Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.6, (2017), Issue 2, pp.73-82

# ON CONTACT CR-SUBMANIFOLDS OF A $\delta$-LORENTZIAN TRANS-SASAKIAN MANIFOLD

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ABSTRACT. The purpose of the present paper is to study contact $CR$-submanifolds of a $\delta$-Lorentzian trans-Sasakian manifold with a quarter symmetric non-metric connection. Also, we investigate the relation between the curvature tensor of $\delta$-Lorentzian trans-Sasakian manifolds and related results with respect to the quarter symmetric non-metric connection and the Levi-Civita connection.

## 1. INTRODUCTION

The notion of $CR$-submanifolds of a Kaehler manifold was introduced by A. Bejancu [1]. Later on, $CR$-submanifolds of a Sasakian manifold were studied by M. Kobayashi [11]. K. Matsumoto introduced the idea of a Lorentzian para-Sasakian structure and studied several of its properties [6]. J. A. Oubina defined and studied a new class of almost contact metric manifold known as trans-Sasakian manifold, which includes $\alpha$-Sasakian, $\beta$-Kenmotsu and cosymplectic structures [5]. M. H. Shahid have studied $CR$-submanifolds of a trans-Sasakian manifold ([9],[10]).

Recently, Lorentzian trans-Sasakian manifolds were studied by the authors (see [15],[16]) and they obtained several interesting results on the manifold. In [14], S. M., Bhati introduced the notion of weakly Ricci $\phi$-symmetric $\delta$-Lorentzian trans-Sasakian manifolds and studied characteristic properties of locally $\phi$-Ricci symmetric and $\phi$-recurrent spaces.

A linear connection $\bar{\nabla}$ in a Riemannian manifold $\bar{M}$ is said to be a quarter-symmetric connection [13] if its torsion tensor $T$ of the connection $\bar{\nabla}$

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - \bar{\nabla}_{[X,Y]}$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where $\eta$ is a 1-form and $\phi$ is a $(1,1)$ tensor field. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = -\eta(Y)g(\phi X, Z) - \eta(Z)g(Y,\phi X)$$

2010 *Mathematics Subject Classification.* 53C05, 53C25, 53C50 .
*Key words and phrases.* $CR$-submanifolds, $\delta$-Lorentzian trans-Sasakian manifold, quarter symmetric non-metric connection, curvature tensor,
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for all $X, Y, Z \in T(\bar{M})$, where $T(\bar{M})$ is the Lie algebra of vector fields of the manifold $\bar{M}$, then $\bar{\nabla}$ is said to be a quarter-symmetric non-metric connection. If we put $\phi X = X$ and $\phi Y = Y$, then the quarter-symmetric non-metric connection reduces to the semi-symmetric non-metric connection [7]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection. A quarter-symmetric non-metric connection have been studied by various authors such as A. K. Mondal and U. C. De [3], R. Nivas and G. Verma [12] and C. Özgür, M. Ahmad and A. Haseeb [4].

The paper is organized as follows : In Section 2, we give a brief introduction of $\delta$-Lorentzian trans-Sasakian manifolds. Section 3, deals with the study of $CR$-submanifolds of a $\delta$-Lorentzian trans-Sasakian manifold with a quarter-symmetric non-metric connection. In Section 4, we obtain the integrability conditions for the distributions $D$ and $D^{\perp}$ on $CR$-submanifolds of a $\delta$-Lorentzian trans-Sasakian manifold with a quarter-symmetric non-metric connection. In Section 5, we investigate the relation between the curvature tensor of $\delta$-Lorentzian trans-Sasakian manifolds and related results with a quarter symmetric non-metric connection and the Levi-Civita connection.

## 2. $\delta$-LORENTZIAN TRANS-SASAKIAN MANIFOLDS

Let $\bar{M}$ be an almost contact metric manifold of dimension $(2n+1)$ with an almost contact metric structure $(\phi, \xi, \eta, g)$ consisting of a $(1,1)$ tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ satisfying [15]

$$\phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = -1, \quad \phi \circ \xi = 0, \quad \eta \circ \phi = 0. \tag{2.2}$$

An almost contact metric manifold $\bar{M}$ is called a $\delta$-almost contact metric manifold if

$$g(\xi, \xi) = -\delta, \tag{2.3}$$

$$\eta(X) = \delta g(X, \xi), \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) + \delta\eta(X)\eta(Y) \tag{2.5}$$

for all vector fields $X, Y \in T(\bar{M})$, where $\delta^2 = 1$ so that $\delta = \pm 1$.

A $\delta$-almost contact metric manifold is called a $\delta$-Lorentzian trans-Sasakian manifold if

$$(\bar{\bar{\nabla}}_X \phi)Y = \alpha[g(\phi X, Y)\xi - \delta\eta(Y)X] + \beta[g(\phi X, Y)\xi - \delta\eta(Y)\phi X] \tag{2.6}$$

for some smooth functions $\alpha$ and $\beta$ on $\bar{M}$ and $\delta = \pm 1$.

If $\delta = 1$, then the $\delta$-Lorentzian trans-Sasakian manifold is usual Lorentzian trans-Sasakian manifold of type $(\alpha, \beta)$. In particular, $\delta$-Lorentzian trans-Sasakian structure of the type $(\alpha, \beta)$ is $\delta$-Lorentzian Sasakian, $\delta$-Lorentzian Kenmotsu and $\delta$-Lorentzian cosymplectic manifolds according as $\beta = 0, \alpha = 1$; $\alpha = 0, \beta = 1$ and $\alpha = \beta = 0$ respectively.

For a $\delta$-Lorentzian trans-Sasakian manifold, we also have

$$(\bar{\bar{\nabla}}_X \eta)Y = \alpha g(\phi X, Y) + \beta[g(X, Y) + \delta\eta(X)\eta(Y)], \tag{2.7}$$

$$\bar{\bar{\nabla}}_X \xi = -\alpha\delta\phi X - \beta\delta X - \beta\delta\eta(X)\xi, \tag{2.8}$$

where $\bar{\bar{\nabla}}$ denotes the operator of the covariant differentiation with respect to the Lorentzian metric $g$.

## 3. $CR$-SUBMANIFOLDS OF $\delta$-LORENTZIAN TRANS-SASAKIAN MANIFOLDS

**Definition.** An $m$-dimensional Riemannian submanifold $M$ of a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ is called a CR-submanifold if $\xi$ is tangent to $M$ and there exists on $M$ a differentiable distribution $D : x \to D_x \subset T_x(M)$ such that
$(i)$ the distribution $D_x$ is invariant under $\phi$, i.e., $\phi D_x \subset D_x$ for each $x \in M$;
$(ii)$ the orthogonal complementary distribution $D^{\perp} : x \to D_x^{\perp} \subset T_x(M)$ of the distribution $D$ on $M$ is anti-invarient under $\phi$, i.e., $\phi D_x^{\perp}(M) \subset T_x^{\perp}(M)$ for all $x \in M$, where $T_x(M)$ and $T_x^{\perp}(M)$ are tangent space and normal space of $M$ at $x \in M$ respectively.

If dim $D_x^{\perp} = 0$ $(resp., dimD_x = 0)$, then $CR$-submanifold is called an invariant $(resp., anti-invariant)$. The distribution $D$ $(resp., D^{\perp})$ is called the horizontal $(resp., vertical)$ distribution. The pair $(D, D^{\perp})$ is called $\xi$-horizontal $(resp., \xi - vertical)$ if $\xi_x \in D_x$ $(resp., \xi_x \in D_x^{\perp})$ for $x \in M$.

Any vector field $X$ tangent to $M$ can be decomposed as $[2, 8]$

$$X = PX + QX, \tag{3.1}$$

where $PX$ and $QX$ belong to the distribution $D$ and $D^{\perp}$ respectively.

For any vector field $N$ normal to $M$, we put

$$\phi N = BN + CN, \tag{3.2}$$

where $BN$ (resp., $CN$) is the tangential (resp., normal) component of $\phi N$.

Now, we remark that owing to the existence of the 1-form $\eta$, we can define a quarter symmetric non-metric connection $\bar{\nabla}$ in an almost contact metric manifold by

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta(Y)\phi X \tag{3.3}$$

for any $X, Y \in T(M)$ and $\bar{\bar{\nabla}}$ is the induced connection with respect to $g$ on $M$.
From (2.6), (2.8) and (3.3), we obtain

$$(\bar{\nabla}_X\phi)Y = \alpha[g(X,Y)\xi - \delta\eta(Y)X] + \beta[g(\phi X, Y)\xi - \delta\eta(Y)\phi X] \tag{3.4}$$
$$-\eta(Y)X - \eta(X)\eta(Y)\xi,$$
$$\bar{\nabla}_X\xi = -\alpha\delta\phi X - \beta\delta X - \beta\delta\eta(X)\xi - \phi X. \tag{3.5}$$

For a contact $CR$-submanifold $M$ of a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection, the Gauss and Weingarten formulae are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{3.6}$$

$$\bar{\nabla}_X N = -A_N X + \eta(N)\phi X + \nabla_X^{\perp} N \tag{3.7}$$

for $X, Y \in T(M)$, $N \in T^{\perp}M$, $h$(resp; $A_N$) is the second fundamental (resp; tensor) form of $M$ in $\bar{M}$, $\bar{\nabla}$ and $\nabla$ are the Riemannian and induced Riemannian connections respectively in $\bar{M}$

and $M$ and $\nabla^{\perp}$ is the operator of the normal connection. Moreover, $h$ satisfies the following condition

$$g(A_N X, Y) = g(h(X,Y), N). \tag{3.8}$$

## 4. MAIN RESULTS

**Lemma 1.** *Let $M$ be a contact $CR$-submanifold of a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then*

$$P\nabla_X \phi PY - PA_{\phi QY}X = \phi P\nabla_X Y + \alpha g(X,Y)P\xi + \beta g(\phi X, Y)P\xi \tag{4.1}$$

$$-\eta(Y)PX - \alpha\delta\eta(Y)PX - \beta\delta\eta(Y)\phi PX - \eta(X)\eta(Y)P\xi,$$

$$Q\nabla_X \phi PY - QA_{\phi QY}X = \alpha g(X,Y)Q\xi \ + \beta g(\phi X, Y)Q\xi - \alpha\delta\eta(Y)QX \tag{4.2}$$

$$-\eta(Y)QX + Bh(X,Y) - \eta(X)\eta(Y)Q\xi,$$

$$h(X, \phi PY) + \nabla^{\perp}_X \phi QY = \phi Q\nabla_X Y + Ch(X,Y) - \beta\delta\eta(Y)\phi QX \tag{4.3}$$

*for all $X, Y \in TM$.*

*Proof.* By the direct covariant differentiation of $\phi Y$, we have

$$\bar{\nabla}_X \phi Y = (\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_X Y).$$

By using (3.1), (3.4), (3.6) and (3.7), we get

$$\nabla_X \phi PY + h(X, \phi PY) - A_{\phi QY}X + \nabla^{\perp}_X \phi QY = \alpha[g(X,Y)\xi - \delta\eta(Y)X]$$

$$+\beta[g(\phi X, Y)\xi - \delta\eta(Y)\phi X] - \eta(Y)X - \eta(X)\eta(Y)\xi + \phi\nabla_X Y + \phi h(X,Y)$$

which by using (3.1) and (3.2) takes the form

$$P\nabla_X \phi PY + Q\nabla_X \phi PY - PA_{\phi QY}X - QA_{\phi QY}X + h(X, \phi PY) + \nabla^{\perp}_X \phi QY$$

$$= \phi P\nabla_X Y + \phi Q\nabla_X Y + Bh(X,Y) + Ch(X,Y) + \alpha g(X,Y)P\xi + \alpha g(X,Y)Q\xi$$

$$-\alpha\delta\eta(Y)PX - \alpha\delta\eta(Y)QX - \eta(Y)PX - \eta(Y)QX + \beta g(\phi X, Y)P\xi + \beta g(\phi X, Y)Q\xi$$

$$-\beta\delta\eta(Y)\phi PX - \beta\delta\eta(Y)\phi QX - \eta(X)\eta(Y)P\xi - \eta(X)\eta(Y)Q\xi.$$

By equating horizontal, vertical and normal components, we obtain equations (4.1)-(4.3). □

Now, we calculate the Nijenhuis tensor $N(X,Y)$ on a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection. For this, first we prove the following lemma:

**Lemma 2.** *Let $M$ be a contact $CR$-submanifold of a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then we have*

$$(\bar{\nabla}_{\phi X}\phi)Y = \alpha[g(\phi X, Y)\xi - \delta\eta(Y)\phi X] + \beta[g(X,Y)\xi + \eta(X)\eta(Y)\xi] - \beta\delta\eta(Y)X$$

$$-\eta(Y)\phi X - \beta\delta\eta(X)\eta(Y)\xi$$

*for all $X, Y \in T\bar{M}$.*

*Proof.* By the definition of a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection, we have

$$(\bar{\nabla}_X\phi)Y = \alpha[g(X,Y)\xi - \delta\eta(Y)X] + \beta[g(\phi X,Y)\xi - \delta\eta(Y)\phi X] \\ -\eta(Y)X - \eta(X)\eta(Y)\xi \tag{4.4}$$

which by replacing $X$ by $\phi X$ and using (2.2) takes the form

$$(\bar{\nabla}_{\phi X}\phi)Y = \alpha[g(\phi X,Y)\xi - \delta\eta(Y)\phi X] + \beta[g(X,Y)\xi + \eta(X)\eta(Y)\xi] - \beta\delta\eta(Y)X \\ -\eta(Y)\phi X - \beta\delta\eta(X)\eta(Y)\xi \tag{4.5}$$

for $X, Y \in T\bar{M}$. □

On a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection, the Nijenhuis tensor is given by

$$N(X,Y) = (\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X - \phi(\bar{\nabla}_X\phi)Y + \phi(\bar{\nabla}_Y\phi)X \tag{4.6}$$

for any $X, Y \in T\bar{M}$.
Using (4.5) in (4.6), we get

$$N(X,Y) = -(\alpha\delta+1)\eta(Y)\phi X + (\alpha\delta+1)\eta(X)\phi Y - \beta\delta\eta(Y)X + \beta\delta\eta(X)Y \\ -\phi(\bar{\nabla}_X\phi)Y + \phi(\bar{\nabla}_Y\phi)X \tag{4.7}$$

for any $X, Y \in T\bar{M}$.

Now we prove the following proposition:

**Proposition 3.** *Let $M$ be a $\xi$-vertical contact $CR$-submanifold of a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then the distribution $D$ is integrable if the following conditions are satisfied:*

$$S(X,Y) \in D, \qquad h(X,\phi Y) = h(Y,\phi X) \tag{4.8}$$

*for any $X, Y \in D$.*

*Proof.* The torsion tensor $S(X,Y)$ of the structure $(\phi,\xi,\eta,g,\delta)$ is given by

$$S(X,Y) = N(X,Y) + 2d\eta(X,Y)\xi = N(X,Y) + 2g(\phi X,Y)\xi.$$

Using (4.7), we have

$$S(X,Y) = 2g(\phi X,Y)\xi - (\alpha\delta+1)\eta(Y)\phi X + (\alpha\delta+1)\eta(X)\phi Y - \beta\delta\eta(Y)X \\ +\beta\delta\eta(X)Y - \phi(\bar{\nabla}_X\phi)Y + \phi(\bar{\nabla}_Y\phi)X \tag{4.9}$$

for any $X, Y \in TM$.
By covariant differentiation of $\phi X$ with respect to $Y$ and using (2.1) and (3.6), we have

$$\phi(\bar{\nabla}_Y\phi)X = \phi\nabla_Y\phi X + \phi h(Y,\phi X) - \nabla_Y X - \eta(\nabla_Y X)\xi - h(X,Y). \tag{4.10}$$

From (4.9) and (4.10), we find

$$S(X,Y) = 2g(\phi X,Y)\xi - \phi(\nabla_X\phi Y - \nabla_Y\phi X) - \phi(h(X,\phi Y) - h(Y,\phi X)) \\ +(\nabla_X Y - \nabla_Y X) + \eta(\nabla_X Y - \nabla_Y X)\xi$$

for any $X, Y \in D$ and $\xi \in D^{\perp}$.
If $S(X,Y) \in D$ for $X, Y \in D$, then the last equation reduces to

$$2g(\phi X, Y)\xi - \phi h(X, \phi Y) + \phi h(Y, \phi X) = 0.$$

On equating vertical and normal components, we get

$$h(X, \phi Y) = h(Y, \phi X)$$

for any $X, Y \in D$. □

**Lemma 4.** *Let $M$ be a $\xi$-vertical contact $CR$-submanifold of a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then*

$$\phi P[W, Z] = A_{\phi W}Z - A_{\phi Z}W + (\alpha\delta + 1)[\eta(Z)W - \eta(W)Z] \quad (4.11)$$
$$+\beta\delta[\eta(Z)\phi W - \eta(W)\phi Z + \eta(W)\phi QZ - \eta(Z)\phi QW]$$

*for any W,Z $\in D^{\perp}$.*

*Proof.* From the equation (4.3), we have

$$\nabla^{\perp}_{W}\phi Z = \phi Q\nabla_W Z + Ch(W, Z) - \beta\delta\eta(Z)\phi QW \quad (4.12)$$

for any $W, Z \in D^{\perp}$. Now by covariant differentiation of $\phi Z$, we have

$$\bar{\nabla}_W \phi Z = (\bar{\nabla}_W \phi)Z + \phi\bar{\nabla}_W Z. \quad (4.13)$$

By using (3.4), (3.6) and (3.7) in (4.13), we get

$$-A_{\phi Z}W + \nabla^{\perp}_{W}\phi Z + \eta(\phi Z)W = \alpha[g(W, Z)\xi - \delta\eta(Z)W] + \beta[g(\phi W, Z)\xi - \delta\eta(Z)\phi W]$$
$$-\eta(Z)W - \eta(W)\eta(Z)\xi + \phi(\nabla_W Z + h(W, Z))$$

for $W, Z \in D^{\perp}$. In view of (3.1) and (3.2), the last equation can be written as

$$\nabla^{\perp}_{W}\phi Z = A_{\phi Z}W + \alpha[g(W, Z)\xi - \delta\eta(Z)W] + \beta[g(\phi W, Z)\xi - \delta\eta(Z)\phi W] \quad (4.14)$$
$$-\eta(Z)W - \eta(W)\eta(Z)\xi + \phi P\nabla_W Z + \phi Q\nabla_W Z + Bh(W, Z) + Ch(W, Z).$$

From (4.12) and (4.14), we have

$$\phi P\nabla_W Z = -A_{\phi Z}W - \alpha[g(W, Z)\xi - \delta\eta(Z)W] - \beta[g(\phi W, Z)\xi - \delta\eta(Z)\phi W] + \eta(Z)W$$
$$+\eta(W)\eta(Z)\xi - Bh(W, Z) - \beta\delta\eta(Z)\phi QW$$

which by interchanging $W$ and $Y$ takes the form

$$\phi P\nabla_Z W = -A_{\phi W}Z - \alpha[g(W, Z)\xi - \delta\eta(W)Z] - \beta[g(\phi Z, W)\xi - \delta\eta(W)\phi Z] + \eta(W)Z$$
$$+\eta(W)\eta(Z)\xi - Bh(W, Z) - \beta\delta\eta(W)\phi QZ.$$

From the last two equations, we obtain

$$\phi P[W, Z] = A_{\phi W}Z - A_{\phi Z}W + (\alpha\delta + 1)[\eta(Z)W - \eta(W)Z]$$
$$+\beta\delta[\eta(Z)\phi W - \eta(W)\phi Z + \eta(W)\phi QZ - \eta(Z)\phi QW]$$

for any $W, Z \in D^{\perp}$. □

Hence we can state the following theorem:

**Theorem 5.** *Let $M$ be a $\xi$-vertical contact $CR$-submanifold of a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi X}Y = A_{\phi Y}X,$$

*provided $\alpha\delta = -1$ for any $X,Y \in D^{\perp}$.*

## 5. Curvature tensor of $\delta$-Lorentzian trans-Sasakian manifolds with a quarter symmetric non-metric connection.

We define the curvature tensor of a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection $\bar{\nabla}$ by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z, \tag{5.1}$$

where $\bar{R}$ is the curvature tensor with respect to a quarter symmetric non-metric connection $\bar{\nabla}$. By using (2.2) and (3.3) in (5.1), we have

$$\bar{R}(X,Y)Z = \bar{\bar{R}}(X,Y)Z + (\bar{\bar{\nabla}}_X\eta)(Z)\phi Y + \eta(Z)(\bar{\bar{\nabla}}_X\phi)Y - (\bar{\bar{\nabla}}_Y\eta)(Z)\phi X - \eta(Z)(\bar{\bar{\nabla}}_Y\phi)X \tag{5.2}$$

which on using (2.6) and (2.7) takes the form

$$\begin{aligned}\bar{R}(X,Y)Z = {} & \bar{\bar{R}}(X,Y)Z + \alpha g(\phi X,Z)\phi Y - \alpha g(\phi Y,Z)\phi X + \beta g(X,Z)\phi Y \\ & -\beta g(Y,Z)\phi X + \alpha\delta(\eta(X)Y - \eta(Y)X)\eta(Z) + 2\beta\delta(\eta(X)\phi Y - \eta(Y)\phi X)\eta(Z),\end{aligned} \tag{5.3}$$

where $X,Y,Z \in T(\bar{M})$ and $\bar{\bar{R}}(X,Y)Z = \bar{\bar{\nabla}}_X\bar{\bar{\nabla}}_Y Z - \bar{\bar{\nabla}}_Y\bar{\bar{\nabla}}_X Z - \bar{\bar{\nabla}}_{[X,Y]}Z$ is the curvature tensor with respect to the connection $\bar{\bar{\nabla}}$.

Interchanging $X$ and $Y$ in (5.3), we have

$$\begin{aligned}\bar{R}(Y,X)Z = {} & \bar{\bar{R}}(Y,X)Z + \alpha g(\phi Y,Z)\phi X - \alpha g(\phi X,Z)\phi Y + \beta g(Y,Z)\phi X \\ & -\beta g(X,Z)\phi Y + \alpha\delta(\eta(Y)X - \eta(X)Y)\eta(Z) + 2\beta\delta(\eta(Y)\phi X - \eta(X)\phi Y)\eta(Z).\end{aligned} \tag{5.4}$$

Adding (5.3) and (5.4) and using the fact that $\bar{\bar{R}}(X,Y)Z + \bar{\bar{R}}(Y,X)Z = 0$, we get

$$\bar{R}(X,Y)Z + \bar{R}(Y,X)Z = 0. \tag{5.5}$$

From (5.3) we can write two more following equations by the cyclic permutations of $X$, $Y$ and $Z$, that are

$$\begin{aligned}\bar{R}(Y,Z)X = {} & \bar{\bar{R}}(Y,Z)X + \alpha g(\phi Y,X)\phi Z - \alpha g(\phi Z,X)\phi Y + \beta g(Y,X)\phi Z \\ & -\beta g(Z,X)\phi Y + \alpha\delta(\eta(Y)Z - \eta(Z)Y)\eta(X) + 2\beta\delta(\eta(Y)\phi Z - \eta(Z)\phi Y)\eta(X)\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}\bar{R}(Z,X)Y = {} & \bar{\bar{R}}(Z,X)Y + \alpha g(\phi Z,Y)\phi X - \alpha g(\phi X,Y)\phi Z + \beta g(Z,Y)\phi X \\ & -\beta g(X,Y)\phi Z + \alpha\delta(\eta(Z)X - \eta(X)Z)\eta(Y) + 2\beta\delta(\eta(Z)\phi X - \eta(X)\phi Z)\eta(Y).\end{aligned} \tag{5.7}$$

Adding (5.3), (5.6) and (5.7) and using the Bianchi's first identity $\bar{\bar{R}}(X,Y)Z + \bar{\bar{R}}(Y,Z)X + \bar{\bar{R}}(Z,X)Y = 0$ with respect to $\bar{\bar{\nabla}}$, we get

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0. \tag{5.8}$$

Contracting $X$ in (5.3), we get

$$\bar{S}(Y,Z) = \bar{\bar{S}}(Y,Z) + (\alpha - \beta\psi)g(Y,Z) - (\alpha\psi - \beta)g(\phi Y,Z) + [\alpha(1-\delta(n-1)) - 2\beta\delta\psi]\eta(Y)\eta(Z), \tag{5.9}$$

where $\bar{S}$ and $\bar{\bar{S}}$ are the Ricci tensors of the connections $\bar{\nabla}$ and $\bar{\bar{\nabla}}$, respectively in $\bar{M}$; $\psi = trace\phi$. This gives

$$\bar{Q}Y = \bar{\bar{Q}}Y + (\alpha - \beta\psi)Y - (\alpha\psi - \beta)\phi Y + [\alpha(1-\delta(n-1)) - 2\beta\delta\psi]\eta(Y)\xi, \tag{5.10}$$

where $\bar{Q}$ and $\bar{\bar{Q}}$ are the Ricci operators of the connections $\bar{\nabla}$ and $\bar{\bar{\nabla}}$, respectively in $\bar{M}$ and $g(\bar{Q}X,Y) = \bar{S}(X,Y)$.
Contracting again $Y$ and $Z$ in (5.9), we get

$$\bar{r} = \bar{\bar{r}} + (\alpha - \beta\psi)(2n+1) - (\alpha\psi - \beta)\psi - [\alpha(1-\delta(n-1)) - 2\beta\delta\psi], \tag{5.11}$$

where $\bar{r}$ and $\bar{\bar{r}}$ are the scalar curvatures of the connections $\bar{\nabla}$ and $\bar{\bar{\nabla}}$, respectively in $\bar{M}$.

Thus we have the following theorem:

**Theorem 6.** *For a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection $\bar{\nabla}$*
$(a)$ *The curvature tensor $\bar{R}$ is given by* (5.3),
$(b)$ $\bar{R}(X,Y)Z + \bar{R}(Y,X)Z = 0$,
$(c)$ $\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0$,
$(d)$ *The Ricci tensor $\bar{S}$ is given by* (5.9),
$(e)$ *The scalar curvature $\bar{r}$ is given by* (5.11).

Now let, $\bar{R}(X,Y)Z = 0$, then from (5.3), we have

$$\bar{\bar{R}}(X,Y)Z = \alpha g(\phi Y,Z)\phi X - \alpha g(\phi X,Z)\phi Y - \beta g(X,Z)\phi Y + \beta g(Y,Z)\phi X - \alpha\delta(\eta(X)Y - \eta(Y)X)\eta(Z) - 2\beta\delta(\eta(X)\phi Y - \eta(Y)\phi X)\eta(Z) \tag{5.12}$$

which on contracting gives

$$\bar{\bar{S}}(Y,Z) = (\alpha\psi - \beta)g(\phi Y,Z) - (\alpha - \beta\psi)g(Y,Z) - [\alpha(1-\delta(n-1)) - 2\beta\delta\psi]\eta(Y)\eta(Z) \tag{5.13}$$

and

$$\bar{\bar{r}} = -(\alpha - \beta\psi)(2n+1) + (\alpha\psi - \beta)\psi + [\alpha(1-\delta(n-1)) - 2\beta\delta\psi]. \tag{5.14}$$

Thus we have

**Theorem 7.** *In a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection $\bar{\nabla}$ with vanishing curvature tensor, the Ricci tensor $\bar{\bar{S}}$ and the scalar curvature $\bar{\bar{r}}$ are given respectively by* (5.13) *and* (5.14).

Taking inner product of (5.3) with $U$, we have

$$\bar{R}(X,Y,Z,U) = \bar{\bar{R}}(X,Y,Z,U) + \alpha g(\phi X,Z)g(\phi Y,U) - \alpha g(\phi Y,Z)g(\phi X,U) + \beta g(X,Z)g(\phi Y,U) - \beta g(Y,Z)g(\phi X,U) + \alpha\delta(\eta(X)g(Y,U) - \eta(Y)g(X,U))\eta(Z) + 2\beta\delta(\eta(X)g(\phi Y,U) - \eta(Y)g(\phi X,U))\eta(Z), \tag{5.15}$$

where $\bar{R}(X,Y,Z,U) = g(\bar{R}(X,Y)Z,U)$ and $\bar{\bar{R}}(X,Y,Z,U) = g(\bar{\bar{R}}(X,Y)Z,U)$.

Interchanging $X$ and $Y$ in (5.15), we get

$$\bar{R}(Y,X,Z,U) = \bar{\bar{R}}(Y,X,Z,U) + \alpha g(\phi Y,Z)g(\phi X,U) - \alpha g(\phi X,Z)g(\phi Y,U) \tag{5.16}$$
$$+\beta g(Y,Z)g(\phi X,U) - \beta g(X,Z)g(\phi Y,U) + \alpha\delta(\eta(Y)g(X,U) - \eta(X)g(Y,U))\eta(Z)$$
$$+2\beta\delta(\eta(Y)g(\phi X,U) - \eta(X)g(\phi Y,U))\eta(Z).$$

From (5.15) and (5.16), we get

$$\bar{R}(X,Y,Z,U) + \bar{R}(Y,X,Z,U) = 0, \tag{5.17}$$

where $\bar{\bar{R}}(X,Y,Z,U) + \bar{\bar{R}}(Y,X,Z,U) = 0$.

Again from (5.15) interchanging $Z$ and $U$, we get

$$\bar{R}(X,Y,U,Z) = \bar{\bar{R}}(X,Y,U,Z) + \alpha g(\phi X,U)g(\phi Y,Z) - \alpha g(\phi Y,U)g(\phi X,Z) \tag{5.18}$$
$$+\beta g(X,U)g(\phi Y,Z) - \beta g(Y,U)g(\phi X,Z) + \alpha\delta(\eta(X)g(Y,Z) - \eta(Y)g(X,Z))\eta(U)$$
$$+2\beta\delta(\eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z))\eta(U).$$

From (5.15) and (5.18), we get

$$\bar{R}(X,Y,Z,U) + \bar{R}(X,Y,U,Z) = 0, \tag{5.19}$$

where $\bar{\bar{R}}(X,Y,Z,U) + \bar{\bar{R}}(X,Y,U,Z) = 0$ and $\alpha = \beta = 0$.

Again from (5.15) interchanging pair of slots, we get

$$\bar{R}(Z,U,X,Y) = \bar{\bar{R}}(Z,U,X,Y) + \alpha g(\phi Z,X)g(\phi U,Y) - \alpha g(\phi U,X)g(\phi Z,Y) \tag{5.20}$$
$$+\beta g(Z,X)g(\phi U,Y) - \beta g(U,X)g(\phi Z,Y) + \alpha\delta(\eta(Z)g(U,Y) - \eta(U)g(Z,Y))\eta(X)$$
$$+2\beta\delta(\eta(Z)g(\phi U,Y) - \eta(U)g(\phi Z,Y))\eta(X).$$

From (5.15) and (5.20), we get

$$\bar{R}(X,Y,Z,U) = \bar{R}(Z,U,X,Y), \tag{5.21}$$

where $\bar{\bar{R}}(X,Y,Z,U) = \bar{\bar{R}}(Z,U,X,Y)$ and $\alpha = \beta = 0$.

Thus we have

**Theorem 8.** *The curvature tensor $\bar{R}$ of type $(0,4)$ of a quarter-symmetric non- metric connection $\bar{\nabla}$ in an indefinite trans-Sasakian manifold is*
$(a)$ *Skew symmetric in first two slots given by* $(5.17)$,
$(b)$ *Skew symmetric in last two slots given by* $(5.19)$ *and the manifold is a $\delta$-Lorentzian cosymplectic manifold if and only if $\alpha = \beta = 0$*
$(c)$ *Symmetric in pair of slots given by* $(5.21)$ *and the manifold is a $\delta$-Lorentzian cosymplectic manifold if and only if $\alpha = \beta = 0$.*

## REFERENCES

[1] A. Bejancu, $CR$-submanifolds of a Kaehler manifold I, Proc. Am. Math. Soc. 69(1978), 135-142.
[2] A. Bejancu, Geometry of $CR$-submanifolds, D. Riedel Pub. Co., 1986.
[3] A. K. Mondal and U. C. De, Quarter symmetric non-metric connection on $P$-Sasakian manifolds, ISRN Geometry (2012), Article ID: 659430, 14 pages.
[4] C. Özgür, M. Ahmad and A. Haseeb, Hypersurfaces of an almost $r$-paracontact Riemannian manifold endowed with a quarter symmetric non-metric connection, Kyungpook Math. J., 49(2009), 533-543.
[5] J. A. Oubina, New class of almost contact metric structures, Publ. Math. Debrecen 32(1985), 187-193.
[6] K. Matsumoto, On Lorentzian para-contact manifolds, Bull. of Yamagata Univ. Nat. Sci. 12(1989), 151-156.
[7] K. Yano, On semi-symmetric metric connection, Rev. Roumaine Math. Pures Appl. 15 (1970), 1579-1586.
[8] M. Ahmad, A. Haseeb, J. B. Jun and M. H. Shahid, $CR$-submanifolds and $CR$-products of a Lorentzian para-Sasakian manifold endowed with a quarter-symmetric semi-metric connection, Afr. Mat., 25(2014), 1113-1124.
[9] M. H. Shahid, $CR$-submanifolds of a trans-Sasakian manifold, Indian J. Pure Appl. Math. 22(1991), 1007-1012.
[10] M H. Shahid, $CR$-submanifolds of a trans-Sasakian manifold II, Indian J. Pure Appl. Math. 25(1994), 299-307.
[11] M. Kobayashi, $CR$-submanifold of a Sasakian manifold, Tensor, N.S., 35(1981), 297-307.
[12] R. Nivas and G. Verma, On quarter symmetric non-metric connection in a Riemannian manifold, J. Rajasthan Acad. Phys. Sci., 4(2005), 187-193.
[13] S. Golab, On semi-symmetric and quarter symmetric linear connections, Tensor, N. S., 29(1975), 249-254.
[14] S. M. Bhati, On weakly Ricci $\phi$-symmetic $\delta$-Lorentzian trans-Sasakian manifolds, Bull. Math. Anal. and Appl., 5(2013), 36-43.
[15] S. S. Pujar and V. J. Khairnar, On Lorentzian trans-Sasakian manifolds I, Int. J. of Ultra Sciences of Physical Sciences, 23(2011), 53-66
[16] U. C. De and K. De, On Lorentzian trans-Sasakian manifolds, Commun. Fac. Sci. Univ. Ank. Series A1, 62(2013), 37-51.

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