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SPLITTING OF LIGHTLIKE SUBMANIFOLDS OF PSEUDO-RIEMANNIAN POISSON MANIFOLDS

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ABSTRACT. In this paper we study the geometry of lightlike symplectic foliation on pseudo-Riemannian Poisson manifold. We give a decomposition of cotangent bundle, conformally to image and kernel of Poisson tensor. We prove that characteristic submanifold is totally geodesic under the vanishing condition of lightlike transversal connection. We also show that such submanifold, the coisotrope case is a product of a non degenerated manifold and a simply connex Lie group.

1. INTRODUCTION

In the present paper we continue the investigation of the geometry of lightlike symplectic foliation of pseudo-Riemaniann Poisson manifold, which was introduced in [7]. In the latter paper, the author shows that such submanifold admits a complex structure and its induced connection is metric. In [8], the class of lightlike submanifold of symplectic manifolds, which is Kaehlerian, is introduced.

The notion of compatibility between a Poisson structure and a pseudo-Riemannian metric was introduced by M. Boucetta in [3], and he therein gave proprieties of the non degenerate symplectic leaves of such manifold. He also shows that the leaf is Kaelher. A lot of works were done on pseudo-Lie algebra based on results obtained by Boucetta.

The growing importance of Poisson manifold in mechanic quantization and in an hamiltonian formulation motivates this study (see [2]). Then in this paper we show the way to set a link between lightlike submanifolds theory and a mechanic quantization.

Let $(\overline{M}, \overline{g}, \pi)$ be a Poisson pseudo-Riemannian manifold and (M, π, g) be the lightlike symplectic leaf where the degenerate metric g is the restriction of $\overline{g}$ (or $<>$) in M. In this paper, we give split of cotangent bundle TM^*, we compute geometric elements of lightlike submanifolds and we show that (M, π, g) is totally geodesic under some conditions. Finally, we prove that the coisotrope case is a product.

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The present paper is organized as follow. Section one introduces the topic. Facts in pseudo-Riemannian Poisson manifolds are treated in section two. Results on lightlike submanifolds of Poisson manifold in section three and others results (essentials results) of the paper are presented in section four. Then in section five, some examples were given.

2. SOME FACTS IN PSEUDO-RIEMANNIAN POISSON MANIFOLDS

The basic notions of Poisson manifolds given in this section are from Vaisman's monograph [9]. Let $\overline{M}$ be a smooth manifold, $C^\infty(\overline{M})$ the space of smooth functions, $\chi^q(\overline{M})$ be the space of multivectors field of degre q, $\overline{g}$ the pseudo-Riemannian metric and π the Poisson tensor.

Let $(\overline{M}, \pi)$ be a Poisson manifold endowed with pseudo-Riemannian metric $\overline{g}$. In [3, 4] based on the notion of contravariant connection, M. Boucetta introduced and studied a notion of compatibility between a Poisson structure and the pseudo-Riemannian metric. This notion is introduced by Vaisman [9] and developped in geometry approach by Fernandes [6]. Let $(\overline{M}, \pi)$ be a Poisson manifold and $<,>$ the pseudo-Riemannian metric on $T\overline{M}$. We generalize this metric to $T^*\overline{M}$ by

$$< \alpha, \beta >=< \#_g\alpha, \#_g\beta > \quad \text{for all } \alpha, \beta \in T^*\overline{M} \tag{1}$$

where $\#_{\overline{g}} : T^*\overline{M} \longrightarrow T\overline{M}$ is the isomorphism associated with pseudo-Riemannian metric. Let D be the metric contravariant connection associate to $(\pi, <,>)$ defined in [3, 4] such that:

(1) the metric $<,>$ is parallel with respect to D i.e.,

$$\#_\pi(\alpha). < \beta, \gamma >=< D_\alpha\beta, \gamma > + < \beta, D_\alpha\gamma >; \tag{2}$$

(2) D is torsion-free. One can define D using Koszul formula

$$\begin{aligned} 2 < D_\alpha\beta, \gamma >= \#_\pi(\alpha) < \beta, \gamma > +\#_\pi(\beta) < \alpha, \gamma > -\#_\pi(\gamma) < \alpha, \beta > + \\ + < [\gamma, \alpha]_\pi, \beta > + < [\gamma, \beta]_\pi, \alpha > + < [\alpha, \beta]_\pi, \gamma > . \end{aligned} \tag{3}$$

where $\#_\pi = \pi : T^*\overline{M} \longrightarrow T\overline{M}$ such that $\alpha, \beta \in T^*\overline{M}$, $\beta(\#_\pi\alpha) = \beta(\pi(\alpha)) = \pi(\alpha, \beta)$.

M. Boucetta [3, 4] used contravariant connection due to its compatibility with the Levi's civita connection on pseudo-Riemannian manifold, which will set the Poisson tensor to have a constant rank. However, most interesting Poisson sutructures are degenerated.

Remark. Since D is torsion free, one can obtain (see, [4])

$$0 = -[\pi, \pi]_S(\alpha, \beta, \gamma) = D\pi(\alpha, \beta, \gamma) + D\pi(\beta, \gamma, \alpha) + D\pi(\gamma, \alpha, \beta) \tag{4}$$

$$D\pi(\gamma, \alpha, \beta) = -d\gamma(\pi(\alpha), \pi(\beta)) - \pi(D_\alpha\gamma, \beta) - \pi(\alpha, D_\beta\gamma) \tag{5}$$

$$\pi(D_\alpha\beta) - \pi(D_\beta\alpha) = [\pi(\alpha), \pi(\beta)] \tag{6}$$

Proposition 1. *[4].*
Let $(\overline{M}, \pi, <,>)$ be a Poisson manifold with a pseudo-Riemannian metric on $T^\star\overline{M}$ and let D be the Levi-Civita contravariant connection associated with the couple $(\pi, <,>)$. The following assertions are equivalent:

(1) *The triplet $(\overline{M}, \pi, <,>)$ is a pseudo-Riemannian Poisson manifold.*
(2) *For any $\alpha, \beta \in \Omega^1(\overline{M})$ and any $f \in C^\infty(\overline{M})$,*

$$\pi(D_\alpha df, \beta) + \pi(\alpha, D_\beta df) = 0. \tag{7}$$

(3) *For any* $\alpha, \beta, \gamma \in \Omega^1(\overline{M})$.

$$d\gamma(\pi(\alpha), \pi(\beta)) + \pi(D_\alpha \gamma, \beta) + \pi(\alpha, D_\beta \gamma) = 0. \quad (8)$$

Let $(\overline{M}, \overline{g}, \pi)$ be a pseudo-Riemannian Poisson manifold. Let U be an open set of $\overline{M}$ and

$$\ker \pi_{|U} = \{\alpha \in \Omega^1(U),\ \pi(\alpha) = \#_\pi \alpha = 0\}$$

be a kernel set of π in $T^*\overline{M}$. The image set of π

$$Im(\pi)_{|U} = \{\pi(\alpha) = \#_\pi \alpha,\ \ \alpha \in \Omega^1(U)\}$$

is an integral distribution called the Stefan distribution. If the restriction of metric to $Im(\pi)_{|U}$ is not degenerated, then $Im(\pi)_{|U} = \#_{\overline{g}}(\ker \pi)^\perp$.

We denote M the symplectic leaf such that $\forall x \in U \cap M\ \ T_xM = Im(\pi)(x)$.
Otherwise if the restriction of metric to $Im(\pi)(x)$ is degenerated, then we call (M, g, π) a lightlike symplectic leaf.

Proposition 2. *[5]. Let* $(\overline{M}, \overline{g}, \pi)$ *be a regular pseudo-Riemannian Poisson manifold. Let* D *be the Levi-Civita contravariant connection associated with* $(\overline{g}, \pi)$. *Then If* $\pi(\beta) = 0$ *then* $\forall \alpha \in \Omega^1(\overline{M}),\ \ \pi(D_\alpha \beta) = 0$.

Proof. Let $\alpha, \beta, \gamma \in \Omega^1(P)$ such that $\pi(\beta) = 0$. We have

$$\gamma(\pi(D_\alpha \beta)) = \pi(D_\alpha \beta, \gamma) = \pi(\alpha).\pi(\beta, \gamma) - \pi(D_\alpha \gamma, \beta) = 0 \ \Box$$

3. Lightlike submanifolds of Poisson manifold

The fundamental difference between the theory of lightlike(or degenerate) submanifolds M and the classical theory of submanifolds of pseudo-Riemannian comes from the fact that:

$$RadT_xM = T_xM \cap T_xM^\perp \neq \{0\}\ \ \forall x \in M. \quad (9)$$

is r-dimensional subspace. This relation is interpreted by

$$Rad\pi_x = Im(\pi_x) \cap Im(\pi_x)^\perp \neq \{0\}\ \ \forall x \in M. \quad (10)$$

which is r-dimensional subspace of T_xM.
Let M^m be a r-lightlike submanifolds of pseudo-Riemannian Poisson manifold $(\overline{M}^{m+n}, \overline{g}, \pi)$. According to (10), there exist four kinds of lightlike submanifolds :

- The proper r-lightlike submanifolds, where $0 < r < min(m, n)$. In this case, $Rad(\pi_x) \subsetneq Im(\pi_x)$ and $Rad(\pi_x) \subsetneq Im(\pi_x)^\perp$, where n is dimension of subspace $(Im(\pi_x))^\perp$.
- The coisotropic submanifolds, when $1 < r = n < m$.
  Then $Rad(\pi_x) = Im(\pi_x)^\perp \subsetneq Im(\pi_x)$.
- The isotropic submanifolds case, when $1 < r = m < n$
  Then, $Rad(\pi_x) = Im(\pi_x) \subsetneq Im(\pi_x)^\perp$.
- The totally lightlike submanifolds, when $1 < r = m = n$. Then $Rad(\pi_x) = Im(\pi_x) = Im(\pi_x)^\perp$.

Definition. Let $(\overline{M}, \overline{g}, \pi)$ be a pseudo-Riemannian Poisson manifold. The lightlike submanifold M of $\overline{M}$ is called lightlike symplectic leaf if for any open set $U \subset \overline{M}$, and $\forall x \in U \cap M$, $T_xM = Im(\pi_x)$

Remark. (1) $T_xM^\perp = (Im\pi_x)^\perp$ and $Rad(T_xM) = Rad(\pi_x)$

(2) In this way, we will consider the D as a $\mathcal{F}$-contravariant connection and give the correspondent covariant connection by

$$D_\alpha\beta = \overline{\nabla}_{\#_\pi\alpha}\beta \text{ for all } \alpha,\beta \in T^*\overline{M}$$

and it 's defined by $\forall X \in T\overline{M},\ (\overline{\nabla}_{\#_\pi\alpha}\beta)(X) = (\overline{\nabla}_{\#_\pi\alpha})\beta(X) - \beta(\overline{\nabla}_{\#_\pi\alpha}(X))$.

(3) The induced connection ∇ is given by $\pi(D_\alpha\beta) = \nabla_{\pi(\alpha)}\pi(\beta)$

Lemma 3. *[7] Let (M^{2n}, g_M) be a lightlike submanifold and let w_M be a symplectic form on M. Then we have*

$$J = \begin{cases} J_s, & \text{on } S(TM) \\ J_r, & \text{on } Rad(TM). \end{cases} \tag{11}$$

such that

$$w_M(X,Y) = g_M(J_sX,Y) \ \ \forall X,Y \in S(TM) \tag{12}$$

and

$$J_r(Rad(TM)) = Rad(TM) \tag{13}$$

Remark. Let $g_M = f^*\overline{g}$ be a lightlike metric on M and $\pi_M : T^*M \longrightarrow Im(\pi)$ be an isomorphism. Then we have

$$w_M(X,Y) = \pi(\pi^{-1}(X), \pi^{-1}(Y))$$

which is symplectic form on M. We have also

$$g_M(X,Y) =< \pi^{-1}(X), \pi^{-1}(Y) >$$

and $\nabla^M{}_Y X = \pi(D_\alpha\beta)_{/M}$ where $\pi(\alpha) = X$ and $\pi(\beta) = Y$. Then $\nabla^M g_M = 0$. In using the previous lemma, we have J such that $\nabla^M J = 0$.

Corollary 4. *[7] Let M be a r-proper lightlike or coisotrope lightlike symplectic leaf. Then M is a screen Kaehler manifold.*

4. SPLITTING POISSON MANIFOLDS

Proposition 5. *Let (M, g, w) be a lightlike Stefan submanifolds of pseudo-Riemannian Poisson manifold $(\overline{M}, \pi, \overline{g})$. Let U be an open set in M Then, according these splitting:*

$$T^*\overline{M}_{\setminus U} = T^*M \oplus tr(T^*M)_{\setminus U}, \tag{14}$$

with

$$\ker\pi = tr(T^*M) \tag{15}$$

Thus

$$T^*\overline{M}_{\setminus U} = S(T^*M) \oplus Rad(T^*M) \oplus tr(T^*M)_{\setminus U} \ \ \textit{and} \tag{16}$$

$$tr(T^*M) = S(T^*M^\perp) \perp ltr(T^*M) \tag{17}$$

Proof. Let U be an open set of M and let $\alpha \in \Gamma(T^*\overline{M}_{\setminus U})$, then we have $\alpha = \alpha_1 + \alpha_2$ where $\pi(\alpha_2) = 0$ and $\pi^{-1}(\pi(\alpha)) = \alpha_1$.
Thus $\alpha = \pi^{-1}(\pi(\alpha)) + (\alpha - \pi^{-1}(\pi(\alpha)))$, then $\pi^{-1}(\pi(\alpha)) \in T^*M$ and $(\alpha - \pi^{-1}(\pi(\alpha))) \in \ker \pi = tr(T^*M)$.
As $g_p(u,v) = \tilde{g}(\pi^{-1}(u), \pi^{-1}(v))$, then if $u \in Rad(TM)$, $\pi^{-1}(u) \in Rad(T^*M)$ and if $u \in S(TM)$ then $\pi^{-1}(u) \in S(T^*M)$. □

Remark. $S(T^*M)$, $S(T^*M^{\perp})$ are co-screen distribution, screen co-transversal vector bundle which is a complementary vector bundle of $Rad(T^*M)$ in $T^*M^{\perp}$ and $ltr(T^*M)$ is a lightlike co-transversal vector bundle. For any local dual basis $\{\xi_i^*\}$ of $Rad(T^*M)$ there exists a local dual frame $\{N_i^*\}$ of $ltr(T^*M)$ such that $\overline{g}(\xi_i^*, N_i^*) = \delta_{ij}$, $\overline{g}(N_j^*, N_i^*) = 0$ and $\overline{g}(W_j^*, N_i^*) = 0$ for any local dual frame $\{W_i^*\}$ of $S(T^*M)$.

This decomposition comes from the classic theorem of uncomplete basis in linear algebra. Indeed for any $\alpha \in \Gamma(T^*\overline{M}_{\setminus U})$ and any $\beta \in Rad(T^*M)$ we have

$$\begin{aligned}
\overline{g}(\beta, \alpha - \pi^{-1}(\pi(\alpha))) &= \overline{g}(\beta,\alpha) - \overline{g}(\beta, \pi^{-1}(\pi(\alpha)) \\
&= \overline{g}(\beta,\alpha) - \overline{g}(\pi(\beta), (\pi(\alpha)) \\
&= \overline{g}(\beta,\alpha)
\end{aligned}$$

Thus, $ltr(T^*M)_{\setminus U}$ can be construsted by supposing $\overline{g}(\beta,\alpha) = 1$. Then, $S(T^*M)_{\setminus U}$ is obtained by uncomplete basis theorem.
These relationships are homologous to those of Duggal and Bejancu in the case of the tangent bundle. Then

$$\begin{aligned}
T\overline{M}_{|M} &= TM \oplus tr(TM) \\
&= S(TM) \perp (Rad(TM) \oplus ltr(TM)) \perp S(TM^{\perp}).
\end{aligned} \tag{18}$$

Moreover

$$D_\alpha \beta = \pi^{-1}(\pi_*(D_\alpha \beta)) + h^*(\alpha, \beta) \tag{19}$$

Consider the relationship between contravariant and covariance connection D and $\overline{\nabla}$ on $\overline{M}$ respectively, as described in Remark 3, and ∇ induced Levi-Civita connection on M. Then, relation (18) and dual of relation (19) can be used for describing geometrical objects.
Thus

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \forall \ X, Y \in \Gamma(TM) \tag{20}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla_X^t V \ \ \forall\, X \in \Gamma(TM),\ V \in \Gamma(tr(TM)) \tag{21}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla_X^t V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively.
Suppose that $S(TM^{\perp}) \neq \{0\}$ and consider the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively.

$$\begin{aligned}
L : tr(TM) &\longrightarrow ltr(TM) \\
S : tr(TM) &\longrightarrow S(TM^{\perp}).
\end{aligned}$$

Then the relations (20) and (21) become

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \ \forall \ X, Y \in \Gamma(TM) \tag{22}$$

where $h^l(X,Y) = L(h(X,Y)), \quad h^s(X,Y) = S(h(X,Y)),$
and

$$\overline{\nabla}_X V = -A_V X + D^l_X V + D^s_X V, \ \forall\, X \in \Gamma(TM),\ V \in \Gamma(tr(TM)) \tag{23}$$

where $D^l_X V = L(\nabla^t_X V)$; $D^s_X V = S(\nabla^t_X V)$.

For any $X \in \Gamma(TM)$

$$\nabla^l_X : \Gamma(ltr(TM)) \longrightarrow \Gamma(ltr(TM)), \qquad \nabla^l_X(LV) = D^l_X(LV) \tag{24}$$

and

$$\nabla^s_X : \Gamma(S(TM^\perp)) \longrightarrow \Gamma(S(TM^\perp)), \qquad \nabla^s_X(SV) = D^s_X(SV) \tag{25}$$

for any $V \in \Gamma(tr(TM))$. Then we define:

$$D^l : \Gamma(TM) \times \Gamma(S(TM^\perp)) \longrightarrow \Gamma(ltr(TM)), \quad D^l(X, SV) = D^l_X(SV) \tag{26}$$

$$D^s : \Gamma(TM) \times \Gamma(ltr(TM)) \longrightarrow \Gamma(S(TM^\perp)), \quad D^s(X, LV) = D^s_X(LV) \tag{27}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$.
Thus the relation (23) becomes

$$\overline{\nabla}_X V = -A_V X + D^l(X, SV) + D^s(X, LV) + \nabla^l_X(LV) + \nabla^s_X(SV) \tag{28}$$

The above different geometry objects verify the follow relations:

Remark. The above different geometrical objects verify the follow relations:

$$\overline{g}(h^s(X,Y), W) + \overline{g}(Y, D^l(X,W)) = \overline{g}(A_W X, Y) \tag{29}$$

$$\overline{g}(h^l(X,Y), \xi) + \overline{g}(Y, h^l(X,\xi)) + \overline{g}(Y, \nabla_X \xi) = 0 \tag{30}$$

$$\overline{g}(W, D^s(X,N)) = \overline{g}(A_W X, N) \tag{31}$$

$$\overline{g}(A_N X, N') = \overline{g}(A_{N'} X, Y) \tag{32}$$

$$\overline{g}(A_N X, PY) = \overline{g}(N, \nabla_X PY) \tag{33}$$

$$h^l_i(X, \xi_j) = h^l_j(X, \xi_i) \tag{34}$$

where $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$, $\xi_i \in \Gamma(Rad(TM))$, $W \in \Gamma(S(TM^\perp))$ and h_i are such that

$$h_i(X,Y) = \overline{g}(\overline{\nabla}_X Y, \xi_i) \quad \forall X, Y \in \Gamma(TM). \tag{35}$$

Proposition 6. *Let $(M, S(TM), S(TM^\perp), g)$ be a lightlike submanifold of pseudo-Riemannian Poisson manifold, $(\overline{M}, \overline{g}, \pi)$. Then $Rad(TM)$ is a killing distribution.*

$$L_X g = 0 \ \ \forall X \in Rad(TM)$$

Proof. Let $Y, Z \in Im(\pi)$ and α, β such that $\pi(\alpha) = Y$ and $\pi(\beta) = Z$. On note γ, the 1-form such that $\pi(\gamma) = X$. Then we have

$$\begin{aligned} 0 = \overline{g}([Y, Z], X) &= \overline{g}([\pi(\alpha), \pi(\beta)], \pi(\gamma)) \\ &= \pi^*(\overline{g})([\alpha, \beta], \gamma) \\ &= g([\alpha, \beta], \gamma) \\ &= g(\alpha, D_\beta \gamma) - g(\beta, D_\alpha \gamma) \\ &= L_{\pi(\gamma)} g(\alpha, \beta). \end{aligned}$$ □

Proposition 7. *Let $(M, S(TM), S(TM^\perp), g)$ be a lightlike submanifold of pseudo-Riemannian Poisson manifold, $(\overline{M}, \overline{g}, \pi)$. Then*

$$A_W = 0, \quad A_N = 0, \quad D^s(., N) = 0, \quad \forall W \in S(TM^\perp), \forall N \in ltr(TM) \tag{36}$$

and $h^s = 0$ on $S(TM)$

Proof. Let $\beta \in tr(T^*M)$ such that $\pi(\beta) = 0$ then $\pi(D_\alpha \beta) = 0$.
Thus if $W \in \Gamma(S(TM^\perp))$ or $N \in \Gamma(ltr(TM))$ then $A_W = 0$ or $A_N = 0$ and $D^s(., N) = 0$ according to relation (28). Moreover according to relation (29) we have $\overline{g}(h^s(X, PU), W) = 0\ \forall W \in S(TM^\perp)$, then $h^s(X, PU) = 0$;
$U \in \Gamma(TM)$ and P denote the projection morphism of $\Gamma(TM)$ dans $\Gamma(S(TM))$. □
Denote by $\overline{R}, R, R^l$ and R^s curvature tensors of $\overline{\nabla}, \nabla, \nabla^l$ and ∇^s respectively.

Proposition 8. *According to relations (3.9), (3.10) and (3.11) in [1, pg 172, chap 5], we have*

$$\begin{aligned} \overline{R}(X, Y)Z &= R(X, Y)Z + D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z)) \\ &\quad + (\nabla_X h^s)(Y, Z) - (\nabla_Y h^s)(X, Z) \end{aligned} \tag{37}$$

$$\overline{R}(X, Y)N = R^l(X, Y)N \tag{38}$$

$$\overline{R}(X, Y)W = R^s(X, Y)W + (\overline{\nabla}_X D^l)(Y, W) - (\overline{\nabla}_Y D^l)(X, W) \tag{39}$$

Moreover

$$< \overline{R}(X, Y)PU, PV > \; = \; < R(X, Y)PU, PV > \tag{40}$$

$$< \overline{R}(X, Y)\xi, N > \; = \; < R(X, Y)\xi, N > \tag{41}$$

For any $X, Y, Z, U, V \in \Gamma(TM)$; $\xi \in Rad(TM)$; $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$.

Theorem 9. *Let $f : M \longrightarrow \overline{M}$ be an immersion isometric of lightlike symplectic leaf (M, g, π) into a pseudo-Riemannian Poisson manifold, $(\overline{M}, \overline{g}, \pi)$. Then:*

(1) *(M, g, π) is totally geodesic if and only if $D^l(., W) = 0$,*
(2) *$S(TM)$ and $Rad(TM)$ are integrable*
(3) *$M = M' \times R$ where M' is non degenerate Kaehler manifold and R is a radical submanifold.*

Proof.

(1) As an induced connection is metric, then $h^l = 0$ and with the previous lemma 7, we have $A_W = 0\ \forall W \in \Gamma(S(TM^\perp))$. Thus $h^s = 0$ if and only if $D^l(., W) = 0$.
(2) Using Proposition 7, the relation (29) and results in [1].

(3) For this point, we use corollary 4 and proposition 6.

□

Corollary 10. *Let $f : M \longrightarrow \overline{M}$ be an immersion isometric of coisotrope or totally lightlike symplectic leaf (M, g, π) into a pseudo-Riemannian Poisson manifold, $(\overline{M}, \overline{g}, \pi)$. Then (M, g, π) is totally geodesic.*

Theorem 11. *Let $f : M \longrightarrow \overline{M}$ be an immersion isometric of coisotrope or totally lightlike symplectic leaf (M, g, π) into a pseudo-Riemannian Poisson manifold, $(\overline{M}, \overline{g}, \pi)$. Then $M = M' \times G$ where M' is non degenerate Kaehler manifold and G is a Lie group which we call Radical Lie Group.*

Lemma 12. *Let $f : M \longrightarrow \overline{M}$ be an immersion isometric of coisotrope or totally lightlike symplectic leaf (M, g, π) into a pseudo-Riemannian Poisson manifold, $(\overline{M}, \overline{g}, \pi)$. Then the dual G^* of isotropy algebra $G = \ker \pi$ is a lie algebra.*

Proof. Let us recall the definition of bracket as in [4]. Let U be a regular open set on $\overline{M}$ and $\alpha, \beta \in \ker \pi_{|U}$. Denote $\tilde{\alpha}, \tilde{\beta} \in \Omega^1(\overline{M})$ such that $\alpha = \tilde{\alpha}_{|U}$ and $\beta = \tilde{\beta}_{|U}$. Then we have

$$[\alpha, \beta] = d(\pi(\tilde{\alpha}, \tilde{\beta}))_{|U} \tag{42}$$

Thus $(G = \ker \pi, [.,.])$ is Lie algebra called isotropy algebra.
Moreover G^* is Lie coalgebra. Indeed its bracket is defined by;

$$\alpha([X, Y]) = d\alpha(X \wedge Y); \quad X, Y \in G^*; \text{ and } \alpha \in G \tag{43}$$

and verify Jacobi Identity. For any $X, Y, Z \in G^*$

$$\begin{aligned} d^2\alpha(X \wedge Y \wedge Z) &= \frac{1}{3} d^2\alpha(X \wedge Y \wedge Z + Y \wedge Z \wedge X + Z \wedge X \wedge Y) \\ &= \frac{1}{3} d\alpha([X, Y] \wedge Z + [Y, Z] \wedge X + [Z, X] \wedge Y) \\ &= \frac{1}{3} \alpha([[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]) \end{aligned}$$

Since $d^2\alpha = 0$, then for any $\alpha \in G$ and $X, Y, Z \in G^*$,

$$\alpha([[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]) = 0$$

Thus

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \text{ (Jacobi Identity)} \quad \square$$

Proof of theorem 11 (M, g, π) is coisotrope lightlike submanifold then $S(TM^{\perp}) = \{0\}$; $S(TM^{\perp})^* = \{0\}$ and $\ker \pi_{|O} = G = ltr(TM)^*$ where O is an open set on M.
Let $\{N_1, N_2, \ldots, N_p\}$ and $\{\xi_1, \xi_2, \ldots, \xi_p\}$ be local cordinates of $ltr(TM)$ and $Rad(TM)$ respectively such that $\overline{g}(N_j, \xi_i) = \delta_{ij}$.
We can defined $\{\theta_i = \overline{g}(\xi_i, .) = \xi^{\flat_{\overline{g}}}\}$ local basis of $ltr(TM)^* = \ker \pi_{|O}$ and reciproquevely $\{\xi_i = \theta_i^{\sharp_{\overline{g}}}\}$ where $\flat_{\overline{g}}$; $\sharp_{\overline{g}}$ are musical isomorphism obtained by nondegenerate pseudo-Riemanniann metric. Thus $Rad(TM)$ is Lie algebra according to lemma 12 and with the theorem9 we conclude the proof. □

Remark. Note that it been proved that for every Lie algebra, there exists one and only one simply connected Lie group [10].
This result introduce thus the notion of local G-transformation on $M^{'}$ in lightlike caracteristic submanifolds of Poisson Manifold. Moreover this result show that Poisson manifold is locally a triple product manifolds

5. EXAMPLES

Totally lightlike Poisson leaf
Let $(\mathbb{R}^4, \overline{g}, \overline{\pi})$ be a pseudo-Riemannian Poisson manifold where

$$\overline{g} = -dx^2 - dy^2 + dz^2 + dt^2$$

and

$$\overline{\pi} = \eta(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \sin u \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \cos u \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial t} - \cos u \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \sin u \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t}).$$

We have

$$\begin{aligned} V_1 = \overline{\pi}(dx, .) &= \frac{\partial}{\partial y} + \sin u \frac{\partial}{\partial z} + \cos u \frac{\partial}{\partial t} \\ V_2 = \overline{\pi}(dy, .) &= -\frac{\partial}{\partial x} - \cos u \frac{\partial}{\partial z} + \sin u \frac{\partial}{\partial t} \\ V_3 = \overline{\pi}(dz, .) &= -\sin u \frac{\partial}{\partial x} + \cos u \frac{\partial}{\partial y} + \frac{\partial}{\partial t} \\ V_4 = \overline{\pi}(dt, .) &= -\cos u \frac{\partial}{\partial x} - \sin u \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \end{aligned}$$

and

$$\begin{aligned} V_3 &= \cos u V_1 + \sin u V_2 \\ V_4 &= -\sin u V_1 + \cos u V_2. \end{aligned}$$

Then

$$(Im(\overline{\pi}))^{\perp_g} = Im(\overline{\pi}) = span\{\xi_1, \xi_2\}$$

where

$$\begin{aligned} \xi_1 &= \frac{\partial}{\partial x} + \cos u \frac{\partial}{\partial z} - \sin u \frac{\partial}{\partial t} \\ \xi_2 &= \frac{\partial}{\partial y} + \sin u \frac{\partial}{\partial z} + \cos u \frac{\partial}{\partial t} \end{aligned}$$

The distribution $(Im\pi) = span\{\xi_1, \xi_2\}$ is integral, then there exists a totally lightlike surface S as a leaf of $\mathbb{R}^4$ such that

$$\forall x \in S, \qquad T_x S = Im(\pi_x).$$

An example of S can be given as the immersion isometric defined by

$$\begin{array}{rcl} f: S & \longrightarrow & \mathbb{R}^4_2 \\ (a,b) & \longmapsto & (a,b,F(a,b),G(a,b)) \end{array}$$

where

$$F(a,b) = a\cos u + b\sin u \qquad G(a,b) = -a\sin u + b\cos u$$

.

The construction of lightlike transversal vector bundle $ltr(TS)$ gives :

$$ltr(TS) = span\{N_1, N_2\}$$

where

$$N_1 = -\frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2}\cos u\frac{\partial}{\partial z} - \frac{1}{2}\sin u\frac{\partial}{\partial t}; \quad N_2 = -\frac{1}{2}\frac{\partial}{\partial y} + \frac{1}{2}\sin u\frac{\partial}{\partial z} - \frac{1}{2}\cos u\frac{\partial}{\partial t}$$

The isotropy Lie algebra which is the dual of lightlike tranversal vector bundle is given by

$$ltr(TS)^* = \ker\pi = span\{\theta_1, \theta_2\}$$

where

$$\theta_1 = -dx + \cos u dz - \sin u dt; \quad \theta_2 = -dy + \sin u dz - \cos u dt$$

Thus h^l and h^s are vanishing.
Moreover we have $[\theta_1, \theta_2]_\pi = 0$ and $[\xi_1, \xi_2] = 0$

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