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RANDERS $g.o$. SPACE OF SPECIFIC TYPE

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ABSTRACT. In this paper, we study Randers spaces whose geodesic are orbit from a specific setting.

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1. INTRODUCTION

A connected Riemannian manifold (M, g) is said to be homogeneous if a connected group of isometries G acts transitively on it. Then M can be viewed as a coset space $\frac{G}{H}$ with a G-invariant metrics, where H is the isotropy subgroup at a fixed point o of M. A geodesic $\gamma(t)$ through the origin $o = eH$ is called a homogeneous geodesic if it is an orbit of a one-parameter subgroup of G, i.e

$$\gamma(t) = exp(tZ)(o).$$

The notion of a homogeneous geodesic plays a fundamental role in the theory of homogeneous Riemannian manifold, especially in the study of g.o. spaces i.e., a Riemannian manifold whose geodesic are all homogeneous. In [5] R. A. Marinosci investigated the set of all homogeneous geodesics in a 3-dimensional Lie group. In [4] the second author studied homogeneous geodesics of left invariant Randers metric on a three-dimensional Lie group. In this paper, we will study Randers g.o. space in the specially setting. we show that there is a three-dimensional unimodular Lie group with a left invariant non-Berwaldian Randers metric with navigation data (h, w) is a g.o. space.

2. PRELIMINARIES

In this section, we recall briefly some known facts about Finsler spaces. For details, see [1]. Let M be a n-dimensional C^∞ manifold and $TM = \bigcup_{x \in M} T_xM$ the tangent bundle. If the continuous function $F : TM \to \mathbb{R}_+$ satisfies the conditions that it is C^∞ on $TM \backslash \{0\}$; $F(tu) = tF(u)$ for all $t \geq 0$ and $u \in TM$, i.e, F is positively homogeneous of degree one; and for any tangent vector $y \in T_xM \backslash \{0\}$, the following bilinear symmetric form $g_y : T_xM \times T_xM \to \mathbb{R}$ is positive definite :

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(x, y + su + tv)]|_{s=t=0},$$

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then we say that (M, F) is a Finsler manifold.
Let π^*TM be the pull-back of the tangent bundle TM by $\pi : TM\backslash\{0\} \to M$. Unlike the Levi-Civita connection in Riemannian geometry, there is no unique natural connection in the Finsler case. Among these connections on π^*TM, we choose the *chern connection* whose coefficients are denoted by Γ^i_{jk} (see[1]). This connection is almost g-compatible and has no torsion. Here $g(x,y) = g_{ij}(x,y)dx^i \otimes dx^j = (\frac{1}{2}F^2)_{y^iy^j}dx^i \otimes dx^j$ is the Riemannian metric on the Pulled-back bundle π^*TM.
The Chern connection defines the covariant derivative D_VU of a vector field $U \in \chi(M)$ in the direction $V \in T_pM$. Since, in general, the Chern connection cofficients Γ^i_{jk} in natural coordinates have a directional dependence, we must say explicitly that D_VU is defined with a fixed reference vector. In particular, let $\sigma : [0,r] \to M$ be a smooth curve with velocity field $T = T(t) = \dot{\sigma}(t)$. Suppose that U and W are vector fields defined along σ. We define D_TU with *reference vector* W as

$$D_TU = [\frac{dU^i}{dt} + U^jT^k(\Gamma^i_{jk})_{(\sigma,w)}]\frac{\partial}{\partial x^i}|_{\sigma(t)}.$$

A curve $\sigma : [0,r] \to M$, with velocity $T = \dot{\sigma}$ is a Finslerian *geodesic* if

$$D_T[\tfrac{T}{F(T)}] = 0 \qquad \text{with reference vector } T.$$

We assume that all our geodesics $\sigma(t)$ have been parameterized to have constant Finslerian speed. That is, the length $F(T)$ is constant. These geodesics are characterized by the equation

$$D_TT = 0, \qquad \text{with reference vector } T.$$

Since $T = \frac{d\sigma^i}{dt}\frac{\partial}{\partial x^i}$, this equation says that

$$\frac{d^2\sigma^i}{dt^2} + \frac{d\sigma^j}{dt}\frac{d\sigma^k}{dt}(\Gamma^i_{jk})_{\sigma,T} = 0.$$

If U,V and W are vector fields along a curve σ, which has velocity $T = \dot{\sigma}$, we have the derivative rule

$$\frac{d}{dt}g_W(U,V) = g_W(D_TU,V) + g_W(U,D_TV)$$

whenever D_TU and D_TV are with reference vector W and one of the following conditions holds:

$i)U$ or V is proportional to W, or

$ii)W = T$ and σ is a geodesic.

3. STRUCTURE OF G.O. SPACES

Let G be a connected Lie group with Lie algebra $\mathfrak{g} = T_eG$. We may identify the tangent bundle TG with $G \times \mathfrak{g}$ by means of the diffeomorphism that sends (g, X) to $(L_g)_*X \in T_gG$.

Definition. A Finsler function $F : TG \to \mathbb{R}_+$ will be called G-invariant if F is constant on all G-orbits in $TG = G \times \mathfrak{g}$; that is, $F(g, X) = F(e, X)$ for all $g \in G$ and $X \in \mathfrak{g}$.

The G-invariant Finsler functions on TG may be identified with the Minkowski norms on $\mathfrak{g}$. If $F : TG \to \mathbb{R}_+$ is an G-invariant Finsler function, then we may Conversely, if we are given a Minkowski norm G-invariant Finsler function

Definition. Let G be a connected Lie group, $\mathfrak{g} = T_eG$ its Lie algebra identified with the tangent space at the identity element, $F : \mathfrak{g} \to \mathbb{R}_+$ a Minkowski norm and F the left-invariant Finsler metric induced by F on G. then (G, F) is called a Finsler g.o. space (geodesic orbit space) if every geodesic of (G, F) is the orbit of a one-parameter subgroup of G. That is, there is a $Z \in \mathfrak{g}$ such that $\gamma(t) = exp(tZ).o$, $t \in \mathbb{R}_+$ holds. A tangent vector $X \in T_eG - \{0\}$ is said to be a geodesic vector if the 1-parameter subgroup $t \to exp(tX)$, $t \in \mathbb{R}_+$, is a geodesic of F.

Concerning geodesic vectors , one has the following

Theorem 1. *(see [4]) Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let F be a left-invariant Finsler metric on G. Then $X \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if*

$$g_X(X, [X, Z]) = 0$$

holds for every $Z \in \mathfrak{g}$.

The following result is obvious.

Proposition 2. *Let G be a connected Lie group, $\mathfrak{g} = T_eG$ its Lie algebra identified with the tangent space at the identity element, $F : \mathfrak{g} \to \mathbb{R}_+$ a Minkowski norm and F the left-invariant Finsler metric induced by F on G. Then every geodesic in G is an orbit of a one-parameter of G if and only if every $X \in \mathfrak{g} - \{0\}$, is a geodesic vector.*

4. Randers g.o. space

In this section, we study Randers g.o. space. A Randers metric is built from a Riemannian metric α and a 1-form β: $F = \alpha + \beta$, where α is a Riemannian metric and β is a smooth 1-form on M whose length with respect to α is everywhere less than 1. There is another presentation of such metrics, by the so-called navigation data [6].

$$F(x, y) = \frac{\sqrt{h(y, W)^2 + \lambda h(y, y)}}{\lambda} - \frac{h(y, W)}{\lambda},$$

where h is a Riemannian metric, W is a vector field on M with $h(W, W) < 1$ and $\lambda = 1 - h(W, W)$. The pair (h, W) is called the navigation data of the Randers metric F.
Let G be a three-dimensional connected Lie group endowed with a left invariant Randers metric with navigation data (h, W). It is easy to check that the underlying Riemannian metric h and the vector field W are also left invariant.

Lemma 3. *Let G be a three-dimensional connected Lie group endowed with a left invariant Randers metric with navigation data (h, W). Then an element $y \in \mathfrak{g} - \{0\}$, $\mathfrak{g} = Lie(G)$, is only geodesic vector if and only if the equation*

$$\tilde{a}(X + \frac{y}{\sqrt{\tilde{a}(y, y)}}, [y, z]) = 0 \qquad (I)$$

holds for any $Z \in \mathfrak{g} - \{0\}$, *if and only if*

$$h(y - F(y)W, [y, Z]) = 0 \qquad (II)$$

holds for any $Z \in \mathfrak{g} - \{0\}$, *where* W *is a vector field.*

Proof. *The Proof is just a direct computation. We only give the proof for* (I). *The proof for* (II) *is given in* [5].

Case I

Let G *be an unimodular Lie group. According to a result due to J.Milnor (see [6]) there existan orthonormal basis* $\{e_1, e_2, e_3\}$ *of the Lie algebra* $\mathfrak{g}$ *such that*

$$[e_1, e_2] = \lambda_3 e_3, \qquad [e_2, e_3] = \lambda_1 e_1, \qquad [e_3, e_1] = \lambda_2 e_2,$$

Let F *be a left invariant Randers metric with navigation data* (h, W) *on* G *defined by the Riemannian* h *and the vector field* $W = w_1 e_1 + w_2 e_2 + w_3 e_3$, *where* $0 < h(W, W) < 1$, *i.e.*

$$F(x, y) = \frac{\sqrt{h(y, W)^2 + \lambda h(y, y)}}{\lambda} - \frac{h(y, W)}{\lambda}.$$

We want to describe all geodesic vectors of (G, h).
For $s, t \in \mathbb{R}$

$$F^2(y + su + tv) = \frac{\sqrt{h(y + su + tv, W)^2 + \lambda h(y + su + tv, y + su + tv)}}{\lambda} - \frac{h(y + su + tv, W)}{\lambda}.$$

By definition

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial r \partial s} F^2(y + ru + sv)|_{r=s=0}.$$

so by a direct computation we get

$$\begin{aligned} g_y(u, v) \;=\; & \frac{1}{\lambda^2}[2h(u, W)h(v, w) + \lambda h(u, v) \\ & -h(u, W)\left(\frac{h(v, W)h(y, W) + \lambda h(y, v)}{\sqrt{h(y, W)^2 + \lambda h(y, y)}}\right) \\ & -h(v, W)\left(\frac{h(u, W)h(y, W) + \lambda h(y, u)}{\sqrt{h(y, W)^2 + \lambda h(y, y)}}\right) \\ & -h(y, W)\left(\frac{h(u, W)h(v, W) + \lambda h(u, v)}{\sqrt{h(y, W)^2 + \lambda h(y, y)}}\right) \\ & +h(y, W)\left(\frac{(h(v, W)h(y, W) + \lambda h(v, y))(h(u, W)h(y, W) + \lambda h(y, u))}{((h(y, W)^2 + \lambda h(y, y))^{\frac{3}{2}}}\right)]. \end{aligned} \qquad (4.1)$$

So for all $Z \in \mathfrak{g}$ *we have*

$$g_y(y, [y, Z]) = \frac{F(y)}{\sqrt{h(y, W)^2 + \lambda h(y, y)}} h(y - F(y)W, [y, Z]) \qquad (\star)$$

By using Theorem 3.3 *and* $(\star)$ *a vector* $y = y_1e_1 + y_2e_2 + y_3e_3$ *of* $\mathfrak{g}$ *is a geodesic vector if and only if*

$h([y_1e_1 + y_2e_2 + y_3e_3, e_j], y_1e_1 + y_2e_2 + y_3e_3 - F(y)(w_1e_1 + w_2e_2 + w_3e_3)) = 0$
for each $j = 1, 2, 3$

So we get :

$$\begin{aligned} &y_2y_3(\lambda_2 - \lambda_3) - (y_2\lambda_3w_3 - y_3\lambda_3w_2)F(y) = 0,\\ &y_1y_3(\lambda_3 - \lambda_1) - (y_1\lambda_3w_3 - y_3\lambda_1w_1)F(y) = 0,\\ &y_1y_2(\lambda_1 - \lambda_2) - (y_1\lambda_2w_2 - y_2\lambda_1w_1)F(y) = 0.\end{aligned}$$

If $W = w_1e_1$, $0 < w_1 < 1$ *and* $\lambda_1 = \lambda_2 = \lambda_3 \neq 0$ *we conclude that all geodesic vector* y *are those from the set span* $\{e_1\}$.
Consequently, there is only one homogeneous geodesic.

Case II

Let G be a non-unimodular Lie group. According to a result due to J. Milnor (see [6]) there exists an orthogonal basis $\{e_1, e_2, e_3\}$ *of the Lie algebra* $\mathfrak{g}$ *such that*

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \qquad [e_2, e_3] = 0, \qquad [e_1, e_3] = \gamma e_2 + \delta e_3,$$

where $\alpha, \beta, \gamma, \delta$ *are real numbers such that the matrix*

has trace $\alpha + \delta = 2$ *and* $\alpha\gamma + \beta\delta = 0$. *Let F be a left invariant Randers metric on G with navigation data* (h, W) *defined by the Riemannian metric h and the vector field* $W = w_1e_1 + w_2e_2 + w_3e_3$, $h(W, W) < 1$. *By using Theorem* 3.3 *and* $(\star)$ *a vector* $y = y_1e_1 + y_2e_2 + y_3e_3$ *of* $\mathfrak{g}$ *is a geodesic vector if and only if*
$h([y_1e_1 + y_2e_2 + y_3e_3, e_j], (y_1e_1 + y_2e_2 + y_3e_3) - F(y)(w_1e_1 + w_2e_2 + w_3e_3)) = 0$
for each $j = 1, 2, 3$.
This condition leads to the system of equations

$$\begin{gathered} -y_2^2\alpha - y_3^2\delta - y_2y_3\beta - y_3y_2\gamma + (y_2\alpha w_2 + y_2\beta w_3 + y_3\gamma w_2 + y_3\delta w_3)F(y) = 0,\\ y_1y_2\alpha - y_1y_3\beta - y_2y_3\gamma - y_3^2\delta - (y_1\alpha w_2 + y_1\beta w_3 - y_3\gamma w_2 + y_3\delta w_3)F(y) = 0,\\ y_1y_2\gamma + y_1y_3\delta - (y_1\gamma w_2 + y_1\delta w_3)F(y) = 0.\end{gathered}$$

Putting $W = w_1e_1$ *and* $\alpha = 2$, $\delta = 0$, $\gamma = 0$ *the above equations take the form*

$$\begin{gathered} -2y_2^2 - y_2y_3\beta = 0,\\ 2y_1y_2 + y_1y_2\beta = 0.\end{gathered}$$

So a vector y *of* $\mathfrak{g}$ *is a geodesic vector if and only if :*

$-y \in Span(e_1, e_3) for \beta = 0,$

$-y \in Span(e_1) \bigcup Span(e_3) \bigcup Span(-\frac{\beta}{2}e_2 + e_3) for \beta \neq 0.$

Proposition 4. *Let G be a three-dimensional connected Lie group endowed with a left invariant Randers metric with navigation data (h, W). Let (G, F) be a Randers g.o. space with navigation data (h, W). Then (G, α) and (G, h) are Riemannian g.o. manifold.*

Proof. We prove the (G, h) case, the proof for the other case being similar. It is sufficient to show that for any $y \in I$, $I = \{y \in \mathfrak{g} | F(y) = 1\}$ is the indicatrix of F on $\mathfrak{g}$, there is only one geodesic vector of (G, h).
Since (G, F) is a Randers g.o. space, we have:

$$h(y - W, [y, Z]) = 0$$

holds for any $Z \in \mathfrak{g}$.

Case I

Let G be an unimodular Lie group. According to a result due to J.Milnor (see [6]) there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ such that

$$[e_1, e_2] = \lambda_3 e_3, \qquad [e_2, e_3] = \lambda_1 e_1, \qquad [e_3, e_1] = \lambda_2 e_2,$$

Let F be a left invariant Randers metric with navigation data (h, W) on G defined by the Riemannian metric h and the vector field $W = w_1 e_1 + w_2 e_2 + w_3 e_3$, where $0 < h(W, W) < 1$, i.e.

$$F(y) = \frac{\sqrt{h(y, W)^2 + \lambda h(y, y)}}{\lambda} - \frac{h(y, W)}{\lambda}.$$

We want to describe all geodesic vectors of (G, h).
For $s, t \in \mathbb{R}$

$$F^2(y + su + tv) = \frac{\sqrt{h(y + su + tv, W)^2 + \lambda h(y + su + tv, y + su + tv)}}{\lambda} - \frac{h(y + su + tv, W)}{\lambda}.$$

By definition

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial r \partial s} F^2(y + ru + sv)|_{r=s=0}.$$

so by a direct computation we get

$$\begin{aligned} g_y(u,v) &= \frac{1}{\lambda^2}[2h(u,W)h(v,w)+\lambda h(u,v) \\ &-h(u,W)(\frac{h(v,W)h(y,W)+\lambda h(y,v)}{\sqrt{h(y,W)^2+\lambda h(y,y)}}) \\ &-h(v,W)(\frac{h(u,W)h(y,W)+\lambda h(y,u)}{\sqrt{h(y,W)^2+\lambda h(y,y)}}) \\ &-h(y,W)(\frac{h(u,W)h(v,W)+\lambda h(u,v)}{\sqrt{h(y,W)^2+\lambda h(y,y)}}) \\ &+h(y,W)\left(\frac{(h(v,W)h(y,W)+\lambda h(v,y))(h(u,W)h(y,W)+\lambda h(y,u))}{((h(y,W)^2+\lambda h(y,y))^{\frac{3}{2}}}\right)]. \end{aligned}$$

So for all $Z \in \mathfrak{g}$ we have

$$g_y(y,[y,Z]) = \frac{F(y)}{\sqrt{h(y,W)^2+\lambda h(y,y)}}h(y-F(y)W,[y,Z]) \qquad (\star)$$

By using Theorem 3.3 and $(\star)$ a vector $y = y_1e_1 + y_2e_2 + y_3e_3$ of $\mathfrak{g}$ is a geodesic vector if and only if
$h([y_1e_1 + y_2e_2 + y_3e_3, e_j], y_1e_1 + y_2e_2 + y_3e_3 - (w_1e_1 + w_2e_2 + w_3e_3)) = 0$
for each $j = 1, 2, 3$.

So we get :

$$y_2y_3(\lambda_2-\lambda_3) - (y_2\lambda_3w_3 - y_3\lambda_3w_2) = 0$$

$,y_1y_3(\lambda_3-\lambda_1) - (y_1\lambda_3w_3 - y_3\lambda_1w_1) = 0, y_1y_2(\lambda_1-\lambda_2) - (y_1\lambda_2w_2 - y_2\lambda_1w_1) = 0.$
IfW = w$_1e_1$, $0 < w_1 < 1$ and $\lambda_1 = \lambda_2 = \lambda_3 \neq 0$ we conclude that all geodesic vector y are those from the set span $\{e_1\}$.
Therefore, y is a only geodesic vector of (G,F) .

Case II

Let G be a non-unimodular Lie group. According to a result due to J. Milnor (see [6]) there exists an orthogonal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ such that

$$[e_1,e_2] = \alpha e_2 + \beta e_3, \qquad [e_2,e_3] = 0, \qquad [e_1,e_3] = \gamma e_2 + \delta e_3,$$

where $\alpha, \beta, \gamma, \delta$ are real numbers such that the matrix

has trace $\alpha + \delta = 2$ and $\alpha\gamma + \beta\delta = 0$. Let F be a left invariant Randers metric on G with navigation data (h, W) defined by the Riemannian metric h and the vector field $W = w_1e_1 + w_2e_2 + w_3e_3$, $h(W,W) < 1$. By using Theorem 3.3 and $(\star)$ a vector $y = y_1e_1 + y_2e_2 + y_3e_3$ of $\mathfrak{g}$ is a geodesic vector if and only if
$h([y_1e_1 + y_2e_2 + y_3e_3, e_j], (y_1e_1 + y_2e_2 + y_3e_3) - (w_1e_1 + w_2e_2 + w_3e_3)) = 0$

for each $j = 1, 2, 3$.
This condition leads to the system of equations

$$-y_2^2\alpha - y_3^2\delta - y_2y_3\beta - y_3y_2\gamma + (y_2\alpha w_2 + y_2\beta w_3 + y_3\gamma w_2 + y_3\delta w_3) = 0,$$
$$y_1y_2\alpha - y_1y_3\beta - y_2y_3\gamma - y_3^2\delta - (y_1\alpha w_2 + y_1\beta w_3 - y_3\gamma w_2 + y_3\delta w_3) = 0,$$
$$y_1y_2\gamma + y_1y_3\delta - (y_1\gamma w_2 + y_1\delta w_3) = 0.$$

Putting $W = w_1e_1$ and $\alpha = 2$, $\delta = 0$, $\gamma = 0$ the above equations take the form

$$-2y_2^2 - y_2y_3\beta = 0,$$
$$2y_1y_2 + y_1y_2\beta = 0.$$

So a vector y of $\mathfrak{g}$ is a geodesic vector if and only if :

$-y \in Span(e_1, e_3) for \beta = 0$

$-y \in Span(e_1) \bigcup Span(e_3) \bigcup Span(-\frac{\beta}{2}e_2 + e_3) for \beta \neq 0$ □

This implies that (G, F) is a g.o. space.

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