ON CONTACT CR-SUBMANIFOLDS OF A $\delta$-LORENTZIAN TRANS-SASAKIAN MANIFOLD

ABDUL HASEEB*, MOBIN AHMAD, AND MOHD. DANISH SIDDIQI

ABSTRACT. The purpose of the present paper is to study contact CR-submanifolds of a $\delta$-Lorentzian trans-Sasakian manifold with a quarter symmetric non-metric connection. Also, we investigate the relation between the curvature tensor of $\delta$-Lorentzian trans-Sasakian manifolds and related results with respect to the quarter symmetric non-metric connection and the Levi-Civita connection.

1. INTRODUCTION

The notion of CR-submanifolds of a Kaehler manifold was introduced by A. Bejancu [1]. Later on, CR-submanifolds of a Sasakian manifold were studied by M. Kobayashi [11]. K. Matsumoto introduced the idea of a Lorentzian para-Sasakian structure and studied several of its properties [6]. J. A. Oubina defined and studied a new class of almost contact metric manifold known as trans-Sasakian manifold, which includes $\alpha$-Sasakian, $\beta$-Kenmotsu and cosymplectic structures [5]. M. H. Shahid have studied CR-submanifolds of a trans-Sasakian manifold ([9],[10]).

Recently, Lorentzian trans-Sasakian manifolds were studied by the authors (see [15],[16]) and they obtained several interesting results on the manifold. In [14], S. M., Bhati introduced the notion of weakly Ricci $\phi$-symmetric $\delta$-Lorentzian trans-Sasakian manifolds and studied characteristic properties of locally $\phi$-Ricci symmetric and $\phi$-recurrent spaces.

A linear connection $\overline{\nabla}$ in a Riemannian manifold $\overline{M}$ is said to be a quarter-symmetric connection [13] if its torsion tensor $T$ of the connection $\overline{\nabla}$

$$T(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - \overline{\nabla}_{[X,Y]}$$

satisfies

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where $\eta$ is a 1-form and $\phi$ is a $(1,1)$ tensor field. If moreover, a quarter-symmetric connection $\overline{\nabla}$ satisfies the condition

$$(\overline{\nabla}_X g)(Y, Z) = -\eta(Y)g(\phi X, Z) - \eta(Z)g(Y, \phi X)$$
for all $X, Y, Z \in T(\tilde{M})$, where $T(\tilde{M})$ is the Lie algebra of vector fields of the manifold $\tilde{M}$, then $\nabla$ is said to be a quarter-symmetric non-metric connection. If we put $\phi X = X$ and $\phi Y = Y$, then the quarter-symmetric non-metric connection reduces to the semi-symmetric non-metric connection [7]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection. A quarter-symmetric non-metric connection have been studied by various authors such as A. K. Mondal and U. C. De [3], R. Nivas and G. Verma [12] and C. Özgür, M. Ahmad and A. Haseeb [4].

The paper is organized as follows : In Section 2, we give a brief introduction of $\delta$-Lorentzian trans-Sasakian manifolds. Section 3, deals with the study of CR-submanifolds of a $\delta$-Lorentzian trans-Sasakian manifold with a quarter-symmetric non-metric connection. In Section 4, we obtain the integrability conditions for the distributions $D$ and $D^\perp$ on CR-submanifolds of a $\delta$-Lorentzian trans-Sasakian manifold with a quarter-symmetric non-metric connection. In Section 5, we investigate the relation between the curvature tensor of $\delta$-Lorentzian trans-Sasakian manifolds and related results with a quarter symmetric non-metric connection and the Levi-Civita connection.

2. $\delta$-LORENTZIAN TRANS-SASAKIAN MANIFOLDS

Let $\tilde{M}$ be an almost contact metric manifold of dimension $(2n + 1)$ with an almost contact metric structure $(\phi, \xi, \eta, g)$ consisting of a $(1, 1)$ tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ satisfying [15]

$$\phi^2 X = X + \eta(X)\xi, \quad (2.1)$$

$$\eta(\xi) = -1, \quad \phi \circ \xi = 0, \quad \eta \circ \phi = 0. \quad (2.2)$$

An almost contact metric manifold $\tilde{M}$ is called a $\delta$-almost contact metric manifold if

$$g(\xi, \xi) = -\delta, \quad (2.3)$$

$$\eta(X) = \delta g(X, \xi), \quad (2.4)$$

$$g(\phi X, \phi Y) = g(X, Y) + \delta \eta(X)\eta(Y) \quad (2.5)$$

for all vector fields $X, Y \in T(\tilde{M})$, where $\delta^2 = 1$ so that $\delta = \pm 1$.

A $\delta$-almost contact metric manifold is called a $\delta$-Lorentzian trans-Sasakian manifold if

$$(\tilde{\nabla}_X \phi) Y = \alpha [g(\phi X, Y)\xi - \delta \eta(Y)X] + \beta [g(\phi X, Y)\xi - \delta \eta(Y)\phi X] \quad (2.6)$$

for some smooth functions $\alpha$ and $\beta$ on $\tilde{M}$ and $\delta = \pm 1$.

If $\delta = 1$, then the $\delta$-Lorentzian trans-Sasakian manifold is usual Lorentzian trans-Sasakian manifold of type $(\alpha, \beta)$. In particular, $\delta$-Lorentzian trans-Sasakian structure of the type $(\alpha, \beta)$ is $\delta$-Lorentzian Sasakian, $\delta$-Lorentzian Kenmotsu and $\delta$-Lorentzian cosymplectic manifolds according as $\beta = 0, \alpha = 1$; $\alpha = 0, \beta = 1$ and $\alpha = \beta = 0$ respectively.

For a $\delta$-Lorentzian trans-Sasakian manifold, we also have

$$(\tilde{\nabla}_X \eta) Y = \alpha g(\phi X, Y) + \beta [g(X, Y) + \delta \eta(X)\eta(Y)], \quad (2.7)$$

$$\tilde{\nabla}_X \xi = -\alpha \delta \phi X - \beta \delta X - \beta \delta \eta(X)\xi, \quad (2.8)$$
where $\nabla$ denotes the operator of the covariant differentiation with respect to the Lorentzian metric $g$.

3. CR-SUBMANIFOLDS OF $\delta$-LORENTZIAN TRANS-SASAKIAN MANIFOLDS

**Definition.** An $m$-dimensional Riemannian submanifold $M$ of a $\delta$-Lorentzian trans-Sasakian manifold $\tilde{M}$ is called a CR-submanifold if $\xi$ is tangent to $M$ and there exists on $M$ a differentiable distribution $D : x \rightarrow D_x \subset T_x(M)$ such that

(i) the distribution $D_x$ is invariant under $\phi$, i.e., $\phi D_x \subset D_x$ for each $x \in M$;

(ii) the orthogonal complementary distribution $D^\perp : x \rightarrow D_x^\perp \subset T_x(M)$ of the distribution $D$ on $M$ is anti-invariant under $\phi$, i.e., $\phi D_x^\perp(M) \subset T_x^\perp(M)$ for all $x \in M$, where $T_x(M)$ and $T_x^\perp(M)$ are tangent space and normal space of $M$ at $x \in M$ respectively.

If $\dim D_x^\perp = 0$ (resp., $\dim D_x = 0$), then CR-submanifold is called an invariant (resp., anti-invariant). The distribution $D$ (resp., $D^\perp$) is called the horizontal (resp., vertical) distribution. The pair $(D, D^\perp)$ is called $\xi$-horizontal (resp., $\xi$-vertical) if $\xi \in D_x$ (resp., $\xi \in D_x^\perp$) for $x \in M$.

Any vector field $X$ tangent to $M$ can be decomposed as [2,8]

$$X = PX + QX,$$

where $PX$ and $QX$ belong to the distribution $D$ and $D^\perp$ respectively.

For any vector field $N$ normal to $M$, we put

$$\phi N = BN + CN,$$

where $BN$ (resp., $CN$) is the tangential (resp., normal) component of $\phi N$.

Now, we remark that owing to the existence of the 1-form $\eta$, we can define a quarter symmetric non-metric connection $\nabla$ in an almost contact metric manifold by

$$\nabla_X Y = \tilde{\nabla}_X Y + \eta(Y)\phi X$$

for any $X, Y \in T(M)$ and $\tilde{\nabla}$ is the induced connection with respect to $g$ on $M$.

From (2.6), (2.8) and (3.3), we obtain

$$\nabla_X \phi Y = \alpha[g(X, Y)\xi - \delta\eta(Y)X] + \beta [g(\phi X, Y)\xi - \delta\eta(Y)\phi X]$$

$$-\eta(Y)X - \eta(X)\eta(Y)\xi,$$

$$\nabla_X \xi = -\alpha \phi X - \beta \delta X - \beta \delta\eta(X)\xi - \phi X.$$  

For a contact CR-submanifold $M$ of a $\delta$-Lorentzian trans-Sasakian manifold $\tilde{M}$ with a quarter symmetric non-metric connection, the Gauss and Weingarten formulae are given respectively by

$$\nabla_X Y = \nabla_X Y + h(X, Y),$$

$$\nabla_X N = -A_N X + \eta(N)\phi X + \nabla_X^\perp N$$

for $X, Y \in T(M)$, $N \in T^\perp M$, $h$ (resp., $A_N$) is the second fundamental (resp; tensor) form of $M$ in $\tilde{M}$, $\tilde{\nabla}$ and $\nabla$ are the Riemannian and induced Riemannian connections respectively in $\tilde{M}$.
Lemma 1. Let $M$ be a contact CR-submanifold of a $\delta$-Lorentzian trans-Sasakian manifold $\tilde{M}$ with a quarter symmetric non-metric connection. Then

$$P\nabla_X\phi Y - PA_{\phi QY}X = \phi P\nabla_XY + \alpha g(X,Y)P\xi + \beta g(\phi X,Y)P\xi$$

(4.1)

$$-\eta(Y)PX - \alpha \delta \eta(Y)PX - \beta \delta \eta(Y)\phi PX - \eta(X)\eta(Y)P\xi,$$

$$Q\nabla_X\phi PY - QA_{\phi QY}X = \alpha g(X,Y)Q\xi + \beta g(\phi X,Y)Q\xi - \alpha \delta \eta(Y)QX$$

(4.2)

$$-\eta(Y)QX + Bh(X,Y) - \eta(X)\eta(Y)Q\xi,$$

$$h(X,\phi PY) + \nabla^\perp_X\phi QY = \phi Q\nabla_XY + Ch(X,Y) - \beta \delta \eta(Y)\phi QX$$

(4.3)

for all $X, Y \in TM$.

Proof. By the direct covariant differentiation of $\phi Y$, we have

$$\nabla_X\phi Y = (\nabla_X\phi)Y + \phi(\nabla_XY).$$

By using (3.1), (3.4), (3.6) and (3.7), we get

$$\nabla_X\phi PY + h(X,\phi PY) - A_{\phi QY}X + \nabla^\perp_X\phi QY = \alpha [g(X,Y)\zeta - \delta \eta(Y)X]$$

$$+ \beta [g(\phi X,Y)\zeta - \delta \eta(Y)\phi X] - \eta(Y)X - \eta(X)\eta(Y)\zeta + \phi \nabla_XY + \phi h(X,Y),$$

which by using (3.1) and (3.2) takes the form

$$P\nabla_X\phi PY + Q\nabla_X\phi PY - PA_{\phi QY}X - QA_{\phi QY}X + h(X,\phi PY) + \nabla^\perp_X\phi QY$$

$$= \phi P\nabla_XY + \phi Q\nabla_XY + Bh(X,Y) + Ch(X,Y) + \alpha g(X,Y)P\xi + \alpha g(X,Y)Q\xi$$

$$- \alpha \delta \eta(Y)PX - \alpha \delta \eta(Y)QX - \eta(Y)PX - \eta(Y)QX + \beta g(\phi X,Y)P\xi + \beta g(\phi X,Y)Q\xi$$

$$- \beta \delta \eta(Y)\phi PX - \beta \delta \eta(Y)\phi QX - \eta(X)\eta(Y)P\xi - \eta(X)\eta(Y)Q\xi.$$

By equating horizontal, vertical and normal components, we obtain equations (4.1)-(4.3). □

Now, we calculate the Nijenhuis tensor $N(X,Y)$ on a $\delta$-Lorentzian trans-Sasakian manifold $\tilde{M}$ with a quarter symmetric non-metric connection. For this, first we prove the following lemma:

Lemma 2. Let $M$ be a contact CR-submanifold of a $\delta$-Lorentzian trans-Sasakian manifold $\tilde{M}$ with a quarter symmetric non-metric connection. Then we have

$$(\nabla_{\phi X}\phi)Y = \alpha [g(\phi X,Y)\zeta - \delta \eta(Y)\phi X] + \beta [g(X,Y)\zeta + \eta(X)\eta(Y)\zeta] - \beta \delta \eta(Y)X$$

$$- \eta(Y)\phi X - \beta \delta \eta(X)\eta(Y)\zeta$$

for all $X, Y \in TM$. 

and $M$ and $\nabla^\perp$ is the operator of the normal connection. Moreover, $h$ satisfies the following condition

$$g(ANX,Y) = g(h(X,Y),N).$$

(3.8)
Using (4.7), we have
\[ \phi_X^{X} \text{ for any } X \]

The torsion tensor \( \mathcal{T} \) for any \( X \) follows (4.9) and (4.10), we find
\[ \mathcal{T}_{\phi X} = \alpha [g(X, Y)\xi - \delta(\eta(Y)X] + \beta [g(X, Y)\xi - \delta(\eta(Y)X] \]

proof. For \( \phi_X^{X} \) manifold \( \mathcal{M} \) with a quarter symmetric non-metric connection, we have
\[ \phi_X^{X} = \alpha [g(X, Y)\xi - \delta(\eta(Y)X] + \beta [g(X, Y)\xi + \eta(\delta)X - \beta\delta(\eta(X)\xi] \]

for \( X, Y \in T\mathcal{M} \).

On a \( \delta \)-Lorentzian trans-Sasakian manifold \( \mathcal{M} \) with a quarter symmetric non-metric connection, the Nijenhuis tensor is given by
\[ N(X, Y) = (\nabla_X \phi)Y - (\nabla_Y \phi)X - \phi(\nabla_X \phi)Y + \phi(\nabla_Y \phi)X \]

for any \( X, Y \in T\mathcal{M} \).

Using (4.5) in (4.6), we get
\[ N(X, Y) = -(\alpha\delta + 1)\eta(\delta)Y\phi X + (\alpha\delta + 1)\eta(\delta)Y\phi X - \beta\delta(\eta(X)\xi + \beta\delta(\eta(X)\xi) \]

for any \( X, Y \in T\mathcal{M} \).

Now we prove the following proposition:

**Proposition 3.** Let \( M \) be a \( \xi \)-vertical contact CR-submanifold of a \( \delta \)-Lorentzian trans-Sasakian manifold \( \mathcal{M} \) with a quarter symmetric non-metric connection. Then the distribution \( D \) is integrable if the following conditions are satisfied:
\[ S(X, Y) \in D, \quad h(X, \phi Y) = h(Y, \phi X) \]

for any \( X, Y \in D \).

proof. The torsion tensor \( S(X, Y) \) of the structure \( (\phi, \xi, \eta, g, \delta) \) is given by
\[ S(X, Y) = N(X, Y) + 2\delta l(\eta(X)\xi) = N(X, Y) + 2g(\phi X, Y)\xi \]

Using (4.7), we have
\[ S(X, Y) = 2g(\phi X, Y)\xi - (\alpha\delta + 1)\eta(\delta)Y\phi X + (\alpha\delta + 1)\eta(\delta)Y\phi X - \beta\delta(\eta(X)\xi + \beta\delta(\eta(X)\xi) \]

for any \( X, Y \in TM \).

By covariant differentiation of \( \phi X \) with respect to \( Y \) and using (2.1) and (3.6), we have
\[ \phi(\nabla_Y \phi)X = \phi \nabla_Y \phi X + \phi h(Y, \phi X) - \nabla_Y X - \eta(\nabla_Y X)\xi - h(X, Y) \]

From (4.9) and (4.10), we find
\[ S(X, Y) = 2g(\phi X, Y)\xi - \phi(\nabla_X \phi Y - \nabla_Y \phi X) - \phi(h(X, \phi Y) - h(Y, \phi X)) + (\nabla_X Y - \nabla_Y X) + \eta(\nabla_X Y - \nabla_Y X)\xi \]
for any \( X, Y \in D \) and \( \xi \in D_\perp \).
If \( S(X, Y) \in D \) for \( X, Y \in D \), then the last equation reduces to
\[
2g(\phi X, Y)\xi - \phi h(X, \phi Y) + \phi h(Y, \phi X) = 0.
\]
On equating vertical and normal components, we get
\[
h(X, \phi Y) = h(Y, \phi X)
\]
for any \( X, Y \in D \). \( \square \)

**Lemma 4.** Let \( M \) be a \( \xi \)-vertical contact CR-submanifold of a \( \delta \)-Lorentzian trans-Sasakian manifold \( \tilde{M} \) with a quarter symmetric non-metric connection. Then

\[
\phi P[W, Z] = A_{\phi W}Z - A_{\phi Z}W + (\alpha \delta + 1)[\eta(Z)W - \eta(W)Z] \quad (4.11)
\]

\[
+ \beta \delta[\eta(Z)\phi W - \eta(W)\phi Z + \eta(W)\phi QZ - \eta(Z)\phi QW]
\]

for any \( W, Z \in D_\perp \).

**Proof.** From the equation (4.3), we have
\[
\nabla_{\overline{W}}\phi Z = \phi Q\nabla_{\overline{W}}Z + Ch(W, Z) - \beta \delta \eta(Z)\phi QW \quad (4.12)
\]
for any \( W, Z \in D_\perp \). Now by covariant differentiation of \( \phi Z \), we have
\[
\nabla_{\overline{W}}\phi Z = (\nabla_{\overline{W}}\phi)Z + \phi \nabla_{\overline{W}}Z. \quad (4.13)
\]
By using (3.4), (3.6) and (3.7) in (4.13), we get
\[
-A_{\phi Z}W + \nabla_{\overline{W}}\phi Z + \eta(\phi Z)W = \alpha [g(W, Z)\xi - \delta \eta(Z)W] + \beta [g(\phi W, Z)\xi - \delta \eta(Z)\phi W]
\]
\[
-\eta(Z)W - \eta(W)\eta(Z)\xi + \phi (\nabla_{\overline{W}}Z + h(W, Z))
\]
for \( W, Z \in D_\perp \). In view of (3.1) and (3.2), the last equation can be written as
\[
\nabla_{\overline{W}}\phi Z = A_{\phi Z}W + \alpha [g(W, Z)\xi - \delta \eta(Z)W] + \beta [g(\phi W, Z)\xi - \delta \eta(Z)\phi W]
\]
\[
-\eta(Z)W - \eta(W)\eta(Z)\xi + \phi P\nabla_{\overline{W}}Z + \phi Q\nabla_{\overline{W}}Z + Bh(W, Z) + Ch(W, Z).
\]
From (4.12) and (4.14), we have
\[
\phi P\nabla_{\overline{W}}Z = -A_{\phi Z}W - \alpha [g(W, Z)\xi - \delta \eta(Z)W] - \beta [g(\phi W, Z)\xi - \delta \eta(Z)\phi W] + \eta(W)Z
\]
\[
+ \eta(W)\eta(Z)\xi - Bh(W, Z) - \beta \delta \eta(Z)\phi QW
\]
which by interchanging \( W \) and \( Y \) takes the form
\[
\phi P\nabla_{\overline{Z}}W = -A_{\phi W}Z - \alpha [g(W, Z)\xi - \delta \eta(W)Z] - \beta [g(\phi W, Z)\xi - \delta \eta(W)\phi Z] + \eta(W)Z
\]
\[
+ \eta(W)\eta(Z)\xi - Bh(W, Z) - \beta \delta \eta(W)\phi QZ.
\]
From the last two equations, we obtain
\[
\phi P[W, Z] = A_{\phi W}Z - A_{\phi Z}W + (\alpha \delta + 1)[\eta(Z)W - \eta(W)Z]
\]
\[
+ \beta \delta[\eta(Z)\phi W - \eta(W)\phi Z + \eta(W)\phi QZ - \eta(Z)\phi QW]
\]
for any \( W, Z \in D_\perp \). \( \square \)

Hence we can state the following theorem:
Theorem 5. Let $\bar{M}$ be a $\xi$-vertical contact CR-submanifold of a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then the distribution $D_\perp$ is integrable if and only if

$$A_{\phi X}Y = A_{\phi Y}X,$$

provided $a\delta = -1$ for any $X,Y \in D_\perp$.

5. CURVATURE TENSOR OF $\delta$-LORENTZIAN TRANS-SASAKIAN MANIFOLDS WITH A QUARTER SYMMETRIC NON-METRIC CONNECTION.

We define the curvature tensor of a $\delta$-Lorentzian trans-Sasakian manifold $\bar{M}$ with a quarter symmetric non-metric connection $\nabla$ by

$$\bar{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,$$  (5.1)

where $\bar{R}$ is the curvature tensor with respect to a quarter symmetric non-metric connection $\nabla$. By using (2.2) and (3.3) in (5.1), we have

$$\bar{R}(X,Y)Z = \bar{R}(X,Y)Z + (\bar{\nabla}_X \eta)(Z)\phi Y + \eta(Z)(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \eta)(Z)\phi X - \eta(Z)(\bar{\nabla}_Y \phi)X$$  (5.2)

which on using (2.6) and (2.7) takes the form

$$\bar{R}(X,Y)Z = \bar{R}(X,Y)Z + \alpha g(\phi X, Z)\phi Y - \alpha g(\phi Y, Z)\phi X + \beta g(X, Z)\phi Y$$  (5.3)

$$-\beta g(Y, Z)\phi X + a\delta(2\eta(X)Y - \eta(Y)X)\eta(Z) + 2\beta\delta(\eta(X)\phi Y - \eta(Y)\phi X)\eta(Z),$$

where $X,Y,Z \in T(\bar{M})$ and $\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z$ is the curvature tensor with respect to the connection $\bar{\nabla}$.

Interchanging $X$ and $Y$ in (5.3), we have

$$\bar{R}(Y,X)Z = \bar{R}(Y,X)Z + \alpha g(\phi Y, Z)\phi X - \alpha g(\phi X, Z)\phi Y + \beta g(Y, Z)\phi X$$  (5.4)

$$-\beta g(X, Z)\phi Y + a\delta(2\eta(Y)X - \eta(X)Y)\eta(Z) + 2\beta\delta(\eta(Y)\phi X - \eta(X)\phi Y)\eta(Z).$$

Adding (5.3) and (5.4) and using the fact that $\bar{R}(X,Y)Z + \bar{R}(Y,X)Z = 0$, we get

$$\bar{R}(X,Y)Z + \bar{R}(Y,X)Z = 0.$$  (5.5)

From (5.3) we can write two more following equations by the cyclic permutations of $X$, $Y$ and $Z$, that are

$$\bar{R}(Y,Z)X = \bar{R}(Y,Z)X + \alpha g(\phi Y, Z)\phi Z - \alpha g(\phi Z, X)\phi Y + \beta g(Y, Z)\phi Z$$  (5.6)

$$-\beta g(Z, X)\phi Y + a\delta(\eta(Y)Z - \eta(Z)Y)\eta(X) + 2\beta\delta(\eta(Y)\phi Z - \eta(Z)\phi Y)\eta(X)$$

and

$$\bar{R}(Z,X)Y = \bar{R}(Z,X)Y + \alpha g(\phi Z, X)\phi Z - \alpha g(\phi X, Y)\phi Z + \beta g(Z, Y)\phi Z$$  (5.7)

$$-\beta g(X, Y)\phi Z + a\delta(\eta(Z)X - \eta(X)Z)\eta(Y) + 2\beta\delta(\eta(Z)\phi Z - \eta(X)\phi Z)\eta(Y).$$

Adding (5.3), (5.6) and (5.7) and using the Bianchi’s first identity $\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0$ with respect to $\bar{\nabla}$, we get

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0.$$  (5.8)
where $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}$ are the Ricci tensors of the connections $\tilde{\nabla}$ and $\tilde{\nabla}$, respectively in $\tilde{M}$; $\psi = \text{trace}\phi$.

This gives

$$\tilde{Q}Y = \tilde{Q}Y + (\alpha - \beta\psi)Y - (\alpha\psi - \beta)\phi Y + [\alpha(1 - \delta(n - 1)) - 2\beta\delta\psi]\eta(Y)\eta(Z),$$

(5.10)

where $\tilde{Q}$ and $\tilde{Q}$ are the Ricci operators of the connections $\tilde{\nabla}$ and $\tilde{\nabla}$, respectively in $\tilde{M}$ and $g(\tilde{Q}X, Y) = \tilde{S}(X, Y)$.

Contracting again $Y$ and $Z$ in (5.9), we get

$$\bar{\nabla} = \bar{\nabla} + (\alpha - \beta\psi)(2n + 1) - (\alpha\psi - \beta)\psi - [\alpha(1 - \delta(n - 1)) - 2\beta\delta\psi],$$

(5.11)

where $\bar{\nabla}$ and $\tilde{\nabla}$ are the scalar curvatures of the connections $\tilde{\nabla}$ and $\tilde{\nabla}$, respectively in $\tilde{M}$.

Thus we have the following theorem:

**Theorem 6.** For a $\delta$-Lorentzian trans-Sasakian manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{\nabla}$

(a) The curvature tensor $\tilde{R}$ is given by (5.3).

(b) $\tilde{R}(X, Y)Z + \tilde{R}(Y, X)Z = 0$.

(c) $\tilde{R}(X, Y)Z + \tilde{R}(Y, Z)X + \tilde{R}(Z, X)Y = 0$.

(d) The Ricci tensor $\tilde{S}$ is given by (5.9).

(e) The scalar curvature $\tilde{\nabla}$ is given by (5.11).

Now let, $\tilde{R}(X, Y)Z = 0$, then from (5.3), we have

$$\tilde{R}(X, Y)Z = \alpha g(\phi Y, Z)\phi X - \alpha g(\phi X, Z)\phi Y - \beta g(Y, Z)\phi X + \beta g(Y, Z)\phi X$$

(5.12)

$$-\alpha\delta(\eta(X)Y - \eta(Y)X)\eta(Z) - 2\beta\delta(\eta(X)\phi Y - \eta(Y)\phi X)\eta(Z)$$

which on contracting gives

$$\tilde{S}(Y, Z) = (\alpha\psi - \beta)\phi Y, Z - (\alpha - \beta\psi)g(Y, Z)$$

(5.13)

$$- [\alpha(1 - \delta(n - 1)) - 2\beta\delta\psi]\eta(Y)\eta(Z)$$

and

$$\bar{\nabla} = -(\alpha - \beta\psi)(2n + 1) + (\alpha\psi - \beta)\psi + [\alpha(1 - \delta(n - 1)) - 2\beta\delta\psi].$$

(5.14)

Thus we have

**Theorem 7.** In a $\delta$-Lorentzian trans-Sasakian manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{\nabla}$ with vanishing curvature tensor, the Ricci tensor $\tilde{S}$ and the scalar curvature $\tilde{\nabla}$ are given respectively by (5.13) and (5.14).

Taking inner product of (5.3) with $U$, we have

$$\tilde{R}(X, Y, Z, U) = \tilde{R}(X, Y, Z, U) + \alpha g(\phi X, Z)\phi Y, U - \alpha g(\phi Y, U)\phi X, U$$

(5.15)

$$+ \beta g(Y, Z)\phi Y, U - \beta g(Y, Z)\phi Y, U + \alpha\delta(\eta(X)g(Y, U) - \eta(Y)g(X, U))\eta(Z)$$

$$+ 2\beta\delta(\eta(X)g(\phi Y, U) - \eta(Y)g(\phi X, U))\eta(Z),$$

where $\tilde{R}(X, Y, Z, U) = g(\tilde{R}(X, Y)Z, U)$ and $\tilde{R}(X, Y, Z, U) = g(\tilde{R}(X, Y)Z, U)$. 

Interchanging $X$ and $Y$ in (5.15), we get
\[
\bar{R}(Y, X, Z, U) = \bar{R}(Y, X, Z, U) + \alpha g(\phi Y, Z)g(\phi X, U) - \alpha g(\phi Y, U)g(\phi X, Z) \quad (5.16)
\]
\[
+ \beta g(Y, Z)g(\phi X, U) - \beta g(X, Z)g(\phi Y, U) + \alpha \delta(\eta(Y)g(X, U) - \eta(X)g(Y, U))\eta(Z) \\
+ 2\beta \delta(\eta(Y)g(\phi X, U) - \eta(X)g(\phi Y, U))\eta(Z).
\]

From (5.15) and (5.16), we get
\[
\bar{R}(X, Y, Z, U) + \bar{R}(Y, X, Z, U) = 0, \quad (5.17)
\]
where $\bar{R}(X, Y, Z, U) + \bar{R}(Y, X, Z, U) = 0$.

Again from (5.15) interchanging $Z$ and $U$, we get
\[
\bar{R}(X, Y, U, Z) = \bar{R}(X, Y, U, Z) + \alpha g(\phi X, U)g(\phi Y, Z) - \alpha g(\phi Y, U)g(\phi X, Z) \quad (5.18)
\]
\[
+ \beta g(X, U)g(\phi Y, Z) - \beta g(Y, U)g(\phi X, Z) + \alpha \delta(\eta(X)g(Y, Z) - \eta(Y)g(X, Z))\eta(U) \\
+ 2\beta \delta(\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z))\eta(U).
\]

From (5.15) and (5.18), we get
\[
\bar{R}(X, Y, Z, U) + \bar{R}(X, Y, U, Z) = 0, \quad (5.19)
\]
where $\bar{R}(X, Y, Z, U) + \bar{R}(X, Y, U, Z) = 0$ and $\alpha = \beta = 0$.

Again from (5.15) interchanging pair of slots, we get
\[
\bar{R}(Z, U, X, Y) = \bar{R}(Z, U, X, Y) + \alpha g(\phi Z, X)g(\phi U, Y) - \alpha g(\phi U, X)g(\phi Z, Y) \quad (5.20)
\]
\[
+ \beta g(Z, X)g(\phi U, Y) - \beta g(U, X)g(\phi Z, Y) + \alpha \delta(\eta(Z)g(U, Y) - \eta(U)g(Z, Y))\eta(X) \\
+ 2\beta \delta(\eta(Z)g(\phi U, Y) - \eta(U)g(\phi Z, Y))\eta(X).
\]

From (5.15) and (5.20), we get
\[
\bar{R}(X, Y, Z, U) = \bar{R}(Z, U, X, Y), \quad (5.21)
\]
where $\bar{R}(X, Y, Z, U) = \bar{R}(Z, U, X, Y)$ and $\alpha = \beta = 0$.

Thus we have

**Theorem 8.** The curvature tensor $\bar{R}$ of type $(0, 4)$ of a quarter-symmetric non-metric connection $\nabla$ in an indefinite trans-Sasakian manifold is

(a) Skew symmetric in first two slots given by (5.17),

(b) Skew symmetric in last two slots given by (5.19) and the manifold is a $\delta$-Lorentzian cosymplectic manifold if and only if $\alpha = \beta = 0$

(c) Symmetric in pair of slots given by (5.21) and the manifold is a $\delta$-Lorentzian cosymplectic manifold if and only if $\alpha = \beta = 0$. 

REFERENCES


DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, JAZAN UNIVERSITY, JAZAN, KINGDOM OF SAUDI ARABIA.
E-mail address: malikhaseeb80@gmail.com, haseeb@jazanu.edu.sa

DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, JAZAN UNIVERSITY, JAZAN, KINGDOM OF SAUDI ARABIA.
E-mail address: mobinahmad@rediffmail.com

DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, JAZAN UNIVERSITY, JAZAN, KINGDOM OF SAUDI ARABIA.
E-mail address: anallintegral@gmail.com