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PROJECTABLE CONFORMAL VECTOR FIELDS ON TANGENT BUNDLE

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ABSTRACT. In this paper, we establish a Lie algebra homomorphism between the Lie algebra of projectable conformal vector fields of (TM, G) and the Lie algebra of homothetic vector fields of (M, g), where G is a special lift of the Riemannian metric g to the tangent space of M.

Keywords: Conformal vector field, lift metric, tangent bundle.[1]

1. INTRODUCTION

Let g be a Riemannian metric on a simply connected manifold M with Levi-Civita connection ∇ and (TM, π, M) be its tangent bundle. The Riemannian metric g has components g_{ij} which are functions of variables x^i on M. Suppose that $(\partial_i, \partial_{\bar{i}})$ be the natural vector fields associated to a natural coordinate (x^i, y^i) on TM. Let $\delta_i = \partial_i - N_i^j \partial_{\bar{j}}$, where N_i^j are the components of the non-linear connection of g. In a local co-frame $(dx^i, \delta y^i)$ dual of adapted non-holonomic frame $(\delta_i, \partial_{\bar{i}})$ on TM, we define a tensor field G as follows

$$G(x, y) = \alpha h_{ij}(x, y) dx^i dx^j + 2\beta h_{ij}(x, y) dx^i \delta y^j + \gamma h_{ij}(x, y) \delta y^i \delta y^j, \tag{1.1}$$

where α, β and γ are real numbers and h_{ij} are given by $h_{ij}(x, y) = \sigma g_{ij}(x)$, where σ is a positive smooth function on TM. One can see that G is a global Riemannian metric on TM if and only if $\alpha\gamma - \beta^2 > 0$. In this case, G is said to be the lift metric of g to TM. This lift metric G, in some sense, is a generalization of those of introduced in [2]- [5].

A vector field X on TM which is π-related to a vector field on M is called projectble. For a vector field $V = V^i \frac{\partial}{\partial x^i}$ on (M, g), its complete lift $V^C := V^i \delta_i + \nabla_0 V^i \partial_{\bar{i}}$ is projectble, where the index 0 denotes contraction with y. It is easy to see that $X = v^h \delta_h + v^{\bar{h}} \partial_{\bar{h}}$ is projectble if and only if v^h depend only on position [5]. In this case, the vector field $\hat{X} := v^i \frac{\partial}{\partial x^i}$ is called the induced vector field of X. Using the relations $[\delta_i, \delta_j] = y^r K_{jir}{}^m \partial_{\bar{m}}$, $[\delta_i, \partial_{\bar{j}}] = \Gamma^m_{ji} \partial_{\bar{m}}$ and integrability of the vertical distribution, we get that the mapping $X \to \hat{X}$ is a surjective Lie algebra homomorphism.

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2. PROJECTBLE CONFORMAL VECTOR FIELDS OF (TM, G)

For shortness, we set $g_1 = h_{ij}dx^i dx^j$, $g_2 = 2h_{ij}dx^i\delta y^j$ and $g_3 = h_{ij}\delta y^i \delta y^j$. To find the conformal vector fields of G, we compute the Lie derivatives $\pounds_X g_1$, $\pounds_X g_2$ and $\pounds_X g_3$ by a lengthy computation.

Lemma 1. *([5]) Let $X = v^h\delta_h + v^{\bar{h}}\partial_{\bar{h}}$ be a projectble vector field on TM. Then the followings hold*

(1) $\pounds_X g_1 = \sigma(2\bar{\varphi}g_{ij} + \pounds_{\hat{X}}g_{ij})dx^i dx^j$,

(2) $\pounds_X g_2 = 2\sigma[-g_{jm}(y^b v^c K_{icb}{}^m - v^{\bar{b}}\Gamma^m_{bi} - \delta_i(v^{\bar{m}}))dx^i dx^j + (\pounds_{\hat{X}}g_{ij} - g_{jm}\nabla_i v^m + g_{jm}\partial_{\bar{i}}(v^{\bar{m}}) + 2\bar{\varphi}g_{ij})dx^j\delta y^i]$,

(3) $\pounds_X g_3 = \sigma[-2g_{mi}(y^b v^c K_{jcb}{}^m - v^{\bar{b}}\Gamma^m_{bj} - \delta_j(v^{\bar{m}}))dx^j\delta y^i + (\pounds_{\hat{X}}g_{ij} - 2g_{jm}\nabla_i v^m + 2g_{jm}\partial_{\bar{i}}(v^{\bar{m}}) + 2\bar{\varphi}g_{ij})\delta y^i\delta y^j]$,

where $\bar{\varphi} := \frac{1}{2}\pounds_X ln(\sigma)$ and $\pounds_{\hat{X}}g_{ij}$ denote the components of $\pounds_{\hat{X}}g$.

Let $X = V^C$ be a conformal vector field of G, i.e., $\pounds_X G = 2\bar{\rho}G$, which, using Lemma 1, is equivalent to the followings

a) $\alpha(\pounds_V g_{ij} - 2\Omega g_{ij}) = \beta[g_{im}(y^b v^c K_{jcb}{}^m - v^{\bar{b}}\Gamma^m_{bj} - \delta_j(v^{\bar{m}}) + g_{jm}(y^b v^c K_{icb}{}^m - v^{\bar{b}}\Gamma^m_{bi} - \delta_i(v^{\bar{m}})]$,

b) $\beta(\pounds_V g_{ij} - 2\Omega g_{ij}) = \beta g_{im}(\nabla_j v^m - \partial_{\bar{j}}v^{\bar{m}}) + \gamma g_{jm}(y^b v^c K_{icb}{}^m - v^{\bar{b}}\Gamma^m_{bi} - \delta_i v^{\bar{m}})$,

c) $2\Omega g_{ij} = g_{mj}\partial_{\bar{i}}(v^{\bar{m}}) + g_{mi}\partial_{\bar{j}}(v^{\bar{m}})$,

where $\Omega := \bar{\rho} - \bar{\varphi}$. Here, we use the fact $\pounds_V g_{ij} = \nabla_i V_j + \nabla_j V_i$. Substituting $v^{\bar{m}} = \nabla_0 V^m$ into (c), then taking the vertical derivative $\partial_{\bar{k}}$, we get $\partial_{\bar{k}}\Omega = 0$. Thus, we have the following.

Lemma 2. *Let $X = V^C$ be a conformal vector field of G with associated function $\bar{\rho}$. Then $\Omega := \bar{\rho} - \bar{\varphi}$ is constant on every fiber of TM, where $\bar{\varphi} := \frac{1}{2}\pounds_X ln(\sigma)$.*

Further, we are going to show that the function Ω is constant on TM. We put $A^m_i := \nabla_i V^m$ and $A_{ji} := g_{mj}A^m_i$. Then, one can rewrite (b) as follows

$$\beta(\pounds_V g_{ij} - 2\Omega g_{ij}) = \gamma y^a(v^c K_{icaj} - g_{mj}\nabla_i A^m_a). \tag{2.1}$$

Lemma 2 implies that the left hand side of (b) depends only on position. Thus, we get

$$v^c K_{icaj} = \nabla_i A_{ja}, \tag{2.2}$$

which yields

$$\nabla_k A_{ij} + \nabla_k A_{ji} = 0. \tag{2.3}$$

On the other hand, rewriting (c), we have

$$2\Omega g_{ij} = A_{ji} + A_{ij}. \tag{2.4}$$

Taking covariant derivative ∇_k from (2.4) and using (2.3) imply that $\nabla_k\Omega = 0$. Here, we use the compatibility of ∇ with g. Summarizing up, we get the following proposition.

Property 1. *Let $X = V^C$ be a conformal vector field of G with associated function $\bar{\rho}$. Then $\Omega := \bar{\rho} - \bar{\varphi}$ is constant on TM, where $\bar{\varphi} := \frac{1}{2}\pounds_X ln(\sigma)$.*

Suppose that $X = V^C$ is a conformal vector field of G. Then (2.4), shows that $\pounds_V g_{ij} = 2\Omega g_{ij}$. It means that V is a homothetic vector field of g. A straightforward computation proves that the converse is true. Hence, we get the following theorem.

Theorem 3. *A vector field V on M is a homothetic of g if and only if V^C is a conformal of G.*

A conformal vector field of G with associated function $\bar{\rho}$ is called conformal affine vector field, if $\Omega := \bar{\rho} - \bar{\varphi}$ depends only on position. In continue, we extend the Proposition 1 to projectble conformal affine vector fields of G. Suppose that $X = v^h\delta_h + v^{\bar{h}}\partial_{\bar{h}}$ is a projectble conformal affine vector field of G with associated function $\bar{\rho}$. Then, (a)-(c) hold, replacing V with $\hat{X}$. Putting $i = j$ in (c), we get $\Omega g_{ii} = g_{mi}\partial_{\bar{i}}(v^{\bar{m}})$. Applying $\partial_{\bar{j}}$ to last relation, we get $\partial_{\bar{j}}\partial_{\bar{i}}(v^{\bar{m}}) = 0$. Therefore, $v^{\bar{m}}$ are as follows:

$$v^{\bar{m}} = D^m_a(x^i)y^a + B^m(x^i). \tag{2.5}$$

Replacing (2.5) into (b), we have

$$\beta(\pounds_{\hat{X}} g_{ij} - 2\Omega g_{ij} - g_{im}(\nabla_j v^m - D^m_j)) + \gamma g_{jm}\nabla_i B^m = \gamma g_{jm} y^a (v^c K_{ica}{}^m - \nabla_i D^m_a). \tag{2.6}$$

The left hand side of (2.6) depends only on position. Thus, we get

$$v^c K_{icaj} = \nabla_i D_{ja}, \tag{2.7}$$

which yields

$$\nabla_k D_{ij} + \nabla_k D_{ji} = 0. \tag{2.8}$$

Plugging (2.5) into (c) and then taking covariant derivative ∇_k and using (2.8), we get Ω is constant.

Theorem 4. *Let X be a projectble conformal affine vector field of G with associated function $\bar{\rho}$. Then $\Omega = \bar{\rho} - \bar{\varphi}$ is constant on TM.*

Suppose that $X = v^h\delta_h + v^{\bar{h}}\partial_{\bar{h}}$ is a projectble conformal vector field of G. Let us put $V := \hat{X}$ and $A_{ij} := \nabla_i v_j$. Comparing (2.2) to (2.7) implies that $\nabla_k(D_{ij} - A_{ij}) = 0$. Hence D_{ij} are in the form

$$D_{ij} = A_{ij} + T_{ij}, \tag{2.9}$$

where T is a parallel $(1,1)$-tensor with respect to g. For the $(1,1)$-tensor T, the natural lift of T is defined by $T^n := T^m{}_0\partial_{\bar{m}}$. The vertical lift of a vector field $B = B^m\frac{\partial}{\partial x^m}$ on the M is given by $B^V := B^m\partial_{\bar{m}}$. Then, we are led to a decomposition of X as follows:

$$X = V^C + T^n + B^V, \tag{2.10}$$

where T and B are given in (2.5) and (2.9).

Corollary 5. *Let X be a projectble conformal affine vector field of G. Then $\hat{X}$ is a homothetic of g if and only if $T = \mu g$ for some constant real number μ.*

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