)$, for all $x \in [a,b]$, it follows

$$R_n^*(F;x) = f\left(\tfrac{x-a}{b-a}\right) - B_n\left(f;\tfrac{x-a}{b-a}\right) = R_n\left(f;\tfrac{x-a}{b-a}\right). \tag{2.4}$$

Now, taking the relations (1.6) and (1.4) into account, we get

$$R_n\left(f;\tfrac{x-a}{b-a}\right) = -\frac{(x-a)(b-x)}{n(b-a)^2}\sum_{k=0}^{n-1} p_{n-1,k}\left(\tfrac{x-a}{b-a}\right)\left[\tfrac{x-a}{b-a},\tfrac{k}{n},\tfrac{k+1}{n};f\right] =$$

$$= -\frac{(x-a)(b-x)}{n}\sum_{k=0}^{n-1} p_{n-1,k}^*(x)\frac{1}{(b-a)^2}\left\{\frac{f\left(\frac{x-a}{b-a}\right)}{\frac{(n(x-a)-k(b-a))(n(x-a)-(k+1)(b-a))}{n^2(b-a)^2}} - \frac{f\left(\frac{k}{n}\right)}{\frac{k(b-a)-n(x-a)}{n^2(b-a)}} + \frac{f\left(\frac{k+1}{n}\right)}{\frac{(k+1)(b-a)-n(x-a)}{n^2(b-a)}}\right\}.$$

Using the equality $f(x) =: F(a + x(b-a))$, for all $x \in [0,1]$, we get

$$R_n^*(F;x) =$$

$$= -\frac{(x-a)(b-x)}{n}\sum_{k=0}^{n-1} p_{n-1,k}^*(x)\left\{\frac{F(x)}{\frac{(n(x-a)-k(b-a))(n(x-a)-(k+1)(b-a))}{n^2}} - \frac{F\left(a+\frac{k(b-a)}{n}\right)}{\frac{k(b-a)^2-n(x-a)(b-a)}{n^2}} + \frac{F\left(a+\frac{(k+1)(b-a)}{n}\right)}{\frac{(k+1)(b-a)^2-n(x-a)(b-a)}{n^2}}\right\} =$$

$$= -\frac{(x-a)(b-x)}{n}\sum_{k=0}^{n-1} p_{n-1,k}^*(x)\left[x, a+\frac{k(b-a)}{n}, a+\frac{(k+1)(b-a)}{n}; F\right],$$

for any $n \in \mathbb{N}$ and $x \in [a,b] \setminus \left\{a + \frac{k(b-a)}{n},\ k = 0,\cdots,n\right\}$. □

We establish the correct form of upper bound estimation for the remainder in classical Bernstein approximation formula (2.1).

Corollary 2.1. Suppose that $F \in C^2[a,b]$ and the divided differences of the second order of F are all bounded on $[a,b]$, then we get

$$|R_n^*(F;x)| \le \frac{(x-a)(b-x)}{2n} M_2[F], \tag{2.5}$$

for any $n \in \mathbb{N}$, where $M_2[F] := \max\limits_{\xi_x \in [a,b]} |F''(\xi_x)|$.

Proof. Taking into account the representation (2.3) of the remainder term in classical Bernstein approximation formula, respectively Theorem 1, it follows

$$|R_n^*(F;x)| = \left|-\frac{(x-a)(b-x)}{n}\sum_{k=0}^{n-1} p_{n-1,k}^*(x)\left[x, a+\frac{k(b-a)}{n}, a+\frac{(k+1)(b-a)}{n}; F\right]\right| \le$$

$$\le \frac{(x-a)(b-x)}{2n}\sum_{k=0}^{n-1} p_{n-1,k}^*(x)|F''(\xi_x)| \le \frac{(x-a)(b-x)}{2n} M_2[F]\sum_{k=0}^{n-1} p_{n-1,k}^*(x) =$$

$$= \frac{(x-a)(b-x)}{2n} M_2[F], \quad \text{for any } n \in \mathbb{N}.$$

□

Remark 2.3. Using the estimation (2.5) and the inequality between arithmetic and geometric mean, also we can establish an upper bound estimation for the remainder in classical Bernstein approximation formula, given by

$$|R_n^*(F;x)| \le \frac{(b-a)^2}{8n} M_2[F]. \tag{2.6}$$

Remark 2.4. Having the representation of the remainder term (2.3), respectively the upper bound estimation (2.5) proved and well defined, it could be integrate the classical Bernstein approximation formula (2.1), in order to get the correct form of classical Bernstein quadrature formula in [4].

If a, b are two real and finite numbers, such that $a < b$, then we can approximate any function F defined on the interval $[a, b]$ by classical Bernstein operator (1.1) and we can evaluate the error in classical Bernstein approximation formula. We give an example in this sense.

Example 2.1. *Taking* $a := 0$, $b := \frac{n}{n+1}$, *then for any real-valued function* $F : [0, \frac{n}{n+1}] \to \mathbb{R}$ *it follows the associated Bernstein operator [3], [11], given by*

$$B_n^{\left[0, \frac{n}{n+1}\right]}(F; x) = \left(\tfrac{n+1}{n}\right)^n \sum_{k=0}^{n} \tbinom{n}{k} x^k \left(\tfrac{n}{n+1} - x\right)^{n-k} F\left(\tfrac{k}{n+1}\right), \tag{2.7}$$

for any $x \in [0, \frac{n}{n+1}]$ *and any* $n \in \mathbb{N}$. *The remainder term associated to the above Bernstein operator admits the following representation*

$$R_n^{\left[0, \frac{n}{n+1}\right]}(F; x) = -\tfrac{x((1-x)n-x)}{n(n+1)} \sum_{k=0}^{n-1} p_{n-1,k}^{\left[0, \frac{n}{n+1}\right]}(x) \left[x, \tfrac{k}{n+1}, \tfrac{k+1}{n+1}; F\right], \tag{2.8}$$

for any $n \in \mathbb{N}$ *and* $x \in \left[0, \frac{n}{n+1}\right] \setminus \left\{ \frac{k}{n+1} \mid k = 0, \cdots, n \right\}$. *Supposing that* $F \in C^2[0, \frac{n}{n+1}]$, *then we get the upper bound estimation*

$$\left| R_n^{\left[0, \frac{n}{n+1}\right]}(F; x) \right| \le \tfrac{x\left(\frac{n}{n+1} - x\right)}{2n} M_2[F] \le \tfrac{n}{8(n+1)^2} M_2[F], \ \textit{for any}\ \ n \in \mathbb{N}. \tag{2.9}$$

Acknowledgement The results presented in this paper were obtained with the support of the Technical University of Cluj-Napoca through the research Contract no. 2011/12.07.2017, Internal Competition CICDI-2017.

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