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ON THE CHANGE OF VARIABLES ASSOCIATED WITH THE HAMILTONIAN STRUCTURE OF THE HARRY DYM EQUATION

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ABSTRACT. In this paper, we investigate the corresponding Hamiltonian structure of the Harry Dym equation under the change of variables $v = u^{-\frac{1}{2}}$ and derive the associated Hamiltonian operators of the transformed equation.

1. INTRODUCTION

The following nonlinear partial differential equation

$$u_t = D_x^3(u^{-\frac{1}{2}}) \tag{1.1}$$

is known as the Harry Dym equation [6]. This equation was obtained by Harry Dym and Martin Kruskal as an evolution equation solvable by a spectral problem based on the string equation instead of Schrödinger equation. This result was reported in [7] and rediscovered independently in [14], [15]. The Harry Dym equation shares many of the properties typical of the soliton equations. It is a completely integrable equation which can be solved by inverse scattering transformation [4], [16], [17]. It has a bi-Hamiltonian structure and an infinite number of conservation laws and infinitely many symmetries [1], [9], [5], [10]. In fact, the Harry Dym equation is one of the most exotic solitonic equations and the hierarchy to which it belongs, has a very rich structure [8].

Under the change of variables $v = u^{-\frac{1}{2}}$, this equation can be written in the equivalent form

$$v_t = -\frac{1}{2}v^3 v_{xxx} \tag{1.2}$$

The aim of this paper is investigating the Hamiltonian structure of the changed Harry Dym equation and determine Hamiltonian operators of the evolution equation (1.2).

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2. Hamiltonian Operators

In this section, we will provide the background definitions and results in Hamiltonian structures in evolution equations that will be used along this paper. Much of it is stated as in [12]. Let $x = (x^1, \cdots, x^p)$ denote the spatial variables, and $u = (u^1, \cdots, u^q)$ the field variables (dependent variables), so each u^α is a function of $x^1, \cdots, x^p$ and the time t. We will be concerned with autonomous systems of evolution equations

$$u_t = K[u], \tag{2.1}$$

in which $K[u] = (K_1[u], \cdots, K_q[u])$ is a q-tuple of differential functions, where the square brackets indicate that each K_α is a function of x, u and finitely many partial derivatives of each u^α with respect to $x^1, \cdots, x^p$. A system of evolution equations is said to be Hamiltonian if it can be written in the form

$$u_t = \mathcal{D}.\mathrm{E}_u(H), \tag{2.2}$$

Here $\mathcal{H}[u] = \int H[u]dx$ is the Hamiltonian functional, and Hamiltonian function $H[u]$ depends on $x, u,$ and the derivatives of the u's with respect to the x's; $\mathrm{E}_u = (\mathrm{E}_1, \cdots, \mathrm{E}_q)$ denotes the Euler operator or variational derivative with respect to u. The Hamiltonian operator $\mathcal{D}$ is a $q \times q$ matrix differential operator, which may depend on both $x, u,$ and derivatives of u (but not on t), and is required to be (formally) skew-adjoint relative to the L^2-inner product $< f, g >= \int f.g dx = \int \sum f^\alpha.g^\alpha dx$, so $\mathcal{D}^* = -\mathcal{D}$, where $*$ denotes the formal L^2 adjoint of a differential operator. In addition, $\mathcal{D}$ must satisfy a nonlinear "Jacobi condition" that the corresponding poisson bracket

$$\{\mathcal{P}, \mathcal{Q}\} = \int \mathrm{E}_u[P].\mathcal{D}\mathrm{E}_u[Q]dx, \tag{2.3}$$

$$\mathcal{P} = \int P[u]dx, \quad \mathcal{Q} = \int Q[u]dx,$$

satisfies the Jacobi identity. In the spatial case that $\mathcal{D}$ is a field-independent skew-adjoint differential operator, meaning that the coefficient of $\mathcal{D}$ do not depend on u or its derivatives (but may depend on x), the Jacobi conditions are automatically satisfied; For more general field-dependent operators, the complicated Jacobi conditions can be considerably simplified by the "functional multi-vector" method which is described in detail in [12]. Multi-vectors are the dual objects of differential forms. To preserve the notational distinction between the two, we use the notation θ_J^α for the uni-vector corresponding to the one-form du_J^α; thus a vertical multi-vector is a finite sum of terms, each of which is the product of a differential function times a wedge product of the basic uni-vectors. The space of functional multi-vectors is the cokernel of the total divergence, so that two vertical multi-vectors determine the same functional multi-vector if and only if they differ by a total divergence. The functional multi-vector determined by $\hat{\Theta}$ is denoted, suggestively by an integral sign: $\Theta = \int \hat{\Theta} dx$.In particular, $\int \hat{\Theta} dx = 0$ if and only if $\hat{\Theta} = \mathrm{Div}\hat{\Psi}$ for some vertical multi-vector $\hat{\Psi}$. This implies that we can integrate functional multi-vectors by parts:

$$\int \hat{\Theta} \wedge (D_i \hat{\Psi}) dx = - \int (D_i \hat{\Theta}) \wedge \hat{\Psi} dx. \tag{2.4}$$

The principal example of a functional bi-vector is that determined by a Hamiltonian differential operator $\mathcal{D}$, which is

$$\Theta_{\mathcal{D}} = \int \theta \wedge \mathcal{D}(\theta) dx. \tag{2.5}$$

Finally, define the formal prolonged vector field

$$\mathrm{pr}\mathbf{v}_{\mathcal{D}\theta} = \sum_{\alpha,J} D_J(\sum_{\beta} \mathcal{D}_{\alpha\beta}\theta^{\beta}) \frac{\partial}{\partial u_J^{\alpha}}, \tag{2.6}$$

which acts on differential functions to produce uni-vectors. We further let $\mathrm{pr}\mathbf{v}_{\mathcal{D}}\theta$ act on vertical multi-vectors by wedging the result of its action on the coefficient differential functions with the product of the θ's. Since $\mathrm{pr}\mathbf{v}_{\mathcal{D}}\theta$ commutes with the total derivative, there is also a well-defined action of $\mathrm{pr}\mathbf{v}_{\mathcal{D}}\theta$ on the space of functional multi-vectors, which essentially amounts to bringing it under the integral sign.

By virtue of the following theorem, one can determine whether or not a differential operator is genuinely Hamiltonian.

Theorem 1. *Let $\mathcal{D}$ be a skew-adjoint differential operator with corresponding bi-vector $\Theta_{\mathcal{D}}$ as above. Then $\mathcal{D}$ is a Hamiltonian operator if and only if*

$$\mathrm{pr}\mathbf{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) = 0. \tag{2.7}$$

The proof that (2.7) is equivalent to the Jacobi identity for the poisson bracket determined by $\mathcal{D}$ can be found in [12].

3. Hamiltonian structure of the Harry Dym equation

The Harry Dym equation admits two Hamiltonian operators

$$\mathcal{D} = 2uD_x + u_x, \quad \mathcal{E} = {D_x}^3, \tag{3.1}$$

and so can be written in Hamiltonian form in two distinct ways [3]. The skew symmetry of these Hamiltonian structures is manifest. The Proof of the Jacobi identiy for this structures as well their compatibility can be shown through the standard method of functional multi vectors. Since the coefficients of the operator $\mathcal{E}$ do not depend on u or its derivatives, then $\mathcal{E}$ is automatically a Hamiltonian operator. For the operator $\mathcal{D}$ it is sufficient to prove that $\mathrm{pr}\mathbf{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) = 0$, where $\Theta_{\mathcal{D}}$ is the functional bi-vector corresponding to the skew-adjoint operator $\mathcal{D}$ and θ is the basic uni-vector corresponding to u.

We can construct the bi-vector associated with the structure $\mathcal{D}$ as

$$\Theta_{\mathcal{D}} = \frac{1}{2} \int \{\theta \wedge \mathcal{D}\theta\}\, dx = \int \{u\theta \wedge \theta_x\}\, dx.$$

In continuation, we apply prolongation relations in order to prove the Jacobi identity.

$$\begin{aligned} \mathrm{pr}\mathbf{v}_{\mathcal{D}^{\cdot}}(u) &= 2u\theta_x + u_x\theta, \\ \mathrm{pr}\mathbf{v}_{\mathcal{D}^{\cdot}}(\Theta_{\mathcal{D}}) &= \int \{(2u\theta_x + u_x\theta) \wedge \theta \wedge \theta_x\}\, dx = 0. \end{aligned} \tag{3.2}$$

Therefore, the Harry Dym equation is bi-Hamiltonian , meaning that it can be written as a Hamiltonian system using any one of the two Hamiltonian operators.

The corresponding Hamiltonian functionals for Hamiltonian operators $\mathcal{E}$ and $\mathcal{D}$ are respectively

$$\mathcal{H}_1 = \int 2u^{\frac{1}{2}}\,dx, \tag{3.3}$$

$$\mathcal{H}_2 = \int \frac{1}{8}u^{-\frac{5}{2}u_x{}^2}\,dx. \tag{3.4}$$

For the Hamiltonian operator $\mathcal{E}$ of the Harry Dym equation, a distinguished functional must satisfy ${D_x}^3\delta\mathcal{P} = 0$ for some functional $\mathcal{P}$. Each such a functional is a constant multiple of the mass $\mathcal{M}[u] = \int u\,dx$. Thus, $\mathcal{M}[u]$ determines a conservation law for the Harry Dym equation relative to $\mathcal{E}$. The Hamiltonian functional $\mathcal{H}_1$ is a distinguished functional for the Hamiltonian operator $\mathcal{D}$ of the Harry Dym equation. Since, it satisfies

$$\mathcal{D}\delta\mathcal{H}_1 = \mathcal{D}\mathrm{E}_u(H_1) = (2uD_x + u_x)u^{-\frac{1}{2}} = 0.$$

Hence, $\mathcal{H}_1$ determines a conservation law for the equation (1.1) relative to $\mathcal{D}$.

Furthermore, these two Hamiltonian operators form a Hamiltonian pair, namely, not only are $\mathcal{E}$ and $\mathcal{D}$ genuine Hamiltonian structures, any arbitrary linear combination of them is as well. In the following, we show that the Hamiltonian operators $\mathcal{E}$ and $\mathcal{D}$ form a Hamiltonian pair. So, it is sufficient to prove that

$$\mathbf{pr v}_{\mathcal{E}`}(\Theta_{\mathcal{D}}) + \mathbf{pr v}_{\mathcal{D}`}(\Theta_{\mathcal{E}}) = 0, \tag{3.5}$$

where $\Theta_{\mathcal{E}}$ and $\Theta_{\mathcal{D}}$ are the bivectors corresponding to the Hamiltonian structures $\mathcal{E}$ and $\mathcal{D}$, respectively. Since $\mathcal{E}$ has constant coefficients, $\mathbf{pr v}_{\mathcal{D}`}(\Theta_{\mathcal{E}}) = 0$.

So we only need to verify $\mathbf{pr v}_{\mathcal{E}`}(\Theta_{\mathcal{D}}) = 0$. where $\mathbf{pr v}_{\mathcal{E}`}(u) = \zeta_x,\ \mathbf{pr v}_{\mathcal{E}`}(v) = \theta_x$. Hence,

$$\mathbf{pr v}_{\mathcal{E}`}(\Theta_{\mathcal{D}}) = \int \{\zeta \wedge \theta_x \wedge \zeta_x + \theta_x \wedge \zeta \wedge \zeta_x\}\,dx = 0. \tag{3.6}$$

Now, according to the compatibility condition between the two poisson structures determined by $\mathcal{D}$ and $\mathcal{E}$, we will be able to recursively construct an infinite hierarchy of symmetries and conservation laws for the equation in the following manner.

According to the theorem [12], if $\mathcal{P}[u]$ is any conserved functional for the equation, then both of the Hamiltonian vector fields $\mathrm{v}_{\mathcal{D}\delta\mathcal{P}}$ and $\mathrm{v}_{\mathcal{E}\delta\mathcal{P}}$ are symmetries. In particular, since both $\mathcal{H}_1$ and $\mathcal{H}_2$ are conserved, not only is the original vector field $\mathrm{v}_{\mathcal{D}\delta\mathcal{H}_2} = \mathrm{v}_{\mathcal{E}\delta\mathcal{H}_1}$ a symmetry of the Harry Dym equation, but so are the two additional vector fields $\mathrm{v}_{\mathcal{D}\delta\mathcal{H}_1}$ and $\mathrm{v}_{\mathcal{E}\delta\mathcal{H}_2}$. The recursion algorithm proceeds on the assumption that one of these new symmetries is a Hamiltonian vector field for the other Hamiltonian structure, so we have $\mathcal{E}\delta\mathcal{H}_2 = \mathcal{D}\delta\mathcal{H}_3$ for some functional $\mathcal{H}_3$. Now, $\mathcal{H}_3$ or some equivalent functional is conserved and so we obtain yet a further symmetry, this time with characteristic $\mathcal{E}\delta\mathcal{H}_3$.

Therefore, after straightforward but tedious calculation, one can obtain the first few Hamiltonian functionals for the Harry Dym equation as follows [3]:

$$\begin{aligned}
\mathcal{H}_1 &= \int 2u^{\frac{1}{2}}\,dx, \\
\mathcal{H}_2 &= \int \frac{1}{8}u^{-\frac{5}{2}}u_x^2\,dx, \\
\mathcal{H}_3 &= \int \frac{1}{16}(\frac{35}{16}u^{-\frac{11}{2}}u_x^4 - u^{-\frac{7}{2}}u_{xx}^2)\,dx, \\
\mathcal{H}_4 &= \int \frac{1}{32}(\frac{5005}{128}u^{-\frac{7}{12}}u_x^6 - \frac{231}{8}u^{-\frac{13}{2}}u_x^2u_{xx}^2 + 5u^{-\frac{11}{2}}u_{xx}^3 + u^{-\frac{9}{2}}u_{xxx}^2)\,dx.
\end{aligned} \tag{3.7}$$

4. Change of variables

Consider general differential substitutions

$$y = P[u], \quad w = Q[u], \tag{4.1}$$

in which y is the new independent variable and w the new dependent variable, and P and Q are differential functions, which therefore are allowed to depend on x, u, and derivatives of u. The goal now is to see how various operators change when subjected to (4.1)

The easiest is the total derivative D_x, whose transformation rule is determined by the chain rule from elementary calculus:

$$D_x = D_xPD_y \qquad D_y = (D_x)^{-1}D_x, \tag{4.2}$$

where $(D_xP)^{-1}$ is just the reciprocal $\frac{1}{D_xP}$. To determine more complicated change of variables, we first need toe recall the Fréchet derivative of a differential function, which is the differential operator D_P defined by the formula [13]

$$D_P(v) = \frac{d}{d\varepsilon}|_{\varepsilon=0}P[u+\varepsilon v]; \tag{4.3}$$

more explicitly,

$$D_P = \sum \frac{\partial P}{\partial u_k}D_x^k. \tag{4.4}$$

Lemma 2. *Let y, w be related to x, u by the differential substitution(4.1), with y the new independent variable. Then*

$$w_t = (D_xP)^{-1}\mathbf{D}(u_t), \tag{4.5}$$

and the Euler operators are related by

$$E_u = \mathbf{D}^*E_w, \tag{4.6}$$

where the operator $\mathbf{D}$ is

$$\mathbf{D} = D_xPD_Q - D_xQD_P. \tag{4.7}$$

Theorem 3. *Let $\mathcal{D}$ be a Hamiltonian operator depending on x and u, and let $y = P[u]$ and $w = Q[u]$ be related to x and u by differential substitution. Then the corresponding Hamiltonian operator in the y, w-variables is*

$$\tilde{\mathcal{D}} = (D_xP)^{-1}\mathbf{D}\mathcal{D}\mathbf{D}^*. \tag{4.8}$$

See [13] for more details.

Note in particular, if the differential substitution does not change the independent variable, so we just have $w = Q[u]$, then (1.2) reduces to the formula

$$\tilde{\mathcal{D}} = D_Q \mathcal{D} D_Q{}^*. \tag{4.9}$$

5. Hamiltonian Formalism of the transformed equation

The change of variables $v = u^{-\frac{1}{2}}$ changes the Harry Dym equation to the evolution equation (1.2). In this section, we discuss its effects on the bi-Hamiltonian structure and then we obtain the corresponding Hamiltonian operators for the equation (1.2).

We have $Q = u^{-\frac{1}{2}}$. Thus, its Frechet derivative is $D_Q = -\frac{1}{2}u^{-\frac{3}{2}}$. So, the adjoint of the Frechet derivative of Q is $D_Q{}^* = -\frac{1}{2}u^{-\frac{3}{2}}$.

For the Hamiltonian operator $\mathcal{D}$ of the Harry Dym equation, the corresponding Hamiltonian operator is

$$\tilde{\mathcal{D}} = (-\frac{1}{2}u^{-\frac{3}{2}})\mathcal{D}(-\frac{1}{2}u^{-\frac{3}{2}}), \tag{5.1}$$

i.e.,

$$\begin{aligned} \tilde{\mathcal{D}} &= (-\frac{1}{2}u^{-\frac{3}{2}})\{2uD_x + u_x\}(-\frac{1}{2}u^{-\frac{3}{2}}) \qquad (5.2) \\ &= \frac{1}{2}u^{-2}D_x - \frac{1}{2}u^{-3}u_x, \end{aligned}$$

According to the change of variables $v = u^{-\frac{1}{2}}$, $u_x = -2v_x v^3$. So, the corresponding Hamiltonian operator $\tilde{\mathcal{D}}$ in x, v-variables is

$$\tilde{\mathcal{D}} = \frac{1}{2}v^4 D_x + v^3 v_x. \tag{5.3}$$

Now, we obtain the corresponding Hamiltoninan functional $\tilde{\mathcal{H}}_2$ relative to $\tilde{\mathcal{D}}$ for the equation (1.2). The Hamiltonian functional $\mathcal{H}_2$ relative to the Hamiltonian operator $\mathcal{D}$ is $\mathcal{H}_2 = \int \frac{1}{8}u^{-\frac{5}{2}}u_x{}^2\, dx$. Therefore, $\tilde{\mathcal{H}}_2 = \int \frac{1}{2}v^{-1}v_x{}^2\, dx$. So, its obvious that the equation (1.2) can be written in Hamiltonian form using the Hamiltonian operator $\tilde{\mathcal{D}}$ and Hamiltonian functional $\tilde{\mathcal{H}}_2$, i.e., $v_t = \tilde{\mathcal{D}}\delta\tilde{\mathcal{H}}_2$.

Now, we apply the theorem for the Hamiltonian operator $\mathcal{E}$ of the Harry Dym equation in order to obtain the corresponding Hamiltonian operator $\tilde{\mathcal{E}}$ for the equation (1.2). By using the formula for the higher order derivatives, we have:

$$\begin{aligned} \tilde{\mathcal{E}} &= (-\frac{1}{2}u^{-\frac{3}{2}})\{D_x{}^3\}(-\frac{1}{2}u^{-\frac{3}{2}}) \\ &= \frac{1}{4}u^{-\frac{3}{2}}\{D_x{}^3u^{-\frac{3}{2}} + 3D_x{}^2u^{-\frac{3}{2}}D_x + 3D_xu^{-\frac{3}{2}}D_x{}^2 + u^{-\frac{3}{2}}D_x{}^3\} \qquad (5.4) \\ &= \frac{1}{4}u^{-\frac{3}{2}}\{(-\frac{105}{8}u^{-\frac{9}{2}}u_x{}^3 + \frac{45}{4}u^{-\frac{7}{2}}u_xu_{xx} - \frac{3}{2}u^{-\frac{5}{2}}u_{xxx}) \\ &\quad +(\frac{45}{4}u^{-\frac{7}{2}}u_x{}^2 - \frac{9}{2}u^{-\frac{5}{2}}u_{xx})D_x - (\frac{9}{2}u^{-\frac{5}{2}}u_x)D_x{}^2 + u^{-\frac{3}{2}}D_x{}^3\}. \end{aligned}$$

Therefore, by using the change of variables and substituting derivatives of u in (5.4), $\tilde{\mathcal{E}}$ takes the form in the x, v-variables

$$\begin{aligned} \tilde{\mathcal{E}} &= (\frac{3}{2}v^3{v_x}^3 + \frac{18}{4}v^4 v_x v_{xx} + \frac{3}{4}v^5 v_{xxx}) + (\frac{18}{4}v^4{v_x}^2 + \frac{9}{4}v^5 v_{xx})D_x \\ &+ \frac{9}{4}v^5 v_x {D_x}^2 + \frac{1}{4}v^6 {D_x}^3. \end{aligned} \tag{5.5}$$

Hence, the corresponding Hamiltonian functional $\tilde{\mathcal{H}}_1$ relative to the Hamiltonian operator $\tilde{\mathcal{E}}$ for the equation (1.2) is

$$\tilde{\mathcal{H}}_1 = \int 2v^{-1}\, dx. \tag{5.6}$$

Thus, its clear that the equation (1.2) can be written in Hamiltonian form using the Hamiltonian operator $\tilde{\mathcal{E}}$ and Hamiltonian functional $\tilde{\mathcal{H}}_1$, i.e.,

$$v_t = \tilde{\mathcal{E}}\delta\tilde{\mathcal{H}}_1 = \tilde{\mathcal{E}}\mathrm{E}_v(\tilde{H}_1). \tag{5.7}$$

Moreover, the Hamiltonian functional $\tilde{\mathcal{H}}_1$ is a distinguished functional for the Hamiltonian operator $\tilde{\mathcal{D}} = \frac{1}{2}v^4 D_x + v^3 v_x$ of the equation (2.1). Since, it must satisfy $\tilde{\mathcal{D}}\delta\tilde{\mathcal{H}}_1 = 0$. We have,

$$\tilde{\mathcal{D}}\delta\tilde{\mathcal{H}}_1 = \tilde{\mathcal{D}}\mathrm{E}_v(\tilde{\mathrm{H}}_1) = (\frac{1}{2}v^4 D_x + v^3 v_x)(-2v^{-2}) = 0. \tag{5.8}$$

So, $\tilde{\mathcal{H}}_1$ is a distinguished functional for the operator $\tilde{\mathcal{D}}$ and therefore, it determines a conservation law for the transformed equation relative to $\tilde{\mathcal{D}}$.

6. Conclusions

In this paper, we discuss on the effect of the change of variables on the bi-Hamiltonian structure of the Harry Dym equation and derive the corresponding Hamiltonian formalism of the transformed equation.

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